Bilinear Forms

V vector spaces.

$B: V \times V \longrightarrow F$ is a <u>bilinear form</u> if $B(v,-)$ and $B(-,w)$ are linear for each fixed $v,w \in W$.

Generalizes idea of inner product.

<u>Ex:</u> $B: M_n(F) \times M_n(F) \longrightarrow F$

$B(A,C) = Tr(AC)$ is bilinear

Associated to B are two maps

$R_B: V \longrightarrow V^*$ $\quad v \longmapsto B(v,-)$

$L_B: V \longrightarrow V^*$ $\quad w \longmapsto B(-,w)$

B is <u>non-degenerate</u> if R_B and L_B are injective. If V is f.d. note this is the same as saying they are isomorphisms.

It's sufficient to work with just R_B if V is f.d.:

Prop: $(R_B)^* = L_B$ under the iso $V^{**} \cong V$.

Proof: $(R_B)^*: V^{**} \longrightarrow V^*$ and we have $V \cong V^{**}$ by $v \to \mathrm{eval}_v$. We get the diagram

$$\begin{tikzcd} V^{**} \arrow[r, "(R_B)^*"] & V^* \\ V \arrow[u, "\mathrm{eval}"] \arrow[ur] & \end{tikzcd}$$

By def, for $\alpha \in V^{**}$ we have $(R_B)^*(\alpha) = \alpha \circ R_B \in V^*$. We have $\alpha = \mathrm{eval}_v$ for some $v \in V$, so for $x \in V$, $(\alpha \circ R_B)(x) = (\mathrm{eval}_v \circ R_B)(x)$
$= \mathrm{eval}_v(R_B(x)) = \mathrm{eval}_v(B(x,-))$
$= B(x,v)$.

That is, $\mathrm{eval}_v \circ R_B$ is the map $V \longrightarrow F$ given by
$x \to B(x,v)$

which agrees with $L_B(v)$. Thus under $v \longleftrightarrow \mathrm{eval}_v$ we have $(R_B)^* = L_B$ ∎

Set $B(V)$ = Bilinear forms on V.

Two points of view for bilinear forms for V f.d:

1.) $\mathcal{B}(V) \cong M_n(F)$ via
$$B \longmapsto [B]_\beta$$

where $[B]_\beta$ has i,j entry $B(v_i, v_j)$ for $\beta = \{v_i\}$ basis of V.

For a matrix $A \in M_n(F)$ we get a bilinear form by

$$B(v,w) = [v]_\beta^t A [w]_\beta.$$

2.) $\mathcal{B}(V) \cong \mathcal{L}(V, V^*)$ via
$$B \longmapsto R_B$$

This says bilinear forms pick out lin. maps from V to V^*.

The first approach uses coordinates, while the second is coordinate free.

Can talk about forms in either language.
e.g.
$[B]_\beta$ invertible $\iff$ B non-degenerate
$\iff$ R_B iso.

Since $V \cong V^*$ there's a natural bilinear form

$$B_N : V \times V^* \to F \quad \text{given by}$$
$$(v, f) \mapsto f(v)$$

R_{B_N} is the map $v \mapsto \mathrm{eval}_v$ and this is an iso, so this form is non-degenerate.

if $T: V \to V$ one can see

$$B_N(v, T^*(f)) = B_N(Tv, f)$$

So that duals are some sort of adjoint

In fact, we can define an adjoint of T

by $T^{adj}: V \to V$
$T^{adj} = L_B^{-1} \circ T^* \circ L_B$

Then $B_W(v, T^{adj}(w)) = B_W(Tv, w)$ for all $v, w \in V$. Compare this with what I did last week!

Last connection: for B symmetric, non-degenerate define for $W \subseteq V$

$$W^\perp = \{ v \in V : B(v,x) = 0 \text{ for all } x \in W \}$$

annihilator

if $v \in W^\perp$ then $R_B(v)(x) = 0$ for all $x \in W$ so $R_B(v)(x) \in W^\circ$

i.e. R_B maps $W^\perp$ to W°. So again:

$$\text{orthogonality} \leftrightsquigarrow \text{duality}$$