Challenge problems

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Unless otherwise stated, $(V, \langle -, - \rangle)$ is an n dimensional inner product space over $F = \mathbb{R}$ or $\mathbb{C}$.

1. (Orthogonal Projections) A linear operator $T : V \to V$ is called a *projection* if $T^2 = T$.

   (a) Suppose that T is a projection. Prove that $V = \operatorname{Im}(T) \oplus \ker(T)$.

   (b) Suppose that T is a projection. Prove that $T = T^*$ if and only if $\operatorname{Im}(T) \perp \ker(T)$.

   Operators satisfying the condition of part (b) are called *orthogonal projections*.

   (c) Prove that if T is a projection, that $\|T(x)\| \leq \|x\|$ for any $x \in V$.

   (d) Let $A, B \in M_n(\mathbb{C})$ be such that $A^2 = A$ and $B^2 = B$. Prove that A and B are similar if and only if $\operatorname{rank}(A) = \operatorname{rank}(B)$.

2. (Isometries) An *isometry* is a linear operator $T : V \to V$ such that $\|T(v)\| = \|v\|$ for all $v \in V$. Prove the following are equivalent:

   (1) T is an isometry.

   (2) $\langle T(x), T(y) \rangle = \langle x, y \rangle$ for all $x, y \in V$.

   (3) $T^{-1} = T^*$.

3. (Polar Decomposition) A linear operator $T : V \to V$ is called *positive* if $\langle T(v), v \rangle \geq 0$ for all $v \in V$. An operator S is called a *square root* of T if $S^2 = T$.

   (a) Prove the following statements are equivalent:

      (1) T is positive.

      (2) T is self-adjoint with non-negative eigenvalues.

      (3) T has a positive square root.

   (b) Prove that a positive operator has a unique *positive* square root.

   We will let $\sqrt{T}$ denote the unique positive square root of T. Our goal is to prove the following:

   Theorem. *Let T be a linear operator on V. Then there is an isometry S such that $T = S\sqrt{T^*T}$.*

   This is an analogue of polar coordinates for linear operators: every operator T can be decomposed into a "scaling" portion given by $\sqrt{T^*T}$ and a "rotation" given by the isometry S. The proof is broken up as follows:

   (c) Prove that $\|T(v)\| = \|\sqrt{T^*T}(v)\|$.

   (d) Define a map $S_1 : \operatorname{Im}(\sqrt{T^*T}) \to \operatorname{Im}(T)$ defined by $S_1(\sqrt{T^*T}(v)) = T(v)$. Prove that S_1 is well-defined, and linear.

   (e) Deduce that S_1 is an isomorphism, so that $\operatorname{Im}(\sqrt{T^*T}) \cong \operatorname{Im}(T)$.

   (f) Find a linear operator $S : V \to V$ such that $S|_{\operatorname{Im}(\sqrt{T^*T})} = S_1$.

   (g) Prove that S is an isometry, and that $T = S\sqrt{T^*T}$, proving the theorem.