Linear Combinations
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Unless otherwise stated, V is a vector space over an arbitrary field.

1. Show that $\{(1,1,0),(0,1,1),(1,0,1)\}$ is a spanning set for $\mathbb{R}^3$.

2. Recall that a matrix $A \in \mathrm{Mat}_{2\times 2}(\mathbb{R})$ is called *symmetric* if $A^t = A$, where A^t denotes the transpose of A. Let $\mathrm{Sym}_2(\mathbb{R})$ be the subspace of symmetric 2×2 matrices. Show that $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ is a spanning set for $\mathrm{Sym}_2(\mathbb{R})$.

3. (a) Let $S_1, S_2 \subset V$ be subsets with $S_1 \subset S_2$. Show that $\mathrm{Span}(S_1) \subset \mathrm{Span}(S_2)$. In particular, if $\mathrm{Span}(S_1) = V$, then deduce that $\mathrm{Span}(S_2) = V$.

   (b) Show that if $\{v_1, v_2, v_3, v_4\}$ is a generating set for V, then $\{v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4\}$ is a generating set for V.

4. Let $S_1, S_2 \subset V$ be subsets.

   (a) Show that $\mathrm{Span}(S_1 \cup S_2) = \mathrm{Span}(S_1) + \mathrm{Span}(S_2)$.

   (b) Show that $\mathrm{Span}(S_1 \cap S_2) \subset \mathrm{Span}(S_1) \cap \mathrm{Span}(S_2)$.

   (c) Show that equality does not necessarily hold in (b) by giving an example of S_1, S_2, V where $\mathrm{Span}(S_1 \cap S_2)$ equals $\mathrm{Span}(S_1) \cap \mathrm{Span}(S_2)$ and one where it does not.

Solutions

1. Let $(x, y, z) \in \mathbb{R}^3$. We want to find real numbers a, b, c such that $a(1,1,0) + b(0,1,1) + c(1,0,1) = (x, y, z)$. This is the same as trying to solve the system of equations $\begin{cases} a+c=x \\ a+b=y \\ b+c=z \end{cases}$.

   If you just solve the system directly, you find $a = \frac{x+y-z}{2}, b = \frac{-x+y+z}{2}, c = \frac{x-y+z}{2}$. Alternatively, you can show the system is consistent using your favorite method from 33A.

2. A matrix $A \in \mathrm{Mat}_{2\times 2}(\mathbb{R})$ is symmetric if it looks like $A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$ for some $a, b, d \in \mathbb{R}$. Therefore, any symmetric matrix is of the form $a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ for some $a, b, d \in \mathbb{R}$, which is precisely what it says for the set in question to span $\mathrm{Sym}_2(\mathbb{R})$.

3. (a) If $x \in \mathrm{Span}(S_1)$, then it's a linear combination of some vectors in S_1. Since $S_1 \subset S_2$, vectors in S_1 *are* vectors in S_2, so that x is a linear combination of vectors in S_2. This says that $x \in \mathrm{Span}(S_2)$, so $\mathrm{Span}(S_1) \subset \mathrm{Span}(S_2)$. For the particular case that $\mathrm{Span}(S_1) = V$, then $V = \mathrm{Span}(S_1) \subset \mathrm{Span}(S_2) \subset V$, so that $V = \mathrm{Span}(S_2)$.

   (b) Note that $v_3 = (v_3 - v_4) + v_4 \in \mathrm{Span}(S_2)$. This then says $v_2 = (v_2 - v_3) + v_3 \in \mathrm{Span}(S_2)$, which then immediately gives $v_1 = (v_1 - v_2) + v_2 \in \mathrm{Span}(S_2)$. Thus, $\mathrm{Span}(S_2)$ contains S_1, so it contains $\mathrm{Span}(S_1)$. The previous part then says $\mathrm{Span}(S_2) = V$.

4. (a) Suppose that $x \in \mathrm{Span}(S_1 \cup S_2)$. Then we can write $x = c_1v_1 + \ldots + c_nv_n$ for some $c_i \in F$ and some $v_i \in S_1 \cup S_2$. Rearrange the vectors so that $v_1, \ldots, v_k \in S_1$ and $v_{k+1}, \ldots, v_n \in S_2$ for some k. Then $x = (c_1v_1 + \ldots + c_kv_k) + (c_{k+1}v_{k+1} + \ldots + c_nv_n)$, where the first term is in $\mathrm{Span}(S_1)$ and the second in $\mathrm{Span}(S_2)$. This says $x \in \mathrm{Span}(S_1) + \mathrm{Span}(S_2)$, so that $\mathrm{Span}(S_1 \cup S_2) \subset \mathrm{Span}(S_1) + \mathrm{Span}(S_2)$.

   Now let $x \in \mathrm{Span}(S_1) + \mathrm{Span}(S_2)$. Then we can write $x = v_1 + v_2$ where v_1 is a linear combination of vectors in S_1 and v_2 is a linear combination of vectors in S_2. It's then clear that x is a linear combination of vectors of that belong to either S_1 or S_2, i.e. a linear combination of vectors in $S_1 \cup S_2$. This says $x \in \mathrm{Span}(S_1 \cup S_2)$, which proves $\mathrm{Span}(S_1 \cup S_2) = \mathrm{Span}(S_1) + \mathrm{Span}(S_2)$.

   (b) If $x \in \mathrm{Span}(S_1 \cap S_2)$, write $x = c_1v_1 + \ldots + c_nv_n$ for some $c_i \in F$ and $v_i \in S_1 \cap S_2$. Then obviously $v_i \in S_1$, so $x \in \mathrm{Span}(S_1)$ and $v_i \in S_2$, so $x \in \mathrm{Span}(S_2)$, which combine to say that $x \in \mathrm{Span}(S_1) \cap \mathrm{Span}(S_2)$. Thus, $\mathrm{Span}(S_1 \cap S_2) \subset \mathrm{Span}(S_1) \cap \mathrm{Span}(S_2)$.

   (c) For the first example, take $S_1 = \{(1,0)\}$, $S_2 = \{(1,0),(0,1)\}$ and $V = \mathbb{R}^2$, so that $\mathrm{Span}(S_1 \cap S_2) = \mathrm{Span}(S_1) \cap \mathrm{Span}(S_2) = \mathrm{Span}\{(1,0)\}$. For the second example, take $S_1 = \{(1,0),(1,1)\}$, $S_2 = \{(0,1),(1,1)\}$ and $V = \mathbb{R}^2$, so that $\mathrm{Span}(S_1 \cap S_2) = \mathrm{Span}\{(1,1)\}$ while $\mathrm{Span}(S_1) \cap \mathrm{Span}(S_2) = \mathbb{R}^2$.