Chapter 2 challenge problems

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Unless otherwise stated, V is a finite dimensional vector space over an arbitrary field F of dimension n.

1. For the following statements, indicate if it is true or false. If it is true, provide a proof and if it is false, justify why.
   (a) Let $T : F^8 \to F^5$ be a linear transformation such that $\ker(T)$ is 3-dimensional. Then T is surjective.
   (b) The vector spaces $\mathrm{Sym}_3(\mathbb{R})$ (symmetric 3×3 matrices) and $W = \left\{X \in M_2(\mathbb{R}) : \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} X = X \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\right\}$ are isomorphic.
   (c) Let V be a vector space (potentially infinite dimensional). If $S, T : V \to V$ are linear and $ST = \mathrm{id}_V$, then $S = T^{-1}$.

2. Give an example of the following concepts, with a brief justification.
   (a) A linear transformation $T : P_3(F) \to P_3(F)$ with $\mathrm{Im}(T) = \ker(T)$.
   (b) A matrix $A \neq I_2 \in M_2(\mathbb{R})$ with $A^5 = I_2$.

4. Let $A, B \in M_n(F)$. Recall that A is *similar* to B if $A = PBP^{-1}$ for some invertible $P \in M_n(F)$.
   (a) Prove that if A is similar to B, then $\mathrm{rank}(A) = \mathrm{rank}(B)$.
   (b) Suppose that $A^2 = A$, $B^2 = B$, and $I_n - (A + B)$ is invertible. Prove that $\mathrm{rank}(A) = \mathrm{rank}(B)$.
   (c) If $AB - BA = A$, prove that A is not invertible. (Hint: use properties of trace).

5. Let $S, T : V \to V$ be linear transformations.
   (a) Prove that $\dim(\ker(ST)) \leq \dim(\ker(S)) + \dim(\ker(T))$.
   (b) (Sylvester's rank inequality) Prove that $\mathrm{rank}(S) + \mathrm{rank}(T) \leq \mathrm{rank}(ST) + n$.

6. Let $T : V \to V$ be a linear transformation.
   (a) Show that if v and $T(v)$ are linearly dependent for all $v \in V$, then $T = c \cdot \mathrm{id}_V$ for some $c \in F$.
   (b) Suppose that $TS = ST$ for all $S \in \mathrm{Hom}_F(V, V)$. Prove that $T = c \cdot \mathrm{id}_V$ for some $c \in F$.

7*. Let $T : V \to V$ be linear and let $W \subset V$ be a T-invariant subspace, i.e. a subspace with $T(W) \subset W$. W is called *T-irreducible* if the only T-invariant subspaces of W are $\{0\}$ and W. Prove that if W is T-irreducible and $\varphi \in \mathrm{Hom}_F(W, W)$ satisfies $\varphi T = T\varphi$, then $\varphi = 0$ or φ is an isomorphism.

8*. A matrix $A \in M_n(F)$ is called *nilpotent* if there exists k such that $A^k = 0$. Prove that if A is nilpotent then $I_n - A$ is invertible. (Hint: think about the power series of $(1 - x)^{-1}$).

9.* Let $W \subset V$ be a subspace. The *annihilator* W° of W is defined by $W^\circ = \{T \in \mathrm{Hom}_F(V, F) : T(x) = 0 \text{ for all } x \in W\}$. Prove that $n = \dim(W) + \dim(W^\circ)$. (Hint: find a basis of W°)