HW 1:

- Will post solutions on Friday
- Proof writing: check all details, unless proved in book, lecture, or discussion.

2.6.12: β is a basis of V

prove that

ψ maps β to β^{**} ← dual basis of dual basis

i.e. check if $\beta = \{v_1, \ldots, v_n\}$

$\beta^* = \{f_1, \ldots, f_n\}$

$\beta^{**} = \{\delta_1, \ldots, \delta_n\}$

$\psi(v_i) = \delta_i$ $\qquad$ δ_i dual vector to f_i.

1.) $T: V \to W$ inj. $\Longrightarrow$

$T^*: W^* \to V^*$ surj.

Use the fact that dual of composition is composition of duals.

Hint: Surj. is the same as having a right inverse.

For #4: Case on $\lambda > 0$, $\lambda = 0$, $\lambda < 0$

More on dual Space

V an F-vector space,

$V^* = \mathcal{L}(V, F)$. Dual Space

V f.d. with basis

$\beta = \{v_1, \ldots, v_n\}$

V^*

$\beta^* = \{f_1, \ldots, f_n\}$ dual basis

$f_i : V \to F$

$$f_i(v_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$V \cong V^*$, but this requires picking a basis.

Rmk: If V f.d. inner prod space,

$V \cong V^*$ is canonical

$v \longmapsto \langle -, v \rangle$

If V is infinite dimensional,

$V \not\cong V^*$

Ex: $\ell^1 = \{ (a_i)_{i=1}^{\infty} : \sum |a_n| < \infty$
$a_i \in \mathbb{R} \}$

$\ell^\infty = \{ (a_i)_{i=1}^{\infty} : \sup_i |a_i| < \infty \}$
$a_i \in \mathbb{R}$

turns out

$(\ell^1)^* = \ell^\infty$ but these are not iso. as $\mathbb{R}$-vector spaces.

Problem: "dual basis" is no longer a spanning set!

However, there is canonical map into V^{**}:

$$\varphi : V \hookrightarrow V^{**}$$

$$v \longmapsto ev_v$$

$$ev_v(f) = f(v)$$

is an injection

For any $T : V \to W$ linear we can define the <u>dual</u> of T,

$T^*: W^* \to V^*$

$T^*(f) = f \circ T$

V^* $\qquad\qquad\qquad\qquad$ W^*

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[dr, "f \circ T"'] & W \arrow[d, "f"] \\ & F \end{tikzcd}$$

<u>Prop:</u> if $T: V \to W$

$S: W \to Z$

$$(S \circ T)^* = T^* \circ S^*$$

<u>Proof:</u>

Let $f : Z \to F$

$(S \circ T)^*(f)$

$= f \circ (S \circ T)$

$(T^* \circ S^*)(f)$

$T^*(S^*(f))$

$= T^*(f \circ S)$

$= (f \circ S) \circ T$

b/c composition is associative, we have

$f \circ (S \circ T) = (f \circ S) \circ T$, so we're done.

When V, W are f.d. w/ bases β, γ

$$[T^*]_{\gamma^*}^{\beta^*} = \left([T]_\beta^\gamma\right)^t.$$

<u>Def:</u> $W \subset V$ subspace

the <u>annihilator</u> of W

$$W^\circ = \{f \in V^* : f(w) = 0 \text{ for all } w \in W\}$$

all functionals that kill W.

$$W^\circ \leq V^*$$

W° is a generalization of an orthogonal complement to an arbitrary vector space.

Prop:

- $U \leq W$, then $W^\circ \leq U^\circ$
- $(U+W)^\circ = U^\circ \cap W^\circ$
- $(V/W)^* \cong W^\circ$

Proof:

Suppose $f \in W^\circ$. Then $f(w) = 0$ for all $w \in W$. Since $U \subseteq W$, $f(u) = 0$ for all $u \in U$.

so $f \in U^\circ$.

Suppose $f \in U^\circ \cap W^\circ$

then $f(u) = 0$ for all $u \in U$

$f(w) = 0$ for all $w \in W$

Since $U+W = \{u+w : \begin{smallmatrix} u \in U \\ w \in W \end{smallmatrix}$

$$f(u+w) = f(u) + f(w)$$
$$= 0 + 0 = 0.$$

$f \in (U+W)^0$

If $f \in (U+W)^0$

$f(x) = 0$ for all $x \in U+W$

$x = u+w$ for some u, w

$U \subset U+W$ $\quad u = u+0$

$W \subset U+W$ $\quad w = 0+w$

So f kills U and W

$\Rightarrow \quad f \in U^\circ \cap W^\circ.$

Define a map

$\Phi : W^\circ \longrightarrow (V/W)^*$

by

$\Phi(f) = \bar{f}$ where

$\bar{f}(v+W) = f(v).$

note that $\bar{f}$ is

well defined because

if $v+W = v'+W$,

then $v = v'+w$ for

some $w \in W$,

and

$\bar{f}(v+W) = f(v)$

$= f(v'+w)$

b/c

$f(w)=0$

as $f \in W^0$

$= f(v')+f(w)$

$= f(v')$

$= \bar{f}(v'+W)$

Claim: Φ is an iso.

Can easily check that Φ is linear.

<u>Inj</u>: Suppose $\Phi(f) = 0$.

So $\bar{f}(v+W) = 0$

for all $v \in V$. Thus

$f(v) = 0$ for all $v \in V$

$\Longrightarrow f = 0$.

Surj:

Pick $f \in (V/W)^*$.

Want to construct g

with

$\Phi(g) = f$.

Define $g(x) = f(x + W)$.

if $w \in W$, then

$w + W = 0 + W$
b/c $w \in W$
$0 + W$ is zero elt. of V/W

$g(w) = f(w + W) = f(0 + W)$
$= 0$

So $g \in W^\circ$. (note that g is also linear).

$\Phi(g) = \bar{g}$

$\bar{g}(v+W) = g(v) = f(v+W)$

So $\bar{g} = f$

$\Phi(g) = f$ as desired. $\square$

<u>Cor:</u> $\dim W^0 = \dim V/W$

$= \dim V - \dim W.$

<u>Rmk:</u> this shows that if V is inner prod. space

$W^0 \cong W^\perp$ b/c same dim.

Didn't get to, but insightful:

Prop: V, W f.d.

$T: V \to W$

$$\ker(T^*) = \operatorname{Im}(T)^0$$

Proof: $g \in \ker(T^*) \iff$

$T^*(g) = 0 \iff g \circ T = 0.$

$\iff (g \circ T)(v) = 0$ for all $v \iff g(T(v)) = 0$ for all v.

$\iff g$ kills $\mathrm{Im}(T)$

$\iff g \in \mathrm{Im}(T)^\circ$. $\square$

If $T : V \to V$ then rank nullity says

$\dim \operatorname{Ker} T^* + \dim \operatorname{Im} T^* = \dim V^*$

$\dim \operatorname{Ker} T + \dim \operatorname{Im} T = \dim V$

Since $\dim V = \dim V^*$

and $\dim (\operatorname{Im} T)^\circ$

$= \dim V / \operatorname{Im}(T)$

$= \dim \operatorname{Ker}(T)$

$\Rightarrow \dim \operatorname{Ker} T^* = \dim \operatorname{Ker} T$

$\dim \operatorname{Im} T^* = \dim \operatorname{Im} T$

So $\operatorname{rank} T^* = \operatorname{rank} T$.

This proves the fact that

"row rank" = "column rank"

from 33A.

More generally, if

$T: V \to W$ then

$\ker(T^*) \cong \operatorname{coker}(T)$

by HW 1.