Inner Products and Orthogonality

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1. Let $(V, \langle -, - \rangle)$ be an inner product space, with basis β. Let $x, y \in V$.
   (a) Show that if $\langle x, z \rangle = 0$ for all $z \in \beta$ then $x = 0$.
   (b) Show that if $\langle x, z \rangle = \langle y, z \rangle$ for all $z \in \beta$ then $x = y$.
   (c) Let $A \in M_n(\mathbb{C})$, and suppose that $\mathrm{tr}(AX) = 0$ for all $X \in M_n(\mathbb{C})$. Show that $A = 0$.
2. Let $(V, \langle -, - \rangle)$ be an inner product space.
   (a) (Parallelogram Law) Prove that $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$
   (b) (Polarization) If V is a real inner product space, prove that $\langle x, y \rangle = \frac{1}{4}\|x+y\|^2 - \frac{1}{4}\|x-y\|^2$
3. Let $(V, \langle -, - \rangle)$ be an inner product space, and let $\beta = \{e_1, \ldots, e_m\}$ be an orthonormal set of vectors. Show that $v \in \mathrm{Span}\{e_1, \ldots, e_m\}$ if and only if $\|v\|^2 = |\langle v, e_1 \rangle|^2 + \ldots + |\langle v, e_m \rangle|^2$.

Solutions

1. (a) Let $\beta = \{v_1, \ldots, v_n\}$ and write $x = c_1v_1+\ldots+c_nv_n$. Then $\langle x, x\rangle = \langle c_1v_1+\ldots+c_nv_n, x\rangle = c_1\langle v_1, x\rangle + \ldots + c_n\langle v_n, x\rangle = 0$. This says $\|x\|^2 = 0$, so that $x = 0$.

   (b) The condition is equivalent to saying $\langle x - y, z\rangle = 0$ for all $z \in \beta$. By the previous part, $x - y = 0$ so $x = y$.

   (c) Let $\langle A, B\rangle = \text{tr}(B^*A)$. Then $\langle A, A\rangle = \text{tr}(A^*A) = \sum_{i,j} |a_{ij}|^2 = 0$, where $A = (a_{ij})$. This says $a_{ij} = 0$ for all i, j so that $A = 0$.

2. (a) Write everything as an inner product : $\|x+y|^2+\|x-y\|^2 = \langle x+y, x+y\rangle+\langle x-y, x-y\rangle = \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle + \langle x, x\rangle - \langle x, y\rangle - \langle y, x\rangle + \langle y, y\rangle = 2\langle x, x\rangle + 2\langle y, y\rangle = 2\|x\|^2 + 2\|y\|^2$.

   (b) Do the same thing as above.

3. If $v \in \text{Span}\{e_1, \ldots, e_m\}$, write $v = c_1e_1 + \ldots + c_mv_m\}$. Then $\langle v, e_i\rangle = c_i\langle e_i, e_i\rangle = c_i\|e_i\|^2 = c_i$ since β is orthonormal. By the Pythagorean theorem, we then have $\|v\|^2 = |\langle v, e_1\rangle|^2 + \ldots + |\langle v, e_m\rangle|^2$. Conversely, suppose $\|v\|^2 = |\langle v, e_1\rangle|^2 + \ldots + |\langle v, e_m\rangle|^2$, and consider $x = v - \langle v, e_1\rangle e_1 - \ldots - \langle v, e_m\rangle e_m$. Then $\langle x, e_i\rangle = 0$ for all i, so $\{e_1, \ldots, e_m, x\}$ is an orthogonal set of vectors. By the Pythagorean theorem, we have $\|v\|^2 = \|x\|^2 + |\langle v, e_1\rangle|^2 + \ldots + |\langle v, e_m\rangle|^2$. By assumption, this says $\|x\|^2 = 0$, so that $x = 0$ says $v = \langle v, e_1\rangle e_1 + \ldots + \langle v, e_m\rangle e_m$, i.e. $v \in \text{Span}\{e_1, \ldots, e_m\}$.