Discussion Review Problems

1. Let V be a vector space, and let $U, W \subset V$ be subspaces. Prove that $(U+W)/(U\cap W) = U/(U\cap W) \oplus W/(U\cap W)$.

2. Let $f, g \neq 0 \in V^*$ be linear functionals on the n dimensional vector space V. Prove that $g = \lambda f$ for some $\lambda \neq 0 \in F$ if and only if $\ker(f) = \ker(g)$.

3. Let $T : V \to V$ be a linear operator on the n dimensional vector space V, and let $W \subset V$ be a T-invariant subspace. We say that W is T-*irreducible* if the only T-invariant subspaces of W and $\{0\}$ and W. Prove that if W is T-irreducible and $f \in \mathcal{L}(W)$ satisfies $f \circ T = T \circ f$, then $f = 0$ or f is an isomorphism.

4. Let $T : V \to V$ be a linear operator on the n dimensional vector space V. We say that T is *nilpotent* if there is a non-negative integer N such that $T^N = 0$.

   (a) Prove that if T is nilpotent, then $T^n = 0$.

   (b) Prove that if T is nilpotent and also diagonalizable, then $T = 0$.

5. Let $T : V \to V$ be an invertible linear operator on the n dimensional vector space V. Prove that $T^{-1} = g(T)$ for some polynomial $g(x) \in F[x]$.

6. Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Consider the linear operator $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(v) = Av$. Compute the minimal polynomial of T.

Solutions

1. Let $x \in (U+W)/(U\cap W)$. Then $x = (u+w)+(U\cap W)$ for some $u \in U$ and $w \in W$, so $x = (u+U\cap W)+(w+U\cap W)$, which says that $(U+W)/(U\cap W) = U/(U\cap W)+W/(U\cap W)$. Now let $x \in U/(U\cap W)\cap W/(U\cap W)$. Then $x = u+(U\cap W) = w+(U\cap W)$ for some $u \in U$ and $w \in W$. This means that $u-w \in U\cap W$, and since $U\cap W$ is a subspace of W, this means that $u \in W$ as well. Thus, we have $x = u+(U\cap W) = U\cap W$. This says their intersection is trivial, so the sum is direct and we conclude that $(U+W)/(U\cap W) = U/(U\cap W)\oplus W/(U\cap W)$ as desired.

2. By rank-nullity, we see that $\dim(\ker(f)) = n-1$ because f is non-zero. Let $\{v_1,\ldots,v_{n-1}\}$ be a basis of $\ker(f)$ and extend this to a basis $\{v_1,\ldots,v_{n-1},v\}$ of V. By definition, we must have $f(v)\neq 0$.

   First, suppose that $g = \lambda f$ for some $\lambda \in F$. Then for each v_i, we have $g(v_i) = \lambda f(v_i) = 0$, so $\ker(f)\subset\ker(g)$. If $x \in \ker(g)\setminus\ker(f)$, write $x = c_1v_1+\ldots+c_{n-1}v_{n-1}+c_nv$ for some $c_i \in F$. Necessarily, we must have $c_n \neq 0$. We then have $f(x) = c_nf(v)$, so $0 = \lambda f(v)$ says $\lambda = 0$, a contradiction. Therefore, $\ker(f) = \ker(g)$. Now, assume that $\ker(f) = \ker(g)$. Then by definition, $\mathrm{Im}(f) = \mathrm{Span}\{f(v)\}$ and $\mathrm{Im}(g) = \mathrm{Span}\{g(v)\}$. Define $\lambda = g(v)/f(v)$ which is non-zero by assumption, so that $\lambda f = g$ since we need only check they are equal on v.

3. Let $x \in \ker(f)$. Then $f(T(x)) = T(f(x)) = T(0) = 0$ by assumption, which says that $T(x)\in\ker(f)$, i.e. $\ker(f)$ is a T-invariant subspace of W. Since W is T-irreducible, we must have $\ker(f) = 0$ or $\ker(f) = W$ by definition. The former says f is an isomorphism, while the latter says that $f = 0$.

4. (a) Since $T^N = 0$ for some N, we see that x^N kills T. Therefore, the minimal polynomial of T is of the form x^k for some $1\leq k\leq N$. Since the minimal polynomial and characteristic polynomial share the same roots, and the characteristic polynomial is of degree n, this forces $c_T(x) = x^n$. The Cayley-Hamilton theorem then says that $T^n = 0$.

   (b) If T is nilpotent and diagonalizable, then the minimal polynomial splits into distinct linear factors. Since the above showed that $m_T(x) = x^k$ for some k, this forces $k = 1$ so that $m_T(x) = x$. This then tells us that $T = 0$ as desired.

5. Let $c_T(x) = a_0+\ldots+x^n$ be the characteristic polynomial of T. By the Cayley-Hamilton theorem, we have $c_T(T) = 0$, so $a_0I_V+\ldots+T^n = 0$. This says $a_0I_V = -(a_1T+\ldots+a_{n-1}T^{n-1}+T^n) = T(-a_1I_V-\ldots-a_{n-1}T^{n-2}-T^{n-1})$. Since a_0 is given by the determinant of T (up to sign) and T is invertible, we see that $a_0\neq 0$. Therefore, $I_V = T(-\dfrac{1}{a_0}(a_1I_V+\ldots+a_{n-1}T^{n-2}+T^{n-1})$, so that T^{-1} is given by $g(T)$ where $g(x) = -\dfrac{1}{a_0}(a_1+\ldots+a_{n-1}x^{n-2}+x^{n-1})$.

6. One can check that $c_T(x) = x^2(x-3)$, so that T has eigenvalues 0 and 3. Note that since $E_0 = \ker(T)$ and $\mathrm{rank}(T) = 1$, we have $\dim E_0 = 2$ so this forces $\mathbb{R}^3 = E_0\oplus E_3$. This tells us T is diagonalizable, so the minimal polynomial is obtained by taking a single linear factor corresponding to each eigenvalue, giving $m(x) = x(x-3)$.