Matrix of a Linear Transformation
Tim Smits

1. For each of the following vectors, find their coordinates relative to the given basis.
   (a) $x = (1,3) \in \mathbb{R}^2$, $\beta = \{(1,1),(1,-1)\}$
   (b) $A = \begin{pmatrix} 2 & -1 \\ -1 & 5 \end{pmatrix} \in M_2(\mathbb{R})$, $\beta = \{E_{11}, E_{12}+E_{21}, E_{22}\}$
   (c) $p = 4 - 3x + 3x^2 \in P_2(\mathbb{R})$, $\beta = \{1, x, \frac{3}{2}x^2 - \frac{1}{2}\}$
2. For each of the following linear transformations T, compute the matrix $[T]_\beta^\gamma$ relative to the listed bases.
   (a) $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(\vec{x}) = \vec{v} \times \vec{x}$ where $v = (1,1,1)$, $\beta = \gamma = \{e_1, e_2, e_3\}$
   (b) $T : P_3(\mathbb{C}) \to \mathbb{C}$ given by $T(p) = p(1) - 2p(0)$, $\beta = \{1, x, x^2, x^3\}$, $\gamma = \{1\}$
   (c) $T : M_2(\mathbb{R}) \to P_3(\mathbb{R})$ given by $T(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = (a+b)+(2d)x+bx^2$, $\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$, $\gamma = \{1, x, x^2\}$
   (d) $T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ given by $T(A) = A - A^t$, $\beta = \gamma = \{E_{11}, E_{12}, E_{21}, E_{22}\}$
   (e) $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by $T(x,y) = (x-y, x, 2x+y)$, $\beta = \{(1,2),(2,3)\}$, $\gamma = \{(1,1,0),(0,1,1),(2,2,3)\}$
3. Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be given by $T(p) = p'' + xp' + (1+x+x^2)p(1)$.
   (a) Compute $[T]_\beta$ where $\beta = \{1, x, x^2\}$.
   (b) Use part (a) to find bases for $\mathrm{Im}(T)$ and $\ker(T)$ (These should not be in terms of coordinates!!)
4. Let $T : V \to V$ satisfy $T^n = 0$, but $T^{n-1} \neq 0$, where $\dim(V) = n$. Let $v \in V$ be a vector such that $T^{n-1}(v) \neq 0$.
   (a) Prove that $\beta = \{T^{n-1}(v), T^{n-2}(v), \ldots, v\}$ is a basis for V.
   (b) Compute $[T]_\beta$.
   (c) Find a matrix $A \in M_3(\mathbb{R})$ such that $A^2 \neq 0$ but $A^3 = 0$.

Solutions

1. (a) $[x]_\beta = (2, -1)$

   (b) $[A]_\beta = (2, -1, 5)$

   (c) $[p]_\beta = (5, -3, 2)$

2. (a) $[T]_\beta = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$

   (b) $[T]_\beta^\gamma = \begin{pmatrix} -1 & 1 & 1 & 1 \end{pmatrix}$

   (c) $[T]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

   (d) $[T]_\beta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

   (e) $[T]_\beta^\gamma = \begin{pmatrix} -7/3 & -11/3 \\ 2 & 3 \\ 2/3 & 4/3 \end{pmatrix}$

3. (a) $[T]_\beta = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix}$

   (b) $\mathrm{Im}(T) = \mathrm{Span}\{1 + x + x^2, 1 + 2x + x^2\}$, $\ker(T) = \mathrm{Span}\{-5 + 2x + x^2\}$

4. (a) Suppose $c_1 T^{n-1}(v) + \ldots + c_{n-1}T(v) + c_n v = 0$. Applying T^{n-1} to both sides shows $c_n T^{n-1}(v) = 0$, so $c_n = 0$. Applying T^{n-2} to both sides shows $c_{n-1}T^{n-1}(v) = 0$, so $c_{n-1} = 0$. Repeating this argument shows that $c_i = 0$ for all i, so β is a basis of V.

   (b) $[T]_\beta = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

   (c) $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.