Quadratic Forms

$Q: V \to F$ with $Q(x) = B(x,x)$
for some symmetric bilinear B

Q is called <u>quadratic form</u>

<u>Note:</u> $Q(ax) = a^2 Q(x)$ so Q is <u>not</u> linear.

$$\left\{ \begin{matrix} \text{Quadratic} \\ \text{Forms} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{Symm. bilinear} \\ \text{Forms} \end{matrix} \right\}$$

$$Q(x) = B(x,x) \longleftarrow B$$

$$Q \longrightarrow B(x,y) = \tfrac{1}{2}\left[B(x+y) - B(x) - B(y)\right]$$

Is a bijection.

Can still talk about matrix of Q, etc.

Over $\mathbb{R}$, theory of quadratic forms is nice.

$[Q]_\beta$ is symmetric $\Longrightarrow$ orthog. diagonalizable by spectral Thm.

if $\beta = \{v_1, \dots, v_n\}$ basis of V

o.n.

and $\gamma = \{w_1, \ldots, w_n\}$ eigenbasis,

let $[x]_\gamma = (c_1, \ldots, c_n)$. Then in γ - coordinates, we can write

$$Q(c_1, \ldots, c_n) = \lambda_1 c_1^2 + \cdots + \lambda_n c_n^2$$

where λ_i is the eigenvalue for w_i.

Ex: $Q(x,y) = x^2 - 2xy + y^2$

$$[Q]_\beta = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

where $\beta = \{e_1, e_2\}$

$$E_0 = \text{Span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$$

$$E_2 = \text{Span}\left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}$$

$$\gamma = \left\{\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}\right\}$$

if $[x]_\gamma = (c_1, c_2)$

$$Q(c_1, c_2) = 2c_2^2$$

If $B_1, B_2 : V \times V \to F$ are two bilinear forms, B_1 and B_2 are called <u>equivalent</u> if there is

$T: V \to V$ iso. s.t.

$B_1(Tv, Tw) = B_2(v, w)$ for all $v, w \in V$.

(Such a T is called an <u>isometry</u>)

In matrix land, this corresponds to a change of basis:

$B_1 \sim B_2 \iff [B_1]_\beta = C^t [B_2]_\beta C$

for some invertible matrix C.

Study bilinear (and quadratic) forms up to equivalence.

For real quadratic forms, nice result,

$[Q]_\beta = D$ for β an eigenbasis

Define

rank Q = # non-zero entries in D

index Q = # pos entries in D

signature Q = # pos − # neg in D

Note: $2 \cdot$ index − rank = signature.

the above numbers are well defined.
(i.e. don't depend on basis)

Thm: Quadratic forms Q_1 and Q_2 are equivalent $\Longleftrightarrow$ same rank and index.

Ex: $Q_1(x,y) = x^2 - 2xy + y^2$

$Q_2(x,y) = 2x^2 + 4xy + 3y^2$

we have with $\beta = \{e_1, e_2\}$

$[Q_1]_\beta = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

$$[\varphi_2]_{\beta} = \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix}^T$$

One can check from this that

rank $\varphi_1 = 1$ index $\varphi_1 = 1$

rank $\varphi_2 = 2$ index $\varphi_2 = 2$

so $\varphi_1 \not\sim \varphi_2$.

How many different equivalence classes are there?

dim $V = n$

possible rank: $0, 1, \dots, n$

For rank $= k$, index can be $0, 1, \dots, k$. So $k+1$ choices for each choice of rank.

$$\sum_{r=0}^{n} r+1 = \frac{(n+1)(n+2)}{2} \text{ choices}$$

has (rank, index)

So $\frac{(n+1)(n+2)}{2}$ equiv. classes.

A rep. for each class is the matrix

$$J_{pr} = \begin{pmatrix} I_p & 0 & 0 \\ 0 & -I_{r-p} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where p = rank
r = index

First example of a canonical form.

Will talk about canonical forms for linear operators later.