Jordan Forms

$T: V \to V$

if c_T splits, can find basis β with

$$[T]_\beta = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_k \end{pmatrix}$$

Jordan Form

$$J_i = \begin{pmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{pmatrix}$$

Jordan blocks

linear operators are classified up to conjugacy by JCF.

Over $\mathbb{C}$, c_T always splits so JCF will always exist.

Recap of Theory:

$$K_\lambda = \{ x \in V : (T-\lambda I)^p x = 0 \text{ for some } p > 0 \}$$

generalized eigenspace of λ

elements are called generalized eigenvectors for λ.

Let $x \in K_\lambda$ and suppose that p is the smallest integer such that $(T-\lambda I)^p x = 0$.

Then $(T-\lambda I)^{p-1} x \neq 0$.

$$(T-\lambda I)\left((T-\lambda I)^{p-1} x\right) = 0$$

$$= T\left((T-\lambda I)^{p-1} x\right) - \lambda (T-\lambda I)^{p-1} x = 0$$

$$\Rightarrow T\left((T-\lambda I)^{p-1} x\right) = \lambda (T-\lambda I)^{p-1} x$$

So $(T-\lambda I)^{p-1} x$ is an eigenvector of eigenvalue λ.

$$\left\{ (T-\lambda I)^{p-1} x, \ldots, (T-\lambda I) x, x \right\}$$

is called a <u>cycle</u>

$(T-\lambda I)^{p-1} x$ is called initial vector

$(T-\lambda I)x$ is called final vector.

Each cycle contains precisely one eigenvector (namely, $(T-\lambda I)^{p-1}x$)

Suppose

$$\gamma = \{(T-\lambda I)^{p-1}x, \ldots, x\}$$

is a cycle. $= \{v_1, \ldots, v_p\}$

$W = \mathrm{Span}(\gamma)$

Then W is T-invariant.

why?

$$T(v_1) = \lambda v_1$$

$$(T-\lambda I) v_i = (T-\lambda I)^{p-i+1} x$$

$$= (T-\lambda I)^{p-(i-1)} x = v_{i-1}$$

$$T v_i = \lambda v_i + v_{i-1}$$

$$[T_W] = \begin{pmatrix} \lambda & 1 & & \\ 0 & \lambda & 1 & \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & & \lambda \end{pmatrix}$$

<u>Thm</u>: Suppose that c_T splits.

Then there is a basis

$B = B_1 \cup B_2 \cup \cdots \cup B_k$

$\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$

β_i are bases of K_{λ_i}

β_i consists of disjoint cycles for λ_i.

Then $[T]_\beta = JCF(T)$.

$$\beta_i = \{ \underbrace{(T-\lambda I)^2 y, (T-\lambda I) y, y}_{\text{length } 3}, \underbrace{(T-\lambda I) x, x}_{\text{length } 2} \}$$

$$\left[\begin{array}{ccc|cc} \lambda & 1 & 0 & & \\ 0 & \lambda & 1 & & \\ 0 & 0 & \lambda & & \\ \hline & & & \lambda & 1 \\ & & & 0 & \lambda \end{array}\right]$$

Jordan blocks

$\updownarrow$

Cycles

Computing JCF

$\lambda_1, \ldots, \lambda_k$ eigenvalues

1.) $\dim(K_\lambda) = \text{alg. mult}(\lambda)$

$K_\lambda = \ker((T-\lambda I)^m)$ $m =$

$\Rightarrow$ $\boxed{\text{Sum of sizes of blocks} = \dim(K_\lambda) = \text{alg. mult}(\lambda)}$

2.) If $JCF(T)$

$$= \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_l \end{pmatrix}$$

Suppose

$J_1, \ldots, J_q$ are blocks for λ.

Then $\dim \ker(J_i - \lambda) =$

$\text{size}(J_i) - 1$.

$$T - \lambda I = \begin{pmatrix} J_1 - \lambda I & & \\ & \ddots & \\ & & J_\ell - \lambda I \end{pmatrix}$$

$\Rightarrow \text{rank}(T - \lambda I) = \dim V - \#$ zero columns

$= \dim V - q$

$\Rightarrow \dim \ker(T - \lambda I) = \dim(E_\lambda)$

$= gm(\lambda) = q$.

$$\boxed{\#\text{ blocks for } \lambda = gm(\lambda)}$$

3.) $m_T \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_\ell \end{pmatrix}$

$$= \begin{pmatrix} m_T(J_1) & & \\ & \ddots & \\ & & m_T(J_\ell) \end{pmatrix}$$

$m_T = (x-\lambda_1)^{e_1} \cdots (x-\lambda_k)^{e_k}$

each J_i has min poly.

$(x-\lambda_i)^{\text{size } J_i}$

So $\boxed{e_i \quad \max \{ \text{sizes of blocks for } \lambda_i \} \\ m_T = \prod (x-\lambda_i)^{e_i}}$

Ex: $T: \mathbb{R}^3 \to \mathbb{R}^3$ $\beta = \{e_1, e_2, e_3\}$

$$[T]_\beta = \begin{pmatrix} 3 & 1 & -2 \\ -1 & 0 & 5 \\ -1 & -1 & 4 \end{pmatrix}$$

$$C_T(x) = -(x-3)(x-2)^2$$

$am(3) = 1 \Rightarrow$ one block of size 1 for $\lambda = 3$

$am(2) = 2$

$\Rightarrow$ 2×2 block

or

2 1×1 blocks

to get #, compute $gm(2)$.

$\dim(E_2) = gm(2) = 1$.

So there is one block for $\lambda = 2$

$$JCF(T) = \begin{pmatrix} \boxed{3} & & \\ & 2 & 1 \\ & 0 & 2 \end{pmatrix}$$

(up to permutation of blocks

Can read off m_T:

$c_T = m_T$ in this case.

Ex: $c_T(x) = (x-2)^3 (x-3)^3$

$m_T(x) = (x-2)^2 (x-3)$

what is $JCF(T)$?

largest block size for $\lambda = 2$ is 2

largest block size for $\lambda = 3$ is 1

Sum of sizes $\lambda = 2$ $= 3$

Sum of sizes $\lambda = 3$ $= 3$

$\Rightarrow$ 2×2 block, 1×1 block $\lambda = 2$

$\Rightarrow$ three 1×1 blocks $\lambda = 3$

$$\begin{pmatrix} 2 & 1 & & & & & \\ 0 & 2 & & & & & \\ & & 2 & & & & \\ & & & 3 & & & \\ & & & & 3 & & \\ & & & & & 3 \end{pmatrix}$$

up to permutation of blocks.

$$\begin{pmatrix} 2 & 1 & & & & & & & & \\ & 2 & 1 & & & & & & & \\ & & 2 & & & & & & & \\ & & & 2 & 1 & & & & & \\ & & & 0 & 2 & & & & & \\ & & & & & 3 & 1 & & & \\ & & & & & 0 & 3 & & & \\ & & & & & & & 4 & & \\ & & & & & & & & 5 \end{pmatrix}$$

$$m_T(x) = (x-2)^3(x-3)^2(x-4)(x-5)$$

What if $c_T(x) = (x-2)^4(x-3)^4$

$m_T(x) = (x-2)^2(x-3)^2$?

Sum of sizes for $\lambda = 2$ $= 4$

have a 2×2 block

$\Rightarrow$ two 2×2 blocks OR

2×2 block, two 1×1 block.

Same for $\lambda = 3$.

4 possible canonical forms up to permutation:

$$\begin{pmatrix} 2 & 1 & & & & & & \\ & 2 & & & & & & \\ & & 2 & 1 & & & & \\ & & & 2 & & & & \\ & & & & 3 & 1 & & \\ & & & & & 3 & & \\ & & & & & & 3 & 1 \\ & & & & & & & 3 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & & & & & & \\ & 2 & & & & & & \\ & & 2 & 1 & & & & \\ & & & 2 & & & & \\ & & & & 3 & 1 & & \\ & & & & & 3 & & \\ & & & & & & 3 & \\ & & & & & & & 3 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & & & & & & \\ & 2 & & & & & & \\ & & 2 & & & & & \\ & & & 2 & & & & \\ & & & & 3 & 1 & & \\ & & & & & 3 & & \\ & & & & & & 3 & 1 \\ & & & & & & & 3 \end{pmatrix} \quad \begin{pmatrix} 2 & 1 & & & & & & \\ & 2 & & & & & & \\ & & 2 & & & & & \\ & & & 2 & & & & \\ & & & & 3 & 1 & & \\ & & & & & 3 & & \\ & & & & & & 3 & \\ & & & & & & & 3 \end{pmatrix}$$