Email: tsmits@math.ucla.edu

OH: Mon 1-2 pm (Even weeks)
Wed 3-4 pm
Th 5-6 pm

SMC: Mon 1-2 (Odd weeks)
Note: this is also open to 115A and 61 students.

Website: Find on CCLE or google my name.

Operations on Vector Spaces

Note: I have a set of 115A notes on my website!

- Intersections
- Products
- Sums

- Quotients

Product: Create a larger vector space where two vector spaces live as subspaces

$$V\times W = \{(v,w) : v\in V,\ w\in W\}$$

$$(v,w)+(v',w') = (v+v',\ w+w')$$

$$c\cdot(v,w) = (cv,\ cw)$$

Sometimes called "External direct sum" and denoted $V\oplus W$.

V and W are identified with $V\times\{0\}$ and $\{0\}\times W$ respectively.

Sums

The sum of subspaces V and W living inside a common vector space Z

is $V+W = \{v+w : v \in V, w \in W\}$

and it's the smallest subspace of Z containing both V and W.

i.e, if $X \subset Z$ and $V \subset X$, and $W \subset X$, then $V+W \subset X$.

if $V \cap W = \{0\}$, then we call $V+W$ a direct sum and instead denote it as $V \oplus W$.

Prop: $V = W_1 + W_2$

$V = W_1 \oplus W_2 \iff v = w_1 + w_2$ uniquely for $w_1 \in W_1$ and $w_2 \in W_2$

for each v.

Proof:

$\Longrightarrow$) Suppose $V = W_1 \oplus W_2$.

$v = w_1 + w_2$ and $v = w_1' + w_2'$

for $w_1, w_1' \in W_1$ and $w_2, w_2' \in W_2$.

$$w_1 + w_2 = w_1' + w_2'$$

$$\underbrace{w_1 - w_1'}_{\in W_1} = \underbrace{w_2' - w_2}_{\in W_2}$$

So $w_1 - w_1' \in W_1 \cap W_2 = \{0\}$

$$w_1 = w_1'$$

and similarly, $w_2 = w_2'$.

$\Leftarrow$ If $v = w_1 + w_2$

uniquely for $w_1 \in W_1$, $w_2 \in W_2$

then clearly we have

$V = W_1 + W_2$.

Let $x \in W_1 \cap W_2$.

$x = x + 0$ b/c $x \in W_1$

$0 + x$ b/c $x \in W_2$

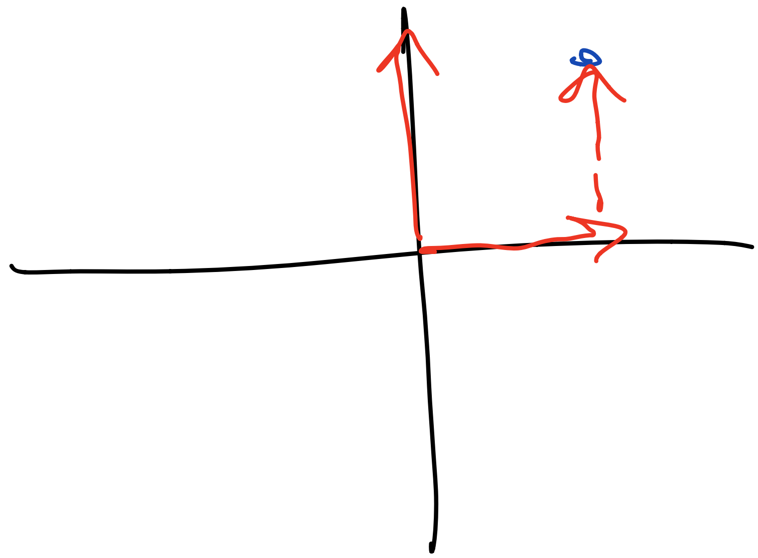
So by uniqueness, this

forces $x = 0$ $\boxtimes$

Ex: $\mathbb{R}^2$, $X =$ x-axis

$Y =$ y-axis

$\mathbb{R}^2 = X \oplus Y$



$\mathbb{R}^3$ $V =$ xy-plane

$W =$ yz-plane

$\mathbb{R}^3 = V + W$, but

the sum is not direct

b/c $V \cap W =$ y-axis.

Quotients

$W \subset V$ subspace. Let $v \in V$

The <u>coset</u> $v + W = \{v + w : w \in W\}$
is the set of translates
of v by elements of W.

The point of cosets is
they break V up into
smaller "pieces":

$$V = \bigcup_{v \in V} v + W$$

In fact, two cosets are either
equal or disjoint, so the

union collapses into a disjoint union.

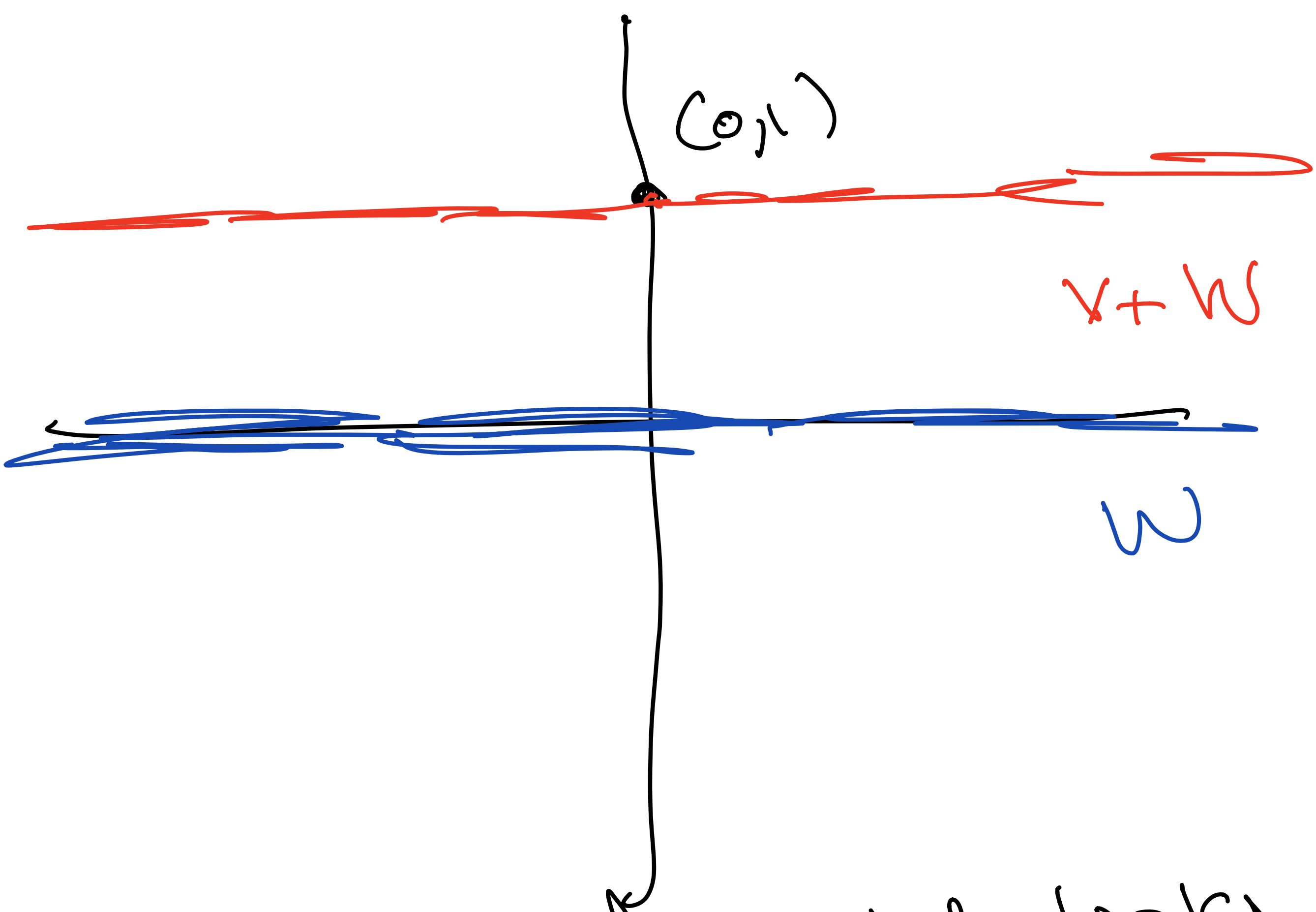
Example: $V = \mathbb{R}^2$

$W = \{(x,0) : x \in \mathbb{R}\}$

x-axis.

$v = (0,1)$.

What is the coset $v+W$?



An element of $v+W$ looks like

$v+w$ for some $w \in W$.

$w = (x, 0)$ for some x.

So an element of $v+W$ looks like $(x, 1)$ for $x \in \mathbb{R}$.

So $v+W = \{(x,1) : x \in \mathbb{R}\}$

$V/W = \{v+W : v \in V\}$

set of all cosets.

$0+W$

Zero element

define

$(v+W)+(v'+W) = (v+v')+W$

$c \cdot (v+W) = cv + W$

these operations turn

V/W into a vector space called the quotient of V by W.

The idea behind the quotient space is it will "crush" vectors in W to a single point.

Prop: Two cosets $v+W$ and $v'+W$ in V/W are equal $\iff v-v' \in W$.

In particular, $v+W = 0+W$
$\iff v \in W$.

Proof: Exercise.

Ex: $V = \mathbb{R}^2$

$W = x\text{-axis}$

$= \{(x,0) : x \in \mathbb{R}\}$

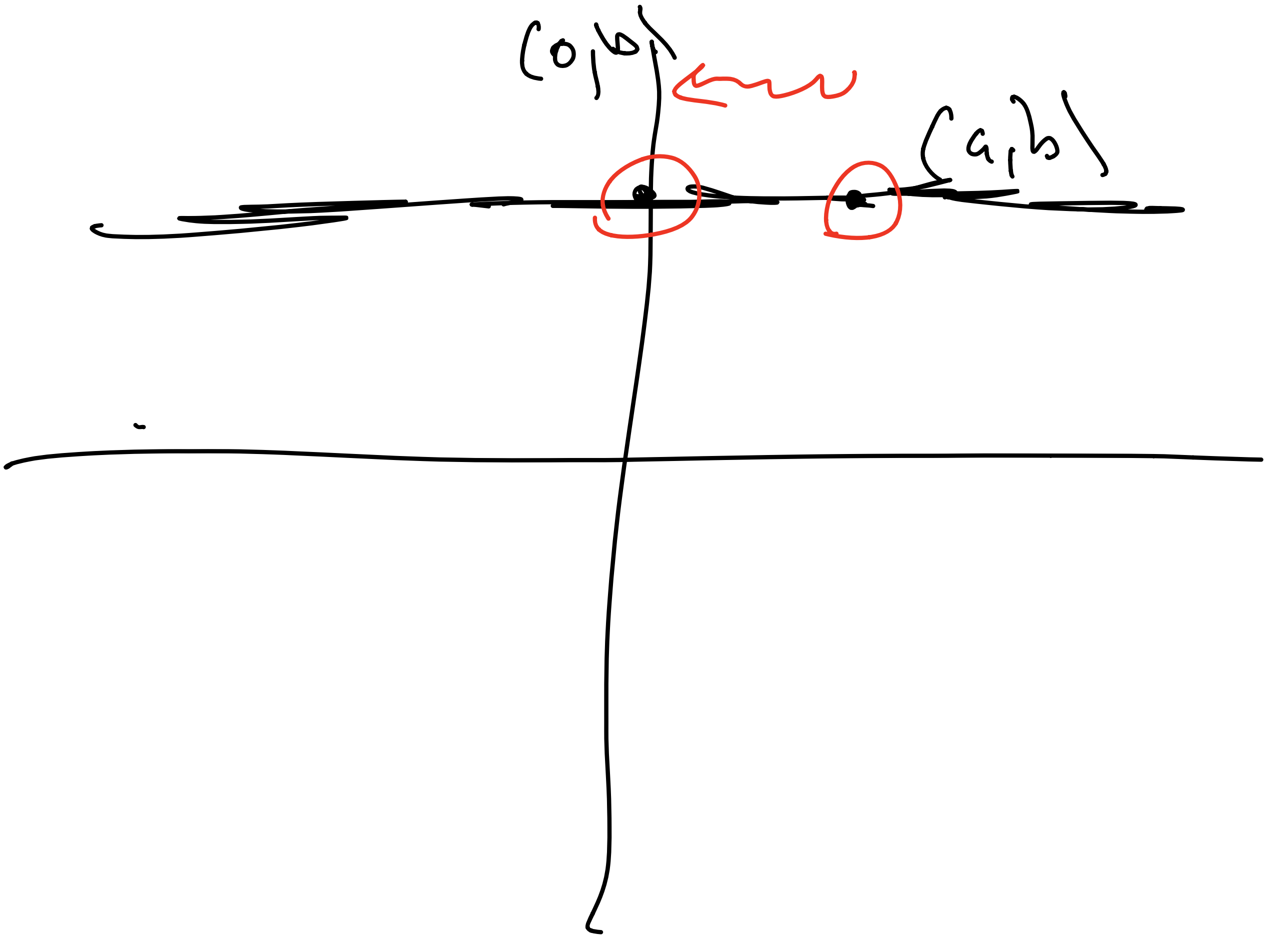
What does V/W look like?

$v = (a,b) \in V.$

$v + W = \{(a+x, b) : x \in \mathbb{R}\}$

In V/W, we have

$v + W = \underline{(0,b) + W}$ because



$(a,b) - (0,b) = (a,0) \in W.$

So $V/W = \{ (0,b) + W : b \in \mathbb{R} \}$

and you can show that

$V/W \cong$ y-axis.

Didn't get to, but useful:

Prop: If $W \subset V$ is a subspace, there is $U \subset V$ subspace such that $V = U \oplus W$.

Such a subspace U is called a complement of W in V, so this prop. says complements always exist.

Prop: If $V \cong U \oplus W$ then $U \cong V/W$, i.e.

quotient Spaces $\longleftrightarrow$ complements

Prop: If $U \oplus W \cong U \oplus Z$ then $W \cong Z$, i.e. complements are unique up to iso.

Prop:

$\dim(U \oplus W) = \dim(U) \oplus \dim(W)$

$\dim(V/W) = \dim(V) - \dim(W)$

direct Sums and quotients

behave nicely with dimension!

Try and prove these
as review!