Axioms for Vector Spaces
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Unless otherwise stated, V is a vector space over an arbitrary field.

1. Prove the following:
   (a) $c \cdot 0 = 0$ for any $c \in F$.
   (b) $(-1) \cdot v = (-v)$ for any $v \in V$.
2. (a) Prove the *cancellation law* for vector spaces: if $u, v, w \in V$ with $u + w = v + w$, then $u = v$.
   (b) Prove that the zero element is unique, i.e. if there are elements $0, 0'$ satisfying axiom VS 3, then $0 = 0'$.
   (c) Prove that additive inverses are unique, i.e. if there are elements w, w' satisfying axiom VS 4, then $w = w'$.
3. For each of the following sets V, determine if the given addition and scalar multiplication operations make V a vector space over $\mathbb{R}$. If it is, prove it. If not, give a counterexample to one of the vector space axioms.
   (a) $V = \mathbb{R}_{>0}$ with operations $a + b = ab$ and $c \cdot a = a^c$.
   (b) $V = \mathbb{R}^2$ with operations $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 b_2)$ and $c \cdot (a_1, a_2) = (ca_1, a_2)$.
   (c) $V = \mathbb{R}^2$ with operations $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$ and $c \cdot (a_1, a_2) = \begin{cases} (0,0) & c = 0 \\ (ca_1, \frac{a_2}{c}) & c \neq 0 \end{cases}$

Solutions

1. (a) Since $0 + 0 = 0$, $c \cdot 0 = c \cdot (0 + 0) = c \cdot 0 + c \cdot 0$. Adding $-(c \cdot 0)$ to both sides says $0 = (c \cdot 0 + c \cdot 0) + -(c \cdot 0) = c \cdot 0 + (c \cdot 0 + -(c \cdot 0)) = c \cdot 0 + 0 = c \cdot 0$.

   (b) We have $0 = 0 \cdot v = (1 + -1) \cdot v = 1 \cdot v + (-1) \cdot v = v + (-1) \cdot v$. Adding $-v$ to both sides (and using associativity again) says $-v = (-1) \cdot v$.

2. (a) If $u + w = v + w$, add $-w$ to both sides to get $(u + w) + (-w) = (v + w) + (-w)$. The left hand side becomes $u + (w + (-w)) = u$, while the right hand side becomes $v + (w + (-w)) = v$ by associativity.

   (b) Suppose there are two 0 elements, 0 and $0'$, so that for any $u \in V$, we have $u + 0 = u$ and $u + 0' = u$. Equating says $u + 0 = u + 0'$, so applying the cancellation law (after flipping addition order) says $0 = 0'$.

   (c) Use the same proof : if there are two additive inverses w, w' of some vector $u \in V$, then $u + w = 0$ and $u + w' = 0$. Equating says $u + w = u + w'$ and applying the cancellation law again (once more, after flipping addition order) says $w = w'$.

3. (a) Weirdly enough, V is a vector space. Let's verify the 8 vector space axioms:

   (i) Let $x, y \in V$. Then $x + y = xy$ and $y + x = yx$, and since multiplication of real numbers is commutative, we have $x + y = xy = yx = y + x$.

   (ii) Let $x, y, z \in V$. Then $(x+y)+z = (xy)z = x(yz) = x+(y+z)$ because multiplication of real numbers is associative.

   (iii) The zero element of V is the real number 1: this is because for any $x \in V$, we have $x + 1 = (x)(1) = x$, because 1 is the multiplicative identity of $\mathbb{R}$.

   (iv) The additive inverse of x is given by $\frac{1}{x}$: this is because for any $x \in V$, we have $x + \frac{1}{x} = (x)(1/x) = 1$.

   (v) For $x \in V$, we have $1 \cdot x = x^1 = x$.

   (vi) For $a, b \in \mathbb{R}$ and $x \in V$, we have $(ab) \cdot x = x^{ab}$, and $a \cdot (b \cdot x) = a \cdot (x^b) = (x^b)^a = x^{ba} = x^{ab}$ by how exponentiation works and because multiplication of real numbers is commutative.

   (vii) For $a \in \mathbb{R}$ and $x, y \in V$, we have $a \cdot (x + y) = a \cdot (xy) = (xy)^a$, and $a \cdot x + a \cdot y = x^a + y^a = (x^a)(y^a) = (xy)^a$ again by how exponents work.

   (viii) For $a, b \in \mathbb{R}$ and $x \in V$, we have $(a + b) \cdot x = x^{a+b} = (x^a)(x^b)$, while $a \cdot x + b \cdot x = x^a + x^b = (x^a)(x^b)$.

   This is a great example to work through and explicitly write down where the various operations are happening (i.e. in the vector space V vs. in the field $\mathbb{R}$), as well as what axioms are being used (field vs. vector space).

   (b) This is not a vector space, because there is no zero element (which we will denote by $\vec{0}$ so there's no confusion): for any $(a, b) \in \mathbb{R}^2$, we have $\vec{0} = 0 \cdot (a, b) = (0, b)$, which says all vectors $(0, b)$ for $b \in \mathbb{R}$ are zero elements. These are obviously distinct elements of $\mathbb{R}^2$, but the previous problem showed that vector spaces have a unique $\vec{0}$ element.

   (c) This is not a vector space; scalar multiplication doesn't work well with addition. We have $(1+2) \cdot (1, 1) = 3 \cdot (1, 1) = (3, \frac{1}{3})$, while $1 \cdot (1, 1) + 2 \cdot (1, 1) = (1, 1) + (2, \frac{1}{2}) = (3, \frac{3}{2})$.

   All you need to disprove the statement is a counter-example, but of course the way you find these counter-examples is that you first try checking if the axioms hold, and then are either unsuccessful in your attempt, or realize from the required condition that they do not hold. I didn't come up with these out of thin air, I verified all the other axioms before the ones I broke on the list!