V w/ countably infinite basis.
For each $i \geq 1$ B_i be a basis of
V. Show V has a basis
$\{w_i\}$ with $w_i \in B_i$ for all $i \geq 1$.

This is false!

$V = F[x]$

$B_i = \{1+x^i, x+x^i, \dots\}$

Claim: it's not possible to form
a basis for V with a vector
from each B_i!

Suppose otherwise: $B = \{w_i\}$
be such a basis.
then $w_i = x^{a_i} + x^{b_i}$ for some

a_i, b_i.

Suppose

$$1 = \sum_{i=1}^{n} c_i \left(x^{a_i} + x^{b_i} \right)$$ for some c_i, all $c_i \neq 0$.

Note that one of the exponents that appears in the sum must be 0: otherwise, plug in $x=0$ to RHS to get $1=0$.

Also note if an exponent appears in one term, it must appear in another in order to cancel out.

How many possible exponents can appear in the sum?

$\leq \frac{2n-1}{2}$ distinct non-zero exponents.

$\frac{2n-1}{2} \leq n-1$, so $\leq n-1$ non-zero exponents that can appear.

Each term in sum comes from B_{a_i} or B_{b_i}

Pidgeon hole says there are two terms in the sum coming from same basis

$\Rightarrow \Leftarrow$

5.

$T: V \to V$ $\quad \dim_{\mathbb{C}} V = 6$

$W_1, W_2 \subset V$

$\dim_{\mathbb{C}} W_1 = \dim_{\mathbb{C}} W_2 = 4$

$\dim_{\mathbb{C}} W_1 \cap W_2 = 2.$

$m_{T_1} = (x-1)(x-2)(x-3)$

$m_{T_2} = x(x-1)$

a.) $m_{T_Z} \mid m_{T_1}$ because $Z \subset W_1$

$m_{T_Z} \mid m_{T_2}$ $\quad Z \subset W_2$

$\Longrightarrow$

$m_{T_Z} \mid (x-1) \Longrightarrow$

$m_{T_Z} = x-1.$

$\Rightarrow \quad T_2 - I = 0$

$\Rightarrow \quad T_2 = I$

b.1 $\quad m_{T_1} \mid m_T \qquad W_1 \subset V$

$m_{T_2} \mid m_T \qquad W_2 \subset V$

$\Rightarrow \quad \underline{x(x-1)(x-2)(x-3)} \mid m_T$

Note that $W_1 + W_2 = V$

for dimension reasons.

$x(x-1)(x-2)(x-3)$ kills

everything in W_1 and

W_2 b/c it contains

m_{T_1} and m_{T_2} as factors

$\Longrightarrow$ kills everything in V

$\Longrightarrow$ $m_T = x(x-1)(x-2)(x-3)$.

(c) What are possibilities for c_T?

$m_Z = x-1$ $\quad \dim_{\mathbb{Q}} Z = 2$

$\Longrightarrow$ $c_Z = (x-1)^2$

$c_Z \mid c_{T_1}$

$$C_Z \mid C_{T_2}$$

$$\Rightarrow \quad C_{T_1} = (x-1)^2(x-2)(x-3)$$

b/c C_{T_1} has degree 4

Possibilities for C_{T_2}:

$$x^2(x-1)^2 \quad \text{OR} \quad x(x-1)^3$$

$$C_{T_1} \mid C_T$$

$$C_{T_2} \mid C_T$$

$$\Rightarrow \quad c_T = x^2(x-1)^2(x-2)(x-3)$$

OR

$$c_T = x(x-1)^3(x-2)(x-3)$$

d.) $m_T = x(x-1)(x-2)(x-3)$

has distinct linear factors

$\Rightarrow$ T diagonalizable.

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4. $V = \text{Span}\{v, Tv, \dots\}$

$V = \text{Span}\{Tv, \dots\}$

$\Leftrightarrow$ T an iso.

Note: inf. dimensional case doesn't happen!

$$v = c_1 T v + \dots + c_k T^k v$$

for some c_i:

So $T^k v \in \operatorname{Span}\{v, Tv, \dots, T^{k-1} v\}$

$$V = \operatorname{Span}\{v, Tv, \dots, T^{k-1} v\}$$

Inner Products and Dual Space

V f.d. inner prod space over $\mathbb{R}$.

There is a canonical iso.

$$\Phi : V \xrightarrow{\sim} V^*$$

$$v \longmapsto \langle -, v \rangle$$

Recall $W < V$

$W^0 < V^*$ $\quad W^0 =$ Annihilator

$$W^0 = \{ f \in V^* : f(w) = 0 \text{ for all } w \in W \}$$

under the above iso.

$$f \longleftrightarrow \langle -, v_f \rangle$$

$$f(w) = \langle w, v_f \rangle$$

$$f(w) = 0 \text{ for all } w \in W$$

$$\Longleftrightarrow$$

$$\langle w, v_f \rangle = 0 \text{ for all } w \in W$$

$\Longleftrightarrow\ v_f \in W^\perp$.

So

$$W^\perp \longleftrightarrow W^0$$

under the iso.

How does dual map and adjoint relate?

$T: V \longrightarrow W$ (V, W fd IPS)

$T^{adj}: W \longrightarrow V$

$$\langle T(x), y\rangle_W = \langle x, T^{adj}(y)\rangle_V$$

$$\begin{tikzcd}
W \arrow[r, "T^{adj}"] \arrow[d, "\Phi"'] \arrow[dr] & V \arrow[d, "\Phi"] \\
W^* \arrow[r, "T^*"'] & V^*
\end{tikzcd}$$

<u>Thm</u>: This diagram commutes

i.e. $\Phi \circ T^{adj} = T^* \circ \Phi$

Proof:

Pick $w \in W$.

$$(T^* \circ \Phi)(w) = T^*(\Phi(w))$$

$$\Phi(w) = f_w \qquad f_w(-) = \langle -, w \rangle$$

$$T^*(f_w) = f_w \circ T$$

What does this do to a vector v?

$$(f_w \circ T)(v) = f_w(T(v))$$

$$= \boxed{\langle T(v), w \rangle_W}$$

$(\Phi \circ T^{adj})(\omega)$

$= \Phi(T^{adj}(\omega))$

$= \langle -, T^{adj}(\omega) \rangle_V$

What does this do to a vector v?

$\boxed{\langle v, T^{adj}(\omega) \rangle_V}$

Red boxes are equal by def.

$\Rightarrow (T^* \circ \Phi)(w) = (\Phi \circ T^{adj})(w)$

as functionals, b/c agree on arbitrary $v \in V$.

$$\Rightarrow \quad T^* \circ \Phi = \Phi \circ T^{adj} \quad \boxtimes$$

Can solve for say, T^{adj}:

$$T^{adj} = \Phi^{-1} \circ T^* \circ \Phi$$

Explicitly, if

β is an o.n. basis of V

γ is an o.n. basis of W

β^* dual bases
γ^*

We'll recover the fact that

$$[T^{adj}]_\gamma^\beta = \left([T]_\beta^\gamma\right)^t$$

$$[T^*]_{\gamma^*}^{\beta^*} = \cancel{=}$$

None of this works over $\mathbb{C}$.

Complex inner product spaces

are weird!