Diagonalization in Action: Fibonacci Numbers
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Definition. The *Fibonacci numbers* are defined recursively through the sequence $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$.

If you write down the first few terms of the Fibonacci sequence, it goes $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$. It's clear the sequence is growing, but how quickly? If you compute the ratio of successive terms of the Fibonacci sequence, it appears to be ≈ 1.618, which suggests that the sequence is growing exponentially. Moreover, is it possible to find an explicit formula for the n-th Fibonacci number? The Fibonacci sequence is an example of a *linear recursive sequence* (i.e. a sequence where the n-th term is a linear combination of previous terms). As the name suggests, since we are dealing with a linear recursive sequence, our problems are therefore something linear algebra is equipped to handle.

Before we can do any linear algebra, we need to find a way to turn our sequence into a matrix. This can be done via the following:

Lemma. For all $n \geq 1$, $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$.

Proof. This is obviously true for $n = 1$. Suppose it's true for $n = k$, that $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^k = \begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix}$. Then $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^k \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} F_{k+1} + F_k & F_{k+1} \\ F_k + F_{k-1} & F_k \end{pmatrix} = \begin{pmatrix} F_{k+2} & F_{k+1} \\ F_{k+1} & F_k \end{pmatrix}$ where the second equality follows from induction hypothesis and the last step from the definition of the Fibonacci sequence. Therefore the statement is true for $n = k + 1$ and therefore all $n \geq 1$ by induction. $\square$

A cute application of the above is the following identity, due to Cassini around 1680 (long before matrices existed!):

Corollary. $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ for all $n \geq 1$.

Proof. For $n \geq 1$, $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$ so taking determinants immediately gives the result. $\square$

Since we've recorded the Fibonacci numbers in a matrix, we can derive a formula for the n-th Fibonacci number by computing the lower left entry of $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n$. Since this requires computing arbitrary powers of some matrix, the natural approach is try diagonalizing to make the problem easy.

Theorem. $F_n = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^n$ *for all* $n \geq 0$.

Proof. First observe the result is obviously true for $n = 0$. By the previous lemma, $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$ for all $n \geq 1$, so that $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}$. The matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ has characteristic polynomial $p_A(x) = x^2 - x - 1$, so that A has eigenvalues $\frac{1 \pm \sqrt{5}}{2}$, and therefore is diagonalizable because these are distinct. One can check that $v_1 = (1+\sqrt{5}, 2)$ and $v_2 = (1-\sqrt{5}, 2)$ are eigenvectors corresponding to $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$ respectively, so that $\beta = \{v_1, v_2\}$ is an eigenbasis of $\mathbb{R}^2$ consisting of eigenvectors of A. With $\mathcal{E}$ the standard basis of $\mathbb{R}^2$, the change of basis matrices are given by $S_\beta^\mathcal{E} = \begin{pmatrix} 1+\sqrt{5} & 1-\sqrt{5} \\ 2 & 2 \end{pmatrix}$ and $S_\mathcal{E}^\beta = \begin{pmatrix} \frac{\sqrt{5}}{10} & \frac{5-\sqrt{5}}{20} \\ \frac{-\sqrt{5}}{10} & \frac{5+\sqrt{5}}{20} \end{pmatrix}$, so the change of basis formula gives us the factorization $A = S_\beta^\mathcal{E} A_\beta S_\mathcal{E}^\beta = \begin{pmatrix} 1+\sqrt{5} & 1-\sqrt{5} \\ 2 & 2 \end{pmatrix} \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{5}}{10} & \frac{5-\sqrt{5}}{20} \\ \frac{-\sqrt{5}}{10} & \frac{5+\sqrt{5}}{20} \end{pmatrix}$, where A_β is the matrix of A with respect to the β basis, i.e. the diagonal matrix of eigenvalues of A. Then $A^n = \begin{pmatrix} 1+\sqrt{5} & 1-\sqrt{5} \\ 2 & 2 \end{pmatrix} \begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{pmatrix} \begin{pmatrix} \frac{\sqrt{5}}{10} & \frac{5-\sqrt{5}}{20} \\ \frac{-\sqrt{5}}{10} & \frac{5+\sqrt{5}}{20} \end{pmatrix}$, so that

$$\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+\sqrt{5} & 1-\sqrt{5} \\ 2 & 2 \end{pmatrix} \begin{pmatrix} \left(\frac{(1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{pmatrix} \begin{pmatrix} \frac{\sqrt{5}}{10} & \frac{5-\sqrt{5}}{20} \\ \frac{-\sqrt{5}}{10} & \frac{5+\sqrt{5}}{20} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Multiplying out the product and computing the bottom entry of the resulting vector, you'll find that $F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$. ☐

The number $\frac{1+\sqrt{5}}{2}$ is denoted by φ, and is called the *golden ratio*. Numerically, we have $\varphi \approx 1.618$. As you might guess from what was said earlier, the following is true:

Corollary. $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \varphi$.

Proof. Let $\phi = \frac{1-\sqrt{5}}{2}$, so that we can write $F_n = \frac{1}{\sqrt{5}}(\varphi^n - \phi^n)$. Then $F_{n+1} = \frac{1}{\sqrt{5}}(\varphi^{n+1} - \phi^{n+1})$, so $\frac{F_{n+1}}{F_n} = \frac{\varphi^{n+1} - \phi^{n+1}}{\varphi^n - \phi^n}$. Since $\phi < 1$, the result immediately follows upon taking limits. ☐

Since $\phi < 1$, as $n \to \infty$ we have $\phi^n \to 0$, so we see $F_n \approx \frac{1}{\sqrt{5}}\varphi^n$. In fact, since we know F_n is an integer, because $\frac{1}{\sqrt{5}}\phi^n < 1/2$, we actually can write $F_n = \left[\frac{1}{\sqrt{5}}\varphi^n\right]$, where $[\cdot]$ is the function that rounds to the nearest integer, which makes the computation of large Fibonacci numbers slightly easier. This also confirms our earlier observation: the Fibonacci sequence is growing at an exponential rate.

As a parting gift, here is an interesting coincidence you can use to wow (or bore) your friends with: the conversion factor from miles to kilometers is 1.60934, which is fairly close to φ. This lets you easily convert miles to kilometers in quantities that are close to Fibonacci numbers. For example, 89 miles is roughly 144 km, because 144 is the number after 89 in the Fibonacci sequence!