DIAGONALIZATION

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After developing the basic theory of linear transformations, we turn our attention to the study of linear operators. As we saw, a change of basis of V from β to β' corresponds to conjugation of the matrix $[T]_\beta$ by some invertible change of basis matrix $S_\beta^{\beta'}$. A natural question is if there is a "best" basis β to pick so that $[T]_\beta$ will be as easy to understand as possible. The best we could hope for in general is that $[T]_\beta$ is a diagonal matrix, so the question becomes if it is possible to find a basis β of V such that $[T]_\beta$ is diagonal. Answering this question will be the primary purpose of this handout.

Throughout this document, V will denote a vector space over an arbitrary field F and $T : V \to V$ will denote a linear operator.

BASIC DEFINITIONS AND EXAMPLES

Definition 0.1. An **eigenvector** of T is a non-zero vector $v \in V$ such that $T(v) = \lambda v$ for some $\lambda \in F$. The number λ is called an **eigenvalue** of T. We sometimes refer to the data (v, λ) as an **eigenpair**.

Our first order of business is to determine what the possible eigenvalues of a linear operator are. When V is finite dimensional, this is quite easily done using the theory of determinants. We remind the reader of the following definition:

Definition 0.2. Let V be an n dimensional vector space. The **determinant** of a linear operator $T : V \to V$ denoted $\det(T)$ is defined as $\det([T]_\beta)$ for any basis β of V.

Elementary properties of the determinant show that similar matrices have the same determinant, so the above definition actually is independent of a choice of basis and so the notation makes sense.

Theorem 0.3. *Let V be an n dimensional vector space. Then $\lambda \in F$ is an eigenvalue of T if and only if $\det(\lambda \cdot I_V - T) = 0$. In terms of matrices, λ is an eigenvalue of T if and only if $\det(\lambda \cdot I_n - [T]_\beta) = 0$ for any basis β of V.*

Proof. Pick a basis β of V. Suppose that $\lambda \in F$ is an eigenvalue of T with eigenvector v. Then $(\lambda \cdot I_V - T)(v) = 0$, i.e. the operator $\lambda \cdot I_V - T$ is not invertible, and from the theory of matrices, this says $[\lambda \cdot I_V - T]_\beta$ is not invertible. Therefore $\det([\lambda \cdot I_V - T]_\beta) = 0$, so by definition this says that $\det(\lambda \cdot I_V - T) = 0$. Conversely, if $\det(\lambda \cdot I_V - T) = 0$, then $\lambda \cdot I_V - T$ is not invertible, so there is some vector v in the kernel of $\lambda \cdot I_V - T$, i.e. a vector v such that $T(v) = \lambda v$ so that λ is an eigenvalue of T. $\square$

The above says that checking if λ is an eigenvalue of T is equivalent to finding a root of the polynomial $\det(x \cdot I_V - T)$. We give this polynomial a name:

Definition 0.4. The polynomial $p_T(x) = \det(x \cdot I_V - T)$ is called the **characteristic polynomial** of T.

Restated in the new definition, we have the following:

Theorem 0.5. *Let V be an n dimensional vector space. $\lambda \in F$ is an eigenvalue of $T : V \to V$ if and only if λ is a root of the characteristic polynomial $p_T(x)$.*

Notice that since the determinant of a linear operator does not depend on a choice of basis, the characteristic polynomial is independent of such a choice as well, and so it is well defined.

Before moving on, we make a few remarks: notice that the definition of an eigenvalue depends on which field we view V as a vector space over. If F is algebraically closed, then every linear operator $T : V \to V$ has an eigenvalue, because then the characteristic polynomial of T necessarily has a root in F. If F is not algebraically closed, it may be possible for an operator to not have an eigenvalue. Additionally, the definition of an eigenvalue makes sense for infinite dimensional vector spaces, even if our criterion for easily finding eigenvalues only works for finite dimensional vector spaces. Some of the examples below will illustrate this.

Example 0.6. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $T(x, y, z) = (y, -5x+4y+z, -x+y+z)$. Then with β the standard basis, we see $[T]_\beta = \begin{pmatrix} 0 & 1 & 0 \\ -5 & 4 & 1 \\ -1 & 1 & 1 \end{pmatrix}$. Then $p_T(x) = (x-1)(x-2)^2$, so T has eigenvalues 1 and 2. One can check T has eigenvectors $v_1 = (1, 1, 2)$ and $v_2 = (1, 2, 1)$ corresponding to the eigenvalues 1 and 2 respectively.

Example 0.7. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a counterclockwise rotation by some angle $\theta \in (0, 2\pi)$. Then T has no eigenvectors, because no vector $v \in \mathbb{R}^2$ is scaled along the same direction by T. Explicitly, with $\beta = \{e_1, e_2\}$ the standard basis of $\mathbb{R}^2$, one can check that $[T]_\beta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$. The characteristic polynomial of T is given by $p_T(x) = x^2 - 2\cos(\theta)x + 1$, and the quadratic formula says this has no real roots.

Example 0.8. Write $V = W \oplus W'$ for some subspaces W, W' of V. Let $P = \pi_W$ be the projection onto W, so that $P^2 = P$. If (λ, v) is an eigenpair for P, then $\lambda^2 v = P^2 v = Pv = \lambda v$, so that $\lambda^2 = \lambda$ says that $\lambda = 0, 1$ are the only possible eigenvalues of P. For any $w \in W$, $P(w) = w$, and for $w' \in W'$, $P(w') = 0$, so w, w' are eigenvectors corresponding to $0, 1$ respectively.

Example 0.9. Let $T : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ be the derivative map, $T(f) = f'$. An eigenvector of T is a function f such that $f' = \lambda f$ for some $\lambda \in \mathbb{R}$. From calculus, we know that such functions are of the form $ce^{\lambda t}$ for some $c \in \mathbb{R}$. This says the exponential functions $ce^{\lambda t}$ are eigenvectors of the derivative operator with eigenvalues $\lambda \in \mathbb{R}$. This is of fundamental importance in the theory of linear differential equations.

Example 0.10. Let $L : F^\infty \to F^\infty$ be the left shift operator, i.e. $L((a_1, a_2, \ldots)) = (a_2, a_3, \ldots)$. An eigenvector of L is a sequence $(a_1, a_2, \ldots)$ such that $(a_2, a_3, \ldots) = \lambda(a_1, a_2, \ldots)$ for some $\lambda \in F$. This says $a_2 = \lambda a_1$, $a_3 = \lambda a_2 = \lambda^2 a_1$, and by induction, that $a_n = \lambda^{n-1} a_1$. Let $\sigma = \{a_n\}$ be a geometric sequence, that is a sequence defined by $a_n = cr^{n-1}$ for some $c, r \in F$. Then we see that an eigenvector of L is a geometric sequence, and any such geometric sequence σ is an eigenvector with eigenvalue r.

Properties of the characteristic polynomial

Before moving on, it will be useful to know some basic properties of the characteristic polynomial $p_T(x)$ of a linear operator T.

Proposition 1. *Let V be an n dimensional vector space and $T : V \to V$ a linear operator. Then the characteristic polynomial $p_T(x)$ is a monic degree n polynomial.*

Proof. Pick a basis β of V, so by definition $p_T(x) = \det(x \cdot I_n - [T]_\beta)$, and write $[T]_\beta = [a_{ij}]$. Then $x \cdot I_n - [T]_\beta$ has entries $x - a_{ii}$ along the diagonal and $-a_{ij}$ elsewhere. Expanding the determinant out using co-factors along the first row, we see that we can write $p_T(x) = (x - a_{11}) \cdots (x - a_{nn}) + q(x)$ where $q(x)$ is a polynomial of degree at most $n - 2$. It's then clear that $p_T(x)$ is a monic degree n polynomial. $\square$

Proposition 2. *Let V be an n dimensional vector space and T a linear operator with characteristic polynomial $p_T(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0$. Then $a_{n-1} = -\operatorname{tr}(T)$ and $a_0 = (-1)^n \det(T)$.*

Proof. Write $p_T(x) = (x - a_{11}) \cdots (x - a_{nn}) + q(x)$ where $q(x)$ has degree at most $n - 2$ as above. Then the coefficient of x^{n-1} comes entirely from the coefficient of x^{n-1} in the product $(x-a_{11}) \ldots (x-a_{nn})$, which is equal to the negative sum of its roots, i.e. $-(a_{11}+\ldots+a_{nn}) = -\operatorname{tr}(T)$. We also see $a_0 = p_T(0) = \det(0 \cdot I_n - T) = \det(-T) = (-1)^n \det(T)$. $\square$

Example 0.11. A particularly useful special case is when V is 2 dimensional, so for a linear operator T we have $p_T(x) = x^2 - \operatorname{tr}(T)x + \det(T)$.

The above result says that the characteristic polynomial $p_T(x)$ record both the trace and determinant of T. Since the roots of $p_T(x)$ are eigenvalues of T, this gives the following important relations:

Corollary 0.12. *Let V be an n dimensional vector space and T a linear operator. Suppose that the eigenvalues of T in some algebraic closure $\overline{F}$ of F are $\lambda_1, \ldots, \lambda_n$ (counted with multiplicity). Then $\operatorname{tr}(T) = \lambda_1 + \ldots + \lambda_n$ and $\det(T) = \lambda_1 \cdots \lambda_n$.*

Proof. In $\overline{F}[x]$, $p_T(x)$ factors as $(x - \lambda_1) \cdots (x - \lambda_n)$. Expanding out the product then says the coefficient of x^{n-1} is $-(\lambda_1 + \ldots + \lambda_n)$ and the constant term is $(-1)^n(\lambda_1 \cdots \lambda_n)$. $\square$

Diagonalization

We now use the theory of eigenvalues to answer our main question.

Definition 0.13. A linear operator T is called *diagonalizable* if there exists a basis β of V such that $[T]_\beta$ is a diagonal matrix.

Necessarily, we see that if such a basis $\beta = \{v_1, \ldots, v_n\}$ exists, if $[T]_\beta = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix}$ that $T(v_i) = \lambda_i v_i$ for all i, so that β is a basis of eigenvectors. This can be rephrased using the language of the previous section as such:

Theorem 0.14. *A linear operator T is diagonalizable if and only if there is a basis β of V consisting of eigenvectors of T.*

In order to determine when such a basis exists, we will utilize the following key result:

Proposition 3. *Suppose $v_1, \dots, v_n$ are eigenvectors of T corresponding to distinct eigenvalues $\lambda_1, \dots, \lambda_n$. Then $\{v_1, \dots, v_n\}$ is linearly independent.*

Proof. We prove this by induction. If $n = 1$ the statement follows immediately since $\{v_1\}$ is linearly independent. Assume that the statement is true for any collection of $n-1$ eigenvectors that correspond to distinct eigenvalues. Suppose that $c_1v_1 + \dots + c_nv_n = 0$, so that applying T says $c_1\lambda_1v_1 + \dots + c_n\lambda_nv_n = 0$. Multiply the first equation by λ_n and subtract to see $c_1(\lambda_1 - \lambda_n)v_1 + \dots + c_{n-1}(\lambda_{n-1} - \lambda_n)v_{n-1} = 0$. By induction hypothesis, the vectors $\{v_1, \dots, v_{n-1}\}$ are linearly independent, and since all eigenvalues are distinct this forces $c_i = 0$ for $1 \le i \le n-1$. This then immediately gives $c_n = 0$, so that $\{v_1, \dots, v_n\}$ is linearly independent. By induction, we are done. $\square$

Corollary 0.15. *Let V be an n dimensional vector space and T a linear operator. If T has n distinct eigenvalues, then T is diagonalizable. If $p_T(x)$ factors into distinct linear factors in $F[x]$, then T is diagonalizable.*

Proof. If T has n distinct eigenvalues, then the associated eigenvectors are a set of n linearly independent vectors in V, hence a basis. Saying that $p_T(x)$ splits into distinct linear factors is the same as saying that T has distinct eigenvalues. $\square$

Example 0.16. The converse to the above statement is not necessarily true. For example, the identity operator I_V is diagonalizable, but has characteristic polynomial $(x-1)^n$.

Proposition 4. *Let V be an n dimensional vector space and T a linear operator. Then if T is diagonalizable, $p_T(x)$ factors into a product of (not necssesarily distinct) linear factors in $F[x]$.*

Proof. Suppose that T is diagonalizable. Let β be a basis of V consisting of eigenvalues $\lambda_1, \dots, \lambda_n$ of T. Then $[T]_\beta = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$, and $\det(xI_n - [T]_\beta) = (x - \lambda_1) \cdots (x - \lambda_n)$ is a product of linear factors in $F[x]$. $\square$

Example 0.17. The converse of the above statement is not necessarily true. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(x, y) = (y, 0)$. Then it's easy to see $p_T(x) = x^2$, so the only eigenvalue of T is 0. However, T is not diagonalizable: rank-nullity says $\dim(\ker(T)) = 1$, so that there is no possible basis of eigenvectors for T.

The above examples show that the characteristic polynomial is not strong enough to detect when an operator is diagonalizable or not, and we saw that what broke was that nothing can be said when the characteristic polynomial has repeated roots. This leads us to the following definitions, which will end up giving a test for diagonalizability.

Definition 0.18. Let λ be an eigenvalue of T. The **eigenspace of** λ, denoted E_λ, is defined as $E_\lambda = \ker(T - \lambda \cdot I_V)$. The **algebraic multiplicity** of λ is its multiplicity as a root of $p_T(x)$. The **geometric multiplicity** of λ is $\dim(E_\lambda)$.

Proposition 5. *The geometric multiplicity of an eigenvalue λ of T is at most its algebraic multiplicity.*

Proof. Suppose that $\dim(E_\lambda) = k$. Pick a basis $\{v_1, \dots, v_k\}$ of E_λ and extend to a basis $\beta = \{v_1, \dots, v_k, w_1, \dots, w_m\}$ of V. Then $[T]_\beta$ is a block matrix given by $[T]_\beta = \begin{pmatrix} \lambda \cdot I_k & A \\ 0 & B \end{pmatrix}$ where 0 is the $m \times k$ zero matrix, and A, B have dimensions $k \times m$ and $m \times m$ respectively. Then $x \cdot I_n - [T]_\beta = \begin{pmatrix} (x-\lambda) \cdot I_k & -A \\ 0 & x \cdot I_m - B \end{pmatrix}$, so $p_T(x) = \det(x \cdot I_n - [T]_\beta) = \det((x-\lambda) \cdot I_k) \det(x \cdot I_m - B) = (x-\lambda)^k g(x)$ where $g(x) = \det(x \cdot I_m - B)$. This says $(x-\lambda)^k$ divides $p_T(x)$, so the algebraic multiplicity of λ is at least k as desired. □

Theorem 0.19. *Let V be an n dimensional vector space. Let T have distinct eigenvalues $\lambda_1, \dots, \lambda_k$ with algebraic multiplicities $e_1, \dots, e_k$, so $p_T(x) = (x-\lambda_1)^{e_1} \cdots (x-\lambda_k)^{e_k}$, and $e_1 + \dots + e_k = n$. Then T is diagonalizable if and only if $V = E_{\lambda_1} \oplus \dots \oplus E_{\lambda_k}$.*

Proof. If T is diagonalizable, then V has a basis of eigenvectors of T, so that $V = E_{\lambda_1} + \dots + E_{\lambda_k}$. Since eigenvectors from different eigenvalues are linearly independent, if $x_1 + \dots + x_k = 0$ with $x_i \in E_{\lambda_i}$, writing x_i in terms of basis vectors of E_{λ_i} shows all $x_i = 0$, so that the sum is direct. Conversely, if $V = E_{\lambda_1} \oplus \dots \oplus E_{\lambda_k}$, then the union of bases for E_{λ_i} is a basis of V. Since each vector in E_{λ_i} is an eigenvector of T, this says V has a basis of eigenvectors for T, i.e. T is diagonalizable. □

Corollary 0.20. *Let T be as above. Then T is diagonalizable if and only if for each eigenvalue λ_i of T the algebraic multiplicity and geometric multiplicity of λ_i are equal.*

Proof. If the algebraic and geometric multiplicity of λ_i are equal for all i, this says $\dim(E_{\lambda_1} \oplus \dots \oplus E_{\lambda_k}) = e_1 + \dots + e_k = n$, so $E_{\lambda_1} \oplus \dots \oplus E_{\lambda_k} = V$ says T is diagonalizable. Conversely if $\dim(E_{\lambda_i}) < e_i$ for some i, then $\dim(E_{\lambda_1} \oplus \dots \oplus E_{\lambda_k}) < n$, so $E_{\lambda_1} \oplus \dots \oplus E_{\lambda_k} \neq V$ says T is not diagonalizable. □

Example 0.21. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $T(x, y, z) = (y, -5x+4y+z, -x+y+z)$ be the example from before. Then we saw $p_T(x) = (x-1)(x-2)^2$, so 1 has algebraic multiplicity 1 and 2 has algebraic multiplicity 2. To check if T is diagonalizable, we need to see if the eigenspace E_2 is 2-dimensional. We see that $T - 2 \cdot I_V$ has matrix representation with respect to the standard basis given by $\begin{pmatrix} -2 & 1 & 0 \\ -5 & 2 & 1 \\ -1 & 1 & -1 \end{pmatrix}$. Notice that $-2(1, 2, 1) + (0, 1, -1) = (-2, -5, -1)$, so that the first column is a linear combination of the other two. The second and third columns are obviously linearly independent, so that $\operatorname{rank}(T - 2 \cdot I_V) = 2$ says $\dim(\ker(T - 2 \cdot I_V)) = 1$, so that T is not diagonalizable.

Example 0.22. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $T(x, y, z) = (-9x + 4y + 4z, -8x + 3y + 4z, -16x + 8y + 7z)$. With $\beta = \{e_1, e_2, e_3\}$ the standard basis of $\mathbb{R}^3$, we have $[T]_\beta = \begin{pmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{pmatrix}$. One can compute that $p_T(x) = (x-3)(x+1)^2$, so that T has eigenvalues 3 and -1 with algebraic multiplicities 1 and 2 respectively. The operator $T + I_V$ has matrix representation $\begin{pmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{pmatrix}$, so $\dim(E_{-1}) = 2$, so that T is diagonalizable. Using row reduction, a basis of E_{-1} is given by $v_1 = (1, 0, 2)$ and $v_2 = (1, 2, 0)$. The operator $T - 3 \cdot I_V$ has matrix representation $\begin{pmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{pmatrix}$, and a basis of E_3 is given by $v_3 = (1, 1, 2)$. Then $\beta' = \{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$ consisting of eigenvectors for T. The change of basis matrix $S_{\beta' \to \beta}$ is the matrix whose columns are these eigenvectors, i.e. $S_{\beta' \to \beta} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 2 & 0 & 2 \end{pmatrix}$, with inverse $S_{\beta \to \beta'} = \begin{pmatrix} 2 & -1 & -1/2 \\ 1 & 0 & -1/2 \\ -2 & 1 & 1 \end{pmatrix}$. The change of basis formula says $[T]_\beta = S_{\beta' \to \beta}[T]_{\beta'} S_{\beta \to \beta'}$, and we see that $[T]_{\beta'} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, which gives a factorization of $[T]_\beta$.