EXAMPLES OF INDUCTION

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The purpose of this handout is to provide some examples of how to prove statements using mathematical induction.

The most common statement of mathematical induction may be stated as follows:

Theorem 0.1. *(Principle of Mathematical Induction) For $n \in \mathbb{N}$, let $P(n)$ be a statement such that*

1. *$P(n_0)$ is true for some n_0*
2. *$P(k)$ is true implies $P(k+1)$ is true for all $k \geq n_0$.*

Then $P(n)$ is true for all $n \geq n_0$.

There is also a "stronger" version of induction:

Theorem 0.2. *(Principle of Strong Induction) For $n \in \mathbb{N}$, let $P(n)$ be a statement such that*

1. *$P(n_0)$ is true for some n_0*
2. *$P(n_0), \ldots, P(k)$ is true implies $P(k+1)$ is true for all $k \geq n_0$.*

Then $P(n)$ is true for all $n \geq n_0$.

It turns out, the two forms of induction are actually equivalent (so that it makes sense to just speak of "induction"), but we'll take this for granted.

The statement of mathematical induction tells us that a proof by induction has two parts: first, check that what we want to prove is true for some specific integer value. This is usually referred to as "checking a base case". the second step in a proof by induction is usually called the "inductive step". In the inductive step, one assumes the statement $P(k)$ is true (or the statements $P(n_0), \ldots, P(k)$ in a proof using strong induction), and uses this to deduce that $P(k+1)$ must be true. The assumption that $P(k)$ is true in the inductive step is called the induction hypothesis. Below are several examples of how induction is commonly used.

Example 0.3. For all $n \geq 1$, $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

Proof. Note that for $n = 1$, the left hand side is $\displaystyle\sum_{i=1}^{1} i = 1$, and the right hand side is $\dfrac{1 \cdot 2}{2} = 1$, so the statement is true for $n = 1$. Suppose it's true for $n = k$, that $\displaystyle\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$. Then for $n = k+1$, we have $\displaystyle\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}$, where the second equality follows from the induction hypothesis. Therefore it's true for $n = k+1$, so by induction, the statement is true for all integers $n \geq 1$. $\square$

Example 0.4. For all $n \geq 4$, $2^n \leq n!$.

Proof. Note that for $n = 4$, the left hand side is 16 while the right hand side is 24, so the statement is true for $n = 4$. Suppose it's true for $n = k \geq 4$, that $2^k \leq k!$. Then $2^{k+1} = 2 \cdot 2^k \leq 2 \cdot k! \leq k \cdot k! = (k+1)!$ by induction hypothesis. Therefore it's true for $n = k+1$, so by induction it holds true for all $n \geq 4$. □

Example 0.5. For all $n \geq 1$, $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$, where F_n is the Fibonacci sequence defined by $F_{n+2} = F_{n+1} + F_n$ and $F_1 = F_2 = 1$.

Proof. For $n = 1$, the left hand side is $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ while the right hand side is also $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ (we define $F_0 = 0$). Suppose it's true for $n = k$, that $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^k = \begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix}$. Then $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^k \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} F_{k+1} + F_k & F_{k+1} \\ F_k + F_{k-1} & F_k \end{pmatrix} = \begin{pmatrix} F_{k+2} & F_{k+1} \\ F_{k+1} & F_k \end{pmatrix}$ where the second equality follows from induction hypothesis and the last step from the definition of the Fibonacci sequence. Therefore the statement is true for $n = k+1$ and therefore all $n \geq 1$ by induction. □

Example 0.6. The sequence $a_n = 2a_{n-1} + 3a_{n-2}$ is odd for $n \geq 3$ where $a_1 = a_2 = 1$.

Proof. We have $a_3 = 2a_2 + 3a_3 = 5$ is odd. Now suppose that it's true for all $1, 2, \cdots, k$ that a_k is odd. Then $a_{k+1} = 2a_k + 3a_{k-1}$ by definition, and by induction hypothesis, a_k and a_{k-1} are odd. Write $a_k = 2m+1$ and $a_{k-1} = 2\ell + 1$ for some integers k and ℓ. Then $a_{k+1} = 2(2m+1) + 3(2\ell+1) = 2(2m+3\ell) + 5$ is odd. Therefore the statement holds for $n = k+1$ and therefore for all $n \geq 3$ by induction. □

Example 0.7. Let $\Gamma(s) = \displaystyle\int_0^\infty t^{s-1} e^{-t}\, dt$. Then $\Gamma(n) = (n-1)!$ for all $n \geq 1$.

Proof. For $n = 1$, we have $\Gamma(1) = \displaystyle\int_0^\infty e^{-t}\, dt = -e^{-t}|_0^\infty = 1$, and $1 = 0!$ so the base case holds. Now suppose that $\Gamma(k) = (k-1)!$ for $n = k$. Using integration by parts, $\Gamma(k+1) = \displaystyle\int_0^\infty t^k e^{-t}\, dt = -t^k e^{-t}|_0^\infty + \int_0^\infty k t^{k-1} e^{-t}\, dt = k \int_0^\infty t^{k-1} e^{-t}\, dt = k\Gamma(k) = k(k-1)! = k!$ by induction hypothesis. Therefore the statement holds for $n = k+1$, so by induction, it holds for all $n \geq 1$. □