HW 6 #2:

$T: V \to V$ normal

m_T factors into deg 1 and 2 irred.

V complex inner product space:

T diagonalizable $\Longrightarrow$

m_T factors into distinct linear factors.

V real inner product space:

$A = [T]$

$m_A = m_T$.

View $A \in M_n(\mathbb{C})$ instead of $M_n(\mathbb{R})$

$$m_A^{\mathbb{C}} = m_A^{\mathbb{R}}$$

$m_A^{\mathbb{C}}$ splits into distinct linear factors in $\mathbb{C}[x]$ b/c A is diagonalizable over $\mathbb{C}$.

$$m_A^{\mathbb{C}} = \underline{(x-\alpha_1)} \cdots \underline{(x-\alpha_k)}$$

$m_A^{\mathbb{C}} = m_A^{\mathbb{R}}$

$\Longrightarrow$ complex roots come in conjugate pairs pair up factor w/ conjugate to get real quadratic factor.

Look at my solutions for full details!!

7.2.6: look at 7.2.7

More Examples of Jordan Forms:

Ex: $A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & -1 & 3 \end{pmatrix}$

$$C_A = (x-3)^2(x-2)^2$$

Sum of block sizes for $\lambda = 2, 3$ is 2.

Either have 2x2 block or two 1x1 blocks.

Compute g.m. to figure out which one.

$$A - 2I : \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

$$\dim(E_2) = \dim(\ker(A - 2I)) = 1$$

$\Longrightarrow$ 1 block for $\lambda = 2$

$$A - 3I : \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\dim(E_3) = \dim(\ker(A - 3I)) = 1$$

$\Longrightarrow$ 1 block for $\lambda = 3$

$$J_A = \begin{pmatrix} \boxed{\begin{matrix} 2 & 1 \\ & 2 \end{matrix}} & \\ & \boxed{\begin{matrix} 3 & 1 \\ \end{matrix}} \end{pmatrix}$$

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How to get a basis that gives J_A?

Construct disjoint cycles for each block.

$(A-2I)^2 v = 0$

$(A-2I)v \neq 0$

$\{(A-2I)v, v\}$

$$(A-2I)^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$c = 0$$
$$-2c + d = 0$$

$$\operatorname{Ker}\left((A-2I)^2\right) = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

Take $v = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$$(A-2I)v = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$ cycle for $\lambda = 2$.

Similarly,

$$(A-3I)^2 = \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} a-2b+c = 0 \\ b-c=0 \end{cases} \Rightarrow a-b=0$$

$$\Rightarrow a=b=c$$

$$\operatorname{Ker}\left((A-3I)^2\right) = \operatorname{Span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\}$$

$$v = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$(A-3I)x = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$\beta = \left\{ \begin{pmatrix} 1 \\ 9 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Rational Canonical Form

$v \in V$ $\qquad T: V \to V$ $\qquad$ k-dim.

$W = \text{Span}\{v, Tv, \ldots, T^{k-1}v\}$ $\leftarrow$

$$C_{T_W} = a_0 + a_1 x + \dots + x^k$$

C-H says that

$$C_{T_W}(T_W) = 0$$

$$a_0 I + \dots + T_W^k = 0$$

$$T_W^k = -a_0 I - \dots - a_{k-1} T_W^{k-1}$$

$$\beta = \{v, Tv, \dots, T^{k-1} v\}$$

$$[T_W]_\beta = \begin{pmatrix} 0 & 0 & & 0 & -a_0 \\ 1 & 0 & & \vdots & \vdots \\ 0 & 1 & \cdots & \vdots & \vdots \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & 1 & -a_{k-1} \end{pmatrix}$$

Companion matrix for the polynomial

$a_0 + \dots + a_{k-1}x^{k-1} + x^k$.

I talked more about companion matrices week 4

RCF: we can find a basis γ of V s.t.

$$[T]_\gamma = \begin{pmatrix} C_1 & & \\ & \ddots & \\ & & C_r \end{pmatrix}$$

C_r is companion matrix for some poly. $g_r(x)$

$$g_1 \mid g_2 \mid \cdots \mid g_r.$$

Turns out: RCF will always exist!!

The polynomials $g_1, g_2, \ldots, g_r$ are called the invariant factors of T.

How to compute RCF?

- m_T is the largest invariant factor
- Product of invariant factors $= c_T$

<u>Ex:</u> $A \in M_4(\mathbb{R})$

$c_A = (x-1)^4$

$m_A = (x-1)^2$

What are possible RCF?

List invariant factors:

$x-2,\ x-2,\ (x-2)^2$

OR

$(x-2)^2,\ (x-2)^2$

$$\begin{pmatrix} \boxed{2} & & & \\ & \boxed{2} & & \\ & & 0 & -4 \\ & & 1 & 4 \end{pmatrix}$$

$$(x-2)^2 = x^2 - 4x + 4$$

$$\begin{pmatrix} \boxed{\begin{matrix} 0 & -4 \\ 1 & 4 \end{matrix}} & \\ & \boxed{\begin{matrix} 0 & -4 \\ 1 & 4 \end{matrix}} \end{pmatrix}$$

How to compute Invariant Factors:

For small matrices, sometimes just forced based off of constraints and what c_T is.

Algorithm:

$$xI - A$$

Use row/column operations to turn $xI - A$ into a diagonal matrix that looks like

$$\begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & g_1 & & & \\ & & & & g_2 & & \\ & & & & & \ddots & \\ & & & & & & g_r \end{pmatrix} \qquad g_1 \mid g_2 \mid \cdots \mid g_r$$

Then $g_1, \ldots, g_r$ are the invariant factors of A.

Ex: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

$$xI - A = \begin{pmatrix} x-1 & -1 & -1 \\ -1 & x-1 & -1 \\ -1 & -1 & x-1 \end{pmatrix}$$

$$\begin{pmatrix} x-1 & -1 & -1 \\ -1 & x-1 & -1 \\ -1 & -1 & x-1 \end{pmatrix} \xrightarrow[-R_2 \to R_2]{R_2 \rightleftarrows R_1} \begin{pmatrix} 1 & 1-x & 1 \\ x-1 & -1 & -1 \\ -1 & -1 & x-1 \end{pmatrix}$$

$$\xrightarrow[R_1+R_3 \to R_3]{(1-x)R_1+R_2 \to R_2} \begin{pmatrix} 1 & 1-x & 1 \\ 0 & x^2-2x & -x \\ 0 & -x & x \end{pmatrix} \xrightarrow[-C_1+C_3 \to C_3]{-C_1+C_2 \to C_2}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & x^2-2x & -x \\ 0 & -x & x \end{pmatrix} \xrightarrow{R_2+R_3 \to R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & x^2-2x & -x \\ 0 & x^2-3x & 0 \end{pmatrix}$$

$$\xrightarrow{(x-2)C_3+C_2 \to C_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -x \\ 0 & x^2-3x & 0 \end{pmatrix}$$

$-C_3 \to C_3$
$C_2 \leftrightarrow C_3$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x^2-3x \end{pmatrix}$$

Invariant factors: $x,\ x^2-3x$

$$\text{RCF} = \begin{pmatrix} \boxed{0} & & \\ & \boxed{\begin{matrix} 0 & 0 \\ 1 & 3 \end{matrix}} \end{pmatrix}$$

RCF: Elementary divisor form.

The RCF above is the usual RCF. There's a different form using elementary divisors instead of <u>invariant factors</u> that your book + lecture do.

Here's how it works:

$G = f_1^{e_1} \cdots f_k^{e_k}$ as a product of irred. polynomials.

<u>Thm:</u> There is a basis β s.t.

$$[T]_\beta = \begin{pmatrix} C_1 & & \\ & \ddots & \\ & & C_r \end{pmatrix}$$

where C_j = Companion matrix of $f_i^{p_i}$ for some i.

The polynomials appearing in this form

are called the elementary divisors of T.

How to find? Use these facts:

1. Product of elementary divisors $= c_T$

2. lcm of elementary divisors $= m_T$

3. For each f_i irred. factor of c_T, # elementary divisors corresponding to f_i $= \dfrac{\dim E_{f_i}}{d_i}$

$E_{f_i} = \ker(f_i(T))$

$d_i = \deg(d_i)$.

Ex: $T: \mathbb{R}^5 \to \mathbb{R}^5 \qquad T(x) = Ax$

$$A = \begin{pmatrix} 0 & 2 & 0 & -4 & 2 \\ 1 & -2 & 0 & 0 & 2 \\ 1 & 0 & 1 & -3 & 2 \\ 1 & -2 & 1 & -1 & 2 \\ 1 & -4 & 3 & -3 & 4 \end{pmatrix}$$

$$C_T = (x^2+2)^2(x-2) = f_1^2 f_2$$

Can only have single elementary divisor corresponding to $x-2$.

$$\{x-2\}$$

Compute that $f_1(A) =$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & -12 & 4 \\ 0 & 0 & 6 & -12 & 6 \\ 0 & 0 & 6 & -12 & 6 \\ 0 & 0 & 12 & -24 & 12 \end{pmatrix}$$

$\dim E_{f_1} = 4$

elementary divisors for $f_1 = 4/2 = 2$.

must be

$\{x^2+2, x^2+2\}$.

So $RCF(T)$

$$= \begin{pmatrix} C_{x^2+2} & & \\ & C_{x^2+2} & \\ & & C_{x-2} \end{pmatrix}$$

Ex: $T: \mathbb{R}^4 \to \mathbb{R}^4$ $T(x) = Ax$

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

$$C_T = (x-2)^4 = f_1^4$$

Can check that $m_T = (x-2)^3$

and that $\dim E_{f_1} = 2$.

So there are $2/1 = 2$

elementary divisors corresponding to f_1. The constraints force them to be $\{x-1, (x-1)^3\}$

So $\quad RCF(A) = \begin{pmatrix} C_{x-1} & \\ & C_{(x-1)^3} \end{pmatrix}.$

How to find a basis that gives RCF? Very similar to JCF.

Companion matrix $\longleftrightarrow$ Cyclic Subspace generated by a vector.

Ex: $A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ as before.

$$RCF(A) = \begin{pmatrix} C_{x-2} & \\ & C_{(x-2)^3} \end{pmatrix}$$

$v_1 \in \ker(T-2I)$. Then

$T(v_1) = 2v_1$ so $W_1 = \operatorname{Span}\{v_1\}$

is T-cyclic with

$$[T|_{W_1}] = [2] = C_{x-2}.$$

v_2 with $(T-2I)^3 = 0$ but
$(T-2I)^2 \neq 0$

Then

$W_2 = \mathrm{Span}\{v, Tv, T^2 v\}$ is

T-cyclic with

$$[T|_{W_2}] = \begin{pmatrix} 0 & 0 & -4 \\ 1 & 0 & 4 \\ 0 & 1 & 3 \end{pmatrix} = C_{(x-2)^3}$$

So $\{v_1, v_2, Av_2, A^2v_2\}$

is a basis that gives RCF.

Make sure to pick $v_1 \notin \mathrm{Span}\{v_2, Av_2, A^2v_2\}$!

Explicity, may choose

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

to get $\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 4 \\ 0 \end{pmatrix} \right\}$

as a basis.

<u>Ex:</u> $A = \begin{pmatrix} 0 & 2 & 0 & -4 & 2 \\ 1 & -2 & 0 & 0 & 2 \\ 1 & 0 & 1 & -3 & 2 \\ 1 & -2 & 1 & -1 & 2 \\ 1 & 4 & 3 & -3 & 4 \end{pmatrix}$

we saw that $RCF(A)$

$$= \begin{pmatrix} C_{x^2+2} & & \\ & C_{x^2+2} & \\ & & C_{x-2} \end{pmatrix}$$

So basis will be of the form

$$\{v_1, Av_1, v_2, Av_2, v_3\}$$

$v_1 \in \operatorname{Ker}(A^2+2I)$

$v_2 \in \operatorname{Ker}(A^2+2I)$ and

$v_2 \notin \operatorname{Span}\{v_1, Av\}$

$v_3 \in \operatorname{Ker}(A-2I)$

$$A^2+2I = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & -12 & 6 \\ 0 & 0 & 6 & -12 & 6 \\ 0 & 0 & 6 & -12 & 6 \\ 0 & 0 & 12 & -24 & 12 \end{pmatrix}$$

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad Av_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad Av_2 = \begin{pmatrix} 2 \\ -2 \\ 0 \\ -2 \\ 4 \end{pmatrix}$$

$$A-2I = \begin{pmatrix} -2 & 2 & 0 & -6 & 2 \\ 1 & -4 & 0 & 0 & 2 \\ 1 & 0 & -1 & -3 & 2 \\ 1 & -2 & 1 & -3 & 2 \\ 1 & -4 & 3 & -3 & 2 \end{pmatrix}$$

$$v_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 2 \end{pmatrix}$$

$$\left\{ \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}2\\-2\\0\\-2\\4\end{pmatrix}, \begin{pmatrix}0\\1\\1\\1\\2\end{pmatrix} \right\}$$

is a basis that gives RCF.

Remark: You can easily get the list of elementary divisors from invariant factors and vice-versa.

Suppose $c_T = f_1^{e_1} \cdots f_k^{e_k}$

Invariant factors

$g_1, \dots, g_r$

$$g_1 = f_1^{a_{11}} \cdots f_k^{a_{1k}}$$

$$g_2 = f_1^{a_{21}} \cdots f_k^{a_{2k}}$$

$$\vdots$$

$$g_r = f_1^{a_{r1}} \cdots f_k^{a_{rk}}$$

elementary divisors for f_i are given by

$$\{ f_i^{a_{1i}}, \dots, f_r^{a_{ri}} \} \quad (a_{ji} \neq 0)$$

Similarly if given elem. divisors

$$\{ f_i^{a_1}, \dots, f_i^{a_r} \} \text{ for each } f_i$$

- order each list so that powers are non-decreasing.

- Add 1's to beginning of list until all lists have same # of elements.

take product of $f_i^{a_i}$ for all i.

<u>Ex:</u> $C_T = (x+1)^2(x-2)^3$

Invariant Factors:

$$\{(x+1)(x-2), (x+1)(x-2)^2,\}$$

$$= \{g_1, g_2\}$$

$$g_1 = (x+1)^1 (x-2)^1$$

$$g_2 = (x+1)^1 (x-2)^2$$

Elem. division

$$= \{x-2, (x-2)^2, x+1, x+1\}$$

Ex: $c_T = x^2(x^2+1)(x+2)^3$

Elem divisors:

$\{x, x, x^2+1, x+2, x+2, x+2\}$

$\{1, x, x\}$

$\{1, 1, x^2+1\}$

$\{x+2, x+2, x+2\}$

Invariant factors:

$$\{(x+2),\ x(x+2),\ x(x^2+1)(x+2)\}$$