Discussion problems
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Unless otherwise stated, V is a finite dimensional vector space over an arbitrary field F of dimension n. The letter T will always denote a linear transformation.

1. Prove there do not exist $A, B \in M_n(F)$ such that $AB - BA = I_n$.

   Solution. Suppose to the contrary, that there are A, B such that $AB - BA = I_n$. Taking traces says $0 = \operatorname{tr}(AB - BA) = \operatorname{tr}(I_n) = n$, a contradiction. Therefore no such A, B exist.

2. Suppose $T : V \to W$ is a surjection. Prove there is $U \subset V$ such that $T|_U : U \to W$ is an isomorphism.

   Solution. Pick a basis $w_1, \ldots, w_k$ of W. By assumption, there are vectors $v_1, \ldots, v_k$ such that $T(v_i) = w_i$. Suppose that $c_1v_1 + \ldots + c_kv_k = 0$. Then $c_1T(v_1) + \ldots + c_kT(v_k) = c_1w_1 + \ldots + c_kw_k = 0$, so all $c_i = 0$ by linear independence of the w_i so $\{v_1, \ldots, v_k\}$ is linearly independent. Set $U = \operatorname{Span}\{v_1, \ldots, v_k\}$. Then $T|_U(v_i) = w_i$ maps a basis of U to a basis of W, and since U and W have the same dimension this says $T|_U$ is an isomorphism.

3. Let $T : V \to V$. Show there is n such that $\ker(T^n) = \ker(T^{n+1})$.

   Solution. Suppose otherwise, that no such n exists. Since $\ker(T^i) \subset \ker(T^{i+1})$ for all i, we get a chain of subspace $\ker(T) \subsetneq \ker(T^2) \subsetneq \ldots$. Since the containment at each stage is proper, this says $\dim\ker(T^i) < \dim\ker(T^{i+1})$ for all i. Since V is finite dimensional, there is some k such that $\dim\ker(T^k) = n$, i.e. $T^k = 0$, but then $T^{k+1} = 0$ so that $\ker(T^k) = \ker(T^{k+1})$, a contradiction.

4. Let $T : V \to V$ be such that $T^2 = 0$. Show that $2 \cdot \operatorname{rank}(T) \leq n$.

   Solution. By rank-nullity, $n = \operatorname{rank}(T) + \dim\ker(T)$. Since $T^2 = 0$, if $y \in \operatorname{Im}(T)$ then $y = T(x)$ for some x so $T(y) = T^2(x) = 0$. This says $\operatorname{Im}(T) \subset \ker(T)$, so $\operatorname{rank}(T) \leq \dim\ker(T)$. This then gives $n = \operatorname{rank}(T) + \dim\ker(T) \geq \operatorname{rank}(T) + \operatorname{rank}(T) = 2 \cdot \operatorname{rank}(T)$ as desired.

5. Let $T, S : V \to V$. Prove that $\operatorname{rank}(ST) \leq \min\{\operatorname{rank}(S), \operatorname{rank}(T)\}$.

   Solution. If $y \in \operatorname{Im}(ST)$, then $y = (ST)(x) = S(T(x))$ for some $x \in V$, so $\operatorname{Im}(ST) \subset \operatorname{Im}(S)$ gives $\operatorname{rank}(ST) \leq \operatorname{rank}(S)$. If $x \in \ker(T)$, then $(ST)(x) = S(T(x)) = 0$, so $\ker(T) \subset \ker(ST)$ gives $\dim\ker(T) \leq \dim\ker(ST)$. By rank-nullity, $n = \operatorname{rank}(ST) + \dim\ker(ST)$ gives $\operatorname{rank}(ST) = n - \dim\ker(ST) \leq n - \dim\ker(T) = \operatorname{rank}(T)$, so $\operatorname{rank}(ST) \leq \operatorname{rank}(T)$ so we are done.

6. Let $T_1, T_2 : V \to V$. Show that $\ker(T_1) = \ker(T_2)$ if and only if there exists $S : V \to V$ invertible such that $T_1 = ST_2$.

   Solution. Suppose that $T_1 = ST_2$ for some invertible S. If $x \in \ker(T_1)$, then $(ST_2)(x) = 0$, so composing with S^{-1} on the left says $T_2(x) = 0$, so $x \in \ker(T_2)$. Similarly if $x \in \ker(T_2)$, then $(ST_2)(x) = T_1(x) = 0$, so $x \in \ker(T_1)$ gives $\ker(T_1) = \ker(T_2)$. Now suppose that $\ker(T_1) = \ker(T_2) = W$. Let $v_1, \ldots, v_k$ be a basis of W. Extend to a basis $v_1, \ldots, v_k, w_1, \ldots, w_m$. Then $T_1(w_1), \ldots, T_1(w_m)$ and $T_2(w_1), \ldots, T_2(w_m)$ must be linearly independent: if $c_1T_j(w_1) + \ldots + c_mT_j(w_m) = 0$, this says $c_1w_1 + \ldots + c_mw_m \in \ker(T_j)$ so $c_1w_1 + \ldots + c_mw_m = d_1v_1 + \ldots + d_kv_k$ for some d_i. Subtracting then forces all $c_i = d_i = 0$. Extend $T_1(w_1), \ldots, T_1(w_m)$ to a basis $T_1(w_1), \ldots, T_1(w_m), e_1, \ldots, e_k$ of V and $T_2(w_1), \ldots, T_2(w_m)$ to a basis $T_2(w_1), \ldots, T_2(w_m), e_1', \ldots, e_k'$. Define $S : V \to V$ by $S(T_2(w_i)) = T_1(w_i)$ and $S(e_i) = e_i'$. Then S maps a basis of V to a basis of V so S is invertible. We have $(ST_2)(w_i) = T_1(w_i)$ and $(ST_2)(v_i) = 0 = T_1(v_i)$, so ST_2 and T_1 agree on a basis of V and therefore are equal.

7. Let $T : V \to V$ satisfy $T^n = 0$ but $T^{n-1} \neq 0$. Let $v \in V$ be such that $T^{n-1}(v) \neq 0$.

   (a) Prove that $\beta = \{T^{n-1}(v), T^{n-2}(v), \ldots, v\}$ is a basis of V, and compute $[T]_\beta$.

   Solution. Suppose $c_1 T^{n-1}(v) + \ldots + c_n v = 0$. Applying T^{n-1} to both sides shows $c_n T^{n-1}(v) = 0$, so $c_n = 0$. Applying T^{n-2} to both sides shows $c_{n-1} T^{n-1}(v) = 0$, so $c_{n-1} = 0$. Repeating this argument shows that $c_i = 0$ for all i, so β is a basis of V. We then see that $[T]_\beta = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

   (b) Find $A \in M_3(\mathbb{R})$ such that $A^2 \neq 0$ but $A^3 = 0$.

   Solution. Take $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

8. Compute the number of invertible matrices A in $M_n(\mathbb{F}_q)$ where $\mathbb{F}_q$ is a finite field of size q.

   Solution. Let $A \in M_n(\mathbb{F}_q)$ be invertible. A matrix is invertible if and only if its columns are linearly independent, so we will count the number of such matrices. There are $q^n - 1$ non-zero vectors in $\mathbb{F}_q^n$, so there are $q^n - 1$ ways to pick the first column v_1 of A. The second column cannot be any of the q multiples of v_1, so there are $q^n - q$ ways to pick the second column v_2. The third column cannot be one of the q^2 linear combinations of v_1 and v_2, so there are $q^n - q^2$ ways to pick v_3. Continuing this, we see there are $(q^n - 1)(q^n - q) \cdots (q^n - q^{n-1})$ possible ways to pick columns $v_1, \ldots, v_n$, and therefore this is the number of invertible matrices.