Midterm 1 Practice
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Unless otherwise stated, V is a finite dimensional vector space over an arbitrary field F of dimension n.

1. For the following statements, indicate if it is true or false. If it is true, provide a proof and if it is false, justify why.
   (a) If U_1, U_2, W are subspaces of V such that $U_1 + W = U_2 + W$, then $U_1 = U_2$.
   (b) $W = \{A \in M_n(F) : AB = BA\}$ is a subspace of $M_n(F)$, where $B \in M_n(F)$ is fixed.
   (c) There is a linear transformation $T : F^5 \to F^2$ with $\ker(T) = \{(x_1, x_2, x_3, x_4, x_5) \in F^5 : x_1 = x_2 \text{ and } x_3 = x_4 = x_5\}$.
2. Give an example of the following concepts, with a brief justification.
   (a) A linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^5$ with $\dim(\ker(T)) = 3$ and $\dim(\operatorname{Im}(T)) = 2$.
   (b) A linear transformation $T : P(\mathbb{R}) \to P(\mathbb{R})$ that is surjective but not injective.
   (c) A basis of $\mathbb{R}^3$ with all basis vectors having entries 1 or -1.
3. Prove that $\{5, t^3 + t^2 + 1, t^3 + t^2 + t, t^3 + t + 2\}$ is a basis of $P_3(\mathbb{R})$.
4. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $T(x, y, z) = (x + y + 3z, -2x + 5y + z)$.
   (a) Prove that T is a linear transformation.
   (b) Find bases for $\ker(T)$ and $\operatorname{Im}(T)$, clearly stating their dimensions.
5. Let $U = \{f \in \mathcal{F}(\mathbb{R}, \mathbb{R}) : f(x) = f(-x) \text{ for all } x \in \mathbb{R}\}$, and $W = \{f \in \mathcal{F}(\mathbb{R}, \mathbb{R}) : f(x) = -f(x) \text{ for all } x \in \mathbb{R}\}$, the subspaces of even and odd functions. Prove that $\mathcal{F}(\mathbb{R}, \mathbb{R}) = U \oplus W$.
6. Let $T : V \to V$ be a linear transformation such that $\operatorname{rank}(T^2) = \operatorname{rank}(T)$, where T^2 means the composition $T \circ T$.
   (a) Show that $\operatorname{Im}(T) \cap \ker(T) = \{0\}$, and that $V = \operatorname{Im}(T) \oplus \ker(T)$.
   (b) Suppose instead that $T^2 = T$, and that V is not necessarily finite dimensional. Prove that $V = \operatorname{Im}(T) \oplus \ker(T)$.

7*. Let $F \subset L$ be a subfield, and let $L \subset K$ be a subfield. Suppose that L has dimension n as an F-vector space, and K has dimension m as an L-vector space. Prove K has dimension mn as an F-vector space.

1. (a) False; Take $V = \mathbb{R}^2$, $U_1 = \mathrm{Span}(\{(1,0)\})$, $U_2 = \mathrm{Span}(\{(0,1)\})$ and $W = \mathrm{Span}(\{(1,1)\})$. Then $U_1 + W = U_2 + W = \mathbb{R}^2$ but $U_1 \neq U_2$.
   (b) True; $B0 = 0B = 0$ so $0 \in W$. If $A, C \in W$ then $(A+C)B = AB + CB = BA + BC = B(A+C)$ so $A+B \in W$. For $c \in F$, $(cA)B = c(AB) = c(BA) = B(cA)$, giving $cA \in W$.
   (c) False; by the dimension theorem $\dim(\ker(T)) + \dim(\mathrm{Im}(T)) = 5$, and we see $\ker(T)$ has dimension 2, so $\dim(\mathrm{Im}(T)) = 3$. However $\mathrm{Im}(T) \subset F^2$ so $\dim(\mathrm{Im}(T)) \leq 2$, so this is impossible.

2. (a) Take $T(x_1, x_2, x_3, x_4, x_5) = (x_1, x_2, 0, 0, 0)$. Basis vectors of $\mathrm{Im}(T)$ are $(1,0,0,0,0)$ and $(0,1,0,0,0)$ so $\dim(\mathrm{Im}(T)) = 2$ and the dimension theorem says $\dim(\ker(T)) = 3$.
   (b) Let $T = \frac{d}{dx}$ be the derivative map. This is not injective, because $T(c) = 0$ for any constant polynomial c, so $\ker(T) \neq \{0\}$. However T is surjective, because given $p \in P(\mathbb{R})$ then $q = \int_0^x p(t)\,dt \in P(\mathbb{R})$ satisfies $T(q) = p$.
   (c) $B = \{(1,1,1), (1,-1,1), (-1,-1,1)\}$ works. It's easy to see $(-1,-1,1) \notin \mathrm{Span}(\{(1,1,1),(1,-1,1)\})$ and the latter is linearly independent, so this is a 3 element linearly independent subset of $\mathbb{R}^3$ and therefore a basis.

3. It's sufficient to check $\mathrm{Span}(\{5, t^3+t^2+1, t^3+t^2+t, t^3+t+2\}) = \mathrm{Span}(\{1, t, t^2, t^3\})$, since then we have a 4 element spanning set in a 4 dimensional space and therefore must be a basis. Set $e_1 = 5$, $e_2 = t^3+t^2+1$, $e_3 = t^3+t^2+t$, and $e_4 = t^3+t+2$. Then $1 = \frac{1}{5}e_1$, $t = e_3 - e_2 + \frac{1}{5}e_1$, $t^2 = e_3 - e_4 + \frac{2}{5}e_1$, and then this gives $t^3 = e_2 - (e_3 - e_4 + \frac{2}{5}e_1) - \frac{1}{5}e_1$. This says $1, t, t^2, t^3 \in \mathrm{Span}(\{5, t^3+t^2+1, t^3+t^2+t, t^3+t+2\})$ so $\mathrm{Span}(\{5, t^3+t^2+1, t^3+t^2+t, t^3+t+2\}) = \mathrm{Span}(\{1, t, t^2, t^3\})$.

4. (a) Set $\vec{x} = (x_1, y_1, z_1)$ and $\vec{y} = (x_2, y_2, z_2)$. Then $\vec{x} + \vec{y} = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$, and $T(\vec{x} + \vec{y}) = (x_1 + x_2 + y_1 + y_2 + 3z_1 + 3z_2, -2x_1 - 2x_2 + 5y_1 + 5y_2 + z_1 + z_2) = (x_1 + y_1 + 3z_1, -2x_1 + 5y_1 + z_1) + (x_2 + y_2 + 3z_2, -2x_2 + 5y_2 + z_2) = T(\vec{x}) + T(\vec{y})$. For $c \in \mathbb{R}$, $c\vec{x} = (cx_1, cy_1, cz_1)$ so $T(c\vec{x}) = (cx_1 + cy_1 + 3cz_1, -2cx_1 + 5cy_1 + cz_1) = c(x_1 + y_1 + 3z_1, -2x_1 + 5y_1 + z_1) = cT(\vec{x})$ and thus T is linear.
   (b) If $T(x,y,z) = (0,0)$ this says $x + y + 3z = 0$ and $-2x + 5y + z = 0$. Solving gives $y = -z$, and $x = -2z$, so $\ker(T) = \mathrm{Span}(\{(-2,-1,1)\})$ is 1 dimensional with basis $\{(-2,-1,1)\}$. The dimension theorem says $3 = \dim(\ker(T)) + \dim(\mathrm{Im}(T))$, so $\dim(\mathrm{Im}(T)) = 2$ and therefore $\mathrm{Im}(T) = \mathbb{R}^2$, so a basis is $\{(1,0),(0,1)\}$.

5. For $f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ write $f(x) = \frac{1}{2}(f(x) + f(-x)) + \frac{1}{2}(f(x) - f(-x))$. The former is an even function while the latter is an odd function, so $\mathcal{F}(\mathbb{R}, \mathbb{R}) = U + W$. If f is both even and odd, $f(x) = f(-x) = -f(x)$ for all x, i.e. $f(x) = 0$, so $U \cap W = \{0\}$ and the sum is direct.

6. (a) If $x \in \mathrm{Im}(T) \cap \ker(T)$, then $x = T(y)$ for some y and $0 = T(x) = T^2(y)$, so $y \in \ker(T^2)$. By the dimension theorem, $\dim(\ker(T)) = \dim(\ker(T^2))$, so because $\ker(T) \subset \ker(T^2)$ this says $\ker(T^2) = \ker(T)$, i.e. $y \in \ker(T)$ so $x = T(y) = 0$. By HW3.5, $\dim(\mathrm{Im}(T) + \ker(T)) = \dim(\mathrm{Im}(T)) + \dim(\ker(T)) - \dim(\mathrm{Im}(T) \cap \ker(T)) = n - 0 = n$ by the dimension theorem and the above, so $\mathrm{Im}(T) + \ker(T) = V$ and therefore $V = \mathrm{Im}(T) \oplus \ker(T)$.
   (b) For $v \in V$, we have $v = T(v) + (v - T(v))$ where $v - T(v) \in \ker(T)$ because $T(v - T(v)) = T(v) - T^2(v) = 0$. This says $V = \mathrm{Im}(T) + \ker(T)$. If $x \in \mathrm{Im}(T) \cap \ker(T)$, then $x = T(y)$ for some y and $T(x) = 0$, so $0 = T(x) = T^2(y) = T(y) = x$, and therefore the sum is direct.

7. Let $\{e_1, \cdots, e_n\}$ be an F-basis of L, and $\{f_1, \cdots, f_m\}$ be a L-basis of K. Then $S = \{e_i f_j\}$ is an F-basis of L. Suppose that $\sum_{j=1}^m \sum_{i=1}^n c_{ij} e_i f_j = 0$, where $c_{ij} \in F$. Then $\sum_{j=1}^m \sum_{i=1}^n c_{ij} e_i f_j = \sum_{j=1}^m \beta_j f_j = 0$ where $\beta_j = \sum_{i=1}^n c_{ij} e_i \in L$. Since the f_j are a basis of K as an L-vector space, each $\beta_j = 0$, so $\sum_{i=1}^n c_{ij} e_i = 0$ for all j. But then the e_i form a basis of L as an F-vector space, so $c_{ij} = 0$ for all i, j, showing S is linearly independent. If $x \in K$, write $x = \sum_{j=1}^m a_j f_j$ where $a_j \in L$. Since $a_j \in L$, write $a_j = \sum_{i=1}^n c_{ij} e_i$ for some $c_{ij} \in F$. This says $x = \sum_{j=1}^m a_j f_j = \sum_{j=1}^m \sum_{i=1}^n c_{ij} e_i f_j \in \mathrm{Span}(S)$. This shows S is a basis of mn elements.