HW 2:

- Lots of people didn't prove things were l.i. or bases - please do so!

HW 3:

#1: $\det\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \det(A)\det(D)$

$$\left(\begin{array}{cccc|cccc}
a_{11} & a_{12} & \cdots & a_{1k} & B_{11} & \cdots & & B_{1\,n-k} \\
\vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\
a_{k1} & & \cdots & a_{kk} & B_{n-k\,1} & & \cdots & B_{n-k\,n-k} \\
\hline
0 & 0 & \cdots & 0 & d_{1\,\cancel{n-k}} & & \cdots & d_{1\,n-k} \\
\vdots & & \ddots & \vdots & \vdots & & & \\
0 & & \cdots & 0 & d_{k1} & & \cdots & d_{k\,n-k}
\end{array}\right)$$

$$a_{11}\det(A_{11})\det(D) - a_{21}\det(A_{21})\det(D)$$

$$+ \dots$$

$$\frac{(a_{11}\det(A_{11}) - a_{21}\det(A_{21}) + a_{31}\det(A_{31})}{\dots \quad \det(D)}$$

$$= \det(A)\det(D)$$

Cyclic Subspaces, C-H, minimal polys:

$T: V \to V \qquad W \subset V$

- W is <u>T-invariant</u> if

$$T(W) \subset W$$

- W is <u>T-cyclic</u> if

$W = \text{Span}\{v, Tv, T^2v, \ldots\}$

for some $v \in V$.

Let's assume that V is finite dimensional.

T_W is restriction of T to W, and this makes sense when W is T-invariant.

If $V = W_1 \oplus \cdots \oplus W_k$
with all W_i T-invariant

$C_T(x) = C_{T_{W_1}}(x) \cdots C_{T_{W_k}}(x)$

<u>Thm:</u> (Cayley-Hamilton Thm)

if V fin. dim. $T: V \to V$

$C_T(T) = 0.$

W = T-cyclic Subspace of V of dimension k.

$W \overset{\text{Span}}{=} \{ v, T_W(v), T_W^2(v), \ldots, T_W^{k-1}(v) \}.$

$= \{ v_1, v_2, v_3, \ldots, v_k \}$

$= \quad \beta$

What does $[T]_\beta$ look like?

$$T(v_1) = v_2$$
$$T(v_2) = v_3$$
$$T(v_3) = v_4$$
$$\vdots$$
$$T(v_k) = \;?$$

$$T(v_k) = T(T^{k-1}(v))$$
$$= T^k(v)$$

$$C_{T_W} = a_0 + a_1 x + a_2 x^2 + \dots + a_k x^k$$

C-H Says that

$$a_0 I + a_1 T + \dots + a_k T^k = 0$$

$$a_k T^k = -a_0 I - a_1 T - \dots - a_{k-1} T^{k-1}$$

Char. poly is <u>monic</u> so $a_k = 1$

$$T^k = -a_0 I - a_1 T \dots - a_{k-1} T^{k-1}$$

$\Rightarrow$

$$T^k(v) = -a_0 v - a_1 T(v) - \dots - a_{k-1} T^{k-1}(v)$$

$$= -a_0 v_1 - a_1 v_2 - \dots - a_{k-1} v_k$$

$$[T]_\beta = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & \vdots & -a_1 \\ 0 & 1 & \cdots & & \vdots \\ \vdots & \vdots & & 0 & \vdots \\ 0 & 0 & & 1 & -a_{k-1} \end{bmatrix}$$

Since char poly. of matrix is same as that as operator in any basis, we've constructed a matrix w/ char poly

$$a_0 + a_1 x + \dots + a_{k-1} x^{k-1} + x^k$$

The above matrix is called a

<u>Companion matrix</u>

Can combine this with earlier result about invariant decompositions to easily construct matrices with specific char. poly.

<u>Ex:</u> 2x2 matrix

A s.t. $A^3 - 3A^2 + 2I = 0$

$x^3 - 3x^2 + 2$

$= (x-1)\boxed{(x^2-2x-2)}$

We'll find A s.t.
Second factor kills A.

$$A = \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix}$$

Minimal Polynomial

$$S = \{ p(x) \in F[x] : p(T) = 0 \}$$

By WOP, S has an element of minimal degree. If we require monic, it's also unique.

This poly is called the <u>minimal poly.</u> of T.

Key Properties:

1. $p(T) = 0$

$\iff m_T \mid p(T)$

In particular, $m_T \mid C_T$.

2. m_T and C_T have same irred factors.

In particular, roots of m_T are your

eigenvalues.

<u>3.</u> T diagonalizable

$\iff m_T$ splits

How to compute

m_T?

<u>Prop</u>: if W is T-inv.

$m_{T_W} \mid m_T$.

Prop: Min. poly of Companion matrix = Char poly.

Ex: $T: \mathbb{R}^4 \to \mathbb{R}^4$

w/ $m_T = (x-7)^2(x-8)^2$

A

$$\left(\begin{array}{cc|cc} 0 & -49 & 0 & 0 \\ 1 & 14 & 0 & 0 \\ \hline 0 & 0 & 0 & -64 \\ 0 & 0 & 1 & 16 \end{array}\right) \quad B$$

2x2 block matrix
w/ companion matrices
as block.

A is companion matrix
of $(x-7)^2$

B is companion matrix of $(x-8)^2$

$\Longrightarrow$ $m_A = (x-7)^2$

$m_B = (x-8)^2$

if m is min. poly of big matrix.

$m_A \mid m$ $\quad$ $m_B \mid m$

$\Rightarrow m_A m_B | m$

b/c m_A, m_B

relately prime.

$(x-7)^2(x-8)^2 | m$

$\deg(m) \leq 4$

by def.

$\Rightarrow$

$$m = (x-7)^2(x-8)^2$$

<u>Ex:</u> $T: \mathbb{R}^4 \to \mathbb{R}^4$

w/

$$c_T = x(x-1)^2(x-3)$$

$$m_T = x(x-1)(x-3)$$

Again, construct block matrix.

$$\begin{array}{cc} A & \\ \left(\begin{array}{cc} \begin{bmatrix}1\end{bmatrix} & \\ & \begin{bmatrix}0&&\\&1&\\&&3\end{bmatrix}\end{array}\right) & B \end{array}$$

$$C_A = x-1$$

$$C_B = x(x-1)(x-3)$$

$\Rightarrow$ Char poly of big matrix is prod. $= x(x-1)^2(x-3)$.

$m_A | m$

$m_B | m$

Don't get $m_{A \cup B} | m$ b/c not rel. prime!

$\Rightarrow x(x-1)(x-3) | m$.

Can check by hand that big matrix satisfies this polynomial.