Challenge problems

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Unless otherwise stated, V is an n dimensional vector space over an arbitrary field F.

1. (Fun with images and kernels) The rank-nullity theorem is perhaps the most useful theorem when trying to prove things about finite dimensional vector spaces. Let $S, T : V \to V$ be linear.

   (a) Prove that $\text{rank}(ST) \leq \min\{\text{rank}(S), \text{rank}(T)\}$.

   (b) Prove that $\dim(\ker(ST)) \leq \dim(\ker(S)) + \dim(\ker(T))$.

   (c) (Sylvester's rank inequality) Prove that $\text{rank}(S) + \text{rank}(T) \leq \text{rank}(ST) + n$.

2. (Nilpotent operators) Let $T : V \to V$ be a linear operator. Then T is called *nilpotent* if $T^k = 0$ for some $k \geq 0$.

   (a) If $T : V \to V$ is a linear operator, show that there is some k such that $\ker(T^k) = \ker(T^{k+1})$.

   (b) If T is nilpotent, show that $T^n = 0$.

   (c) If T is nilpotent, prove that $\text{id}_V - T$ is invertible. (Hint: you might want to think about power series).

3. (The structure of $\text{End}_F(V)$) Recall $\text{End}_F(V) = \text{Hom}_F(V, V)$, the vector space of linear operators on V. We've seen that linear operators do not necessarily commute which each other. Which linear operators commute with everything?

   (a) Let $T \in \text{End}_F(V)$. Suppose that $v, T(v)$ are linearly dependent for all $v \in V$. Prove that $T = c \cdot \text{id}_V$ for some $c \in F$.

   (b) Suppose $T \in \text{End}_F(V)$ satisfies $TS = ST$ for all $S \in \text{End}_F(V)$. Prove that $T = c \cdot \text{id}_V$ for some $c \in F$.

   The following is a special case of a very important result in representation theory known as *Schur's lemma*.

   (c) Let $T : V \to V$ be a linear operator and let $W \subset V$ be a T-invariant subspace, i.e. $T(W) \subset W$. We say that W is T-*irreducible* if the only T-invariant subspaces of W are $\{0\}$ and W. Prove that if W is T-invariant and $\varphi \in \text{End}_F(W)$ satisfies $\varphi T = T\varphi$, then $\varphi = 0$ or φ is an isomorphism.

4. (The dual space). Recall that for a vector space V, the *dual space* V^* of V is defined by $V^* = \text{Hom}_F(V, F)$. Since V and V^* have the same dimension, they are isomorphic. However, this isomorphism is "non-canonical" in the sense that writing down an explicit isomorphism between V and V^* requires picking a basis. The *double dual* of V, denoted V^{**}, is defined as $(V^*)^*$.

   (a) Show that the map $\Phi : V \to V^{**}$ given by $v \to T_v$ where $T_v(\varphi) = \varphi(v)$ is an isomorphism.

   This isomorphism does not require picking a basis, so in some sense it is more "natural". Now let $W \subset V$ be a subspace. The *annihilator* of W, denoted W°, is defined by $W^\circ = \{\varphi \in V^* : \varphi(w) = 0 \text{ for all } w \in W\}$. One can check that W° is a subspace of V^*.

   (b) Prove that if $U \subset W$ then $W^\circ \subset U^\circ$.

(c) Prove that $(U+W)^\circ = U^\circ \cap W^\circ$.

In addition to the basic properties above, the annihilator of a subspace has a very important connection with the quotient space:

(d) Prove that $(V/W)^* \cong W^\circ$.

Let W be a vector space, and let $T : V \to W$ be a linear transformation. The *dual* of T is the linear transformation $T^* : W^* \to V^*$ defined by $T^*(\varphi) = \varphi \circ T$. There is one common matrix operation that you are familiar with that we have not yet seen an algebraic interpretation of: the transpose.

(e) Let $\beta = \{v_1, \ldots, v_n\}$, $\gamma = \{w_1, \ldots, w_m\}$ be bases for V and W, and let β', γ' be the corresponding dual basis for V^* and W^*, i.e. $\beta' = \{\delta_1, \ldots, \delta_n\}$ where $\delta_j(v_i) = 1$ if $i = j$ and 0 otherwise, and likewise for γ'. Prove that $[T^*]_{\gamma'}^{\beta'} = ([T]_\beta^\gamma)^t$.

(f) Prove that $\ker(T^*) = \mathrm{Im}(T)^\circ$.

(g) Deduce that $\mathrm{rank}(T) = \mathrm{rank}(T^*)$, proving the fact that "row rank" = "column rank".

5. (Linear algebra in characteristic p) Let K be a field of characteristic p (i.e. a field where $p \cdot 1 = 0$). Then the set $L = \{0, 1, \ldots, (p-1)\cdot 1\}$ is a subfield of K with $L \cong \mathbb{F}_p$. Recall that any field is a vector space over a subfield, so in particular, any field of characteristic p is a vector space over $\mathbb{F}_p$.

(a) Show every finite field of characteristic p has size p^n for some n.

Now let's suppose that $K = \mathbb{F}_q$ is a finite field of size $q = p^n$ for some n. The *Frobenius map* is the map $\sigma : K \to K$ defined by $\sigma(x) = x^p$.

(b) Show that for $x, y \in K$, that $(x+y)^p = x^p + y^p$ (Hint: binomial theorem).

(c) For $c \in \mathbb{F}_p$, define $f : \mathbb{F}_p \to \mathbb{F}_p$ by $f(x) = cx$. Show that f is bijective, and use this to show that $c^{p-1} = 1$. (Hint: look at the product of all non-zero elements of $\mathbb{F}_p$).

(d) Deduce that σ is $\mathbb{F}_p$-linear, and an isomorphism.

The Frobenius map plays an extremely important role in number theory and the study of fields. One interesting question that we can answer when working over finite fields: if V is an n-dimensional $\mathbb{F}_p$-vector space, how many isomorphisms $T : V \to V$ are there?

(e) Compute the number of invertible matrices in $M_n(\mathbb{F}_p)$.