Subspaces

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Unless otherwise stated, vector spaces are over an arbitrary field.
*Starred problems are optional problems that relate the concepts to other areas of math.

1. Let U, W be subspaces of a vector space V. Show that $U \cap W$ is a subspace of V.

2. Let $U, W \subset V$ be subspaces.
   (a) Show that the union $U \cup W$ need not be a subspace of V. (Hint: look for a counter-example in $\mathbb{R}^2$).
   (b) (Harder) When *is* $U \cup W$ a subspace of V? Come up with necessary and sufficient conditions, and then try to prove they work! (Hint: first, try to figure out how to fix your counter-example from (a). Then try to come up with general conditions).

3. Which of the following sets S are subspaces? If they are, prove it. If they aren't, give a counter-example.
   (a) $S = \{(x, y, z) \in \mathbb{R}^3 : xy = z^2\}$
   (b) $S = \{f \in C(\mathbb{R}) : f(-1) = 0\}$
   (c) $S = \{A \in \text{Mat}_{2\times 2}(\mathbb{R}) : AB = BA\}$ where $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

4. (a) Give an example of non-trivial subspaces U_1, U_2 and a vector space V such that $V = U_1 + U_2$ but $V \neq U_1 \oplus U_2$.
   (b) Prove or disprove: if $U_1, U_2, W \subset V$ are subspaces, if $U_1 + W = U_2 + W$ then $U_1 = U_2$.

5.* Let $C([a, b])$ be the set of continuous functions $f : [a, b] \to \mathbb{R}$ on some interval $[a, b]$ and let $C^\infty([a, b])$ be the subset of *smooth functions*, i.e. functions $f : [a, b] \to \mathbb{R}$ such that f is infinitely differentiable. Then it's easy to see that $C^\infty([a, b])$ is a subspace of $C([a, b])$. Let $S = \{f \in C^\infty([a, b]) : f'' + 2f' + f = 0\}$. Show that S is a subspace of $C^\infty([a, b])$. (This illustrates the extremely important relation between linear algebra and solutions to differential equations).

Solutions

1. We need to check the three conditions for the subspace criterion. First, we have $0 \in U \cap W$, because $0 \in U$ and $0 \in W$ because both U, W are subspaces. Next, suppose that $c \in F$ and $x \in U \cap W$. Then in particular, $x \in U$, so $cx \in U$ because U is a subspace. Similarly, $cx \in W$, so $cx \in U \cap W$. Finally, suppose $x, y \in U \cap W$. Then $x, y \in U$ and $x, y \in W$ by definition, so $x + y \in U$ and $x + y \in W$ because both U, W are subspaces. This shows that $U \cap W$ is a subspace of V.

2. (a) Let $U = \{(x, 0) : x \in \mathbb{R}\}$ and $W = \{(0, y) : y \in \mathbb{R}\}$ be the x and y-axes in $\mathbb{R}^2$ respectively. We have $(1, 0) \in U \cup W$ and $(0, 1) \in U \cup W$, but $(1, 0) + (0, 1) = (1, 1) \notin U \cup W$. This says $U \cup W$ is not a subspace, because it's not closed under addition.

   (b) My claim is that $U \cup W$ is a subspace of V if and only if $U \subset W$ or $W \subset U$ (i.e. one subspace is contained in the other). First, let's prove the backwards direction. Suppose one subspace is contained in the other, say, $U \subset W$ (the argument will be the same in the other case). Then $U \cup W = W$, and W is a subspace by assumption. Conversely, suppose that $U \cup W$ is a subspace. If $U \not\subset W$ and $W \not\subset U$, then there is some $x \in W \setminus U$ and $y \in U \setminus W$. Since $U \cup W$ is a subspace, we must have $x + y \in U \cup W$, so it's either in U or in W. Without loss of generality, suppose that $x + y \in U$. Then $x = (x + y) - y \in U$ because U is a subspace. However, this contradicts that $x \in W \setminus U$. Therefore, $U \subset W$ or $W \subset U$.

3. (a) This is not a subspace, because it's not closed under addition. We have $(1, 1, 1) \in S$ and $(-1, -1, 1) \in S$ but $(0, 0, 2) \notin S$.

   (b) This is a subspace; the 0 function $f(x) = 0$ certainly satisfies $f(-1) = 0$. Suppose that $c \in \mathbb{R}$ and $f \in S$. Then $(cf)(-1) = cf(-1) = c \cdot 0 = 0$, so $cf \in S$. Finally, if $f, g \in S$ we have $(f + g)(-1) = f(-1) + g(-1) = 0 + 0 = 0$, so $f + g \in S$ says S is a subspace.

   (c) This is a subspace; the 0 matrix is in S because $0B = B0 = 0$. If $c \in \mathbb{R}$ and $A \in S$, we have $(cA)B = c(AB) = c(BA) = B(cA)$ by how matrix multiplication works, so $cA \in S$. Finally, if $A, A' \in S$ we have $(A + A')B = AB + A'B = BA + BA' = B(A + A')$, so $A + A' \in S$. This says S is a subspace.

4. (a) Take $V = \mathbb{R}^3$, and $U_1 = \{(x, y, 0) : x, y \in \mathbb{R}\}$, $U_2 = \{(0, y, z) : y, z \in \mathbb{R}\}$ the xy and yz-planes respectively. Then we certainly have $\mathbb{R}^3 = U_1 + U_2$ because any vector (x, y, z) can be written as $(x, y, 0) + (0, 0, z)$. However, this sum is not direct, because $U_1 \cap U_2 = \{(0, y, 0) : y \in \mathbb{R}\}$ is the y-axis.

   (b) Take U_1 and U_2 the xy and yz-planes as above, and take $W = \{(x, 0, z) : x, z \in \mathbb{R}\}$ the xz-plane. Then $U_1 + W = \mathbb{R}^3 = U_2 + W$, but $U_1 \neq U_2$.

5. The 0 function is in S because $0'' + 2 \cdot 0' + 0 = 0 + 0 + 0 = 0$. Suppose that $c \in \mathbb{R}$ and $f \in S$. Then $(cf)'' + 2(cf)' + (cf) = cf'' + 2cf' + cf = c(f'' + 2f' + f) = c \cdot 0 = 0$, so $cf \in S$. If $f, g \in S$, we have $(f + g)'' + 2(f + g)' + (f + g) = f'' + g'' + 2f' + 2g' + f + g = (f'' + 2f' + f) + (g'' + 2g' + g) = 0 + 0 = 0$. Note that we have used the basic fact from calculus that the derivative is *linear*, i.e. $\frac{d}{dx}(cf) = c\frac{d}{dx}f$ and $\frac{d}{dx}(f + g) = \frac{d}{dx}f + \frac{d}{dx}g$. (Which as we will see, is the key property that lets us bring in linear algebra).