Eigenvalues

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1. The characteristic polynomial $p_T(x)$ of an operator $T : V \to V$ is defined in terms of an arbitrary choice of basis β. Prove that this definition is well-defined by showing that if $A, B \in M_n(F)$ are similar matrices, the characteristic polynomials of A and B are equal.

2. Prove that $A \in M_n(F)$ is invertible if and only if 0 is not an eigenvalue of A.

3. Let $D : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be the differential operator $D(p) = p'$, and set $T = D^2 + D + I$. Let $\beta = \{1, x, x^2\}$ be the standard basis of $P_2(\mathbb{R})$.

   (a) Find the eigenvalues of T and bases of the corresponding eigenspaces.

   (b) Is T diagonalizable? If so, find matrices S, D such that $[T]_\beta = SDS^{-1}$.

4. Let $T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ be given by $T(A) = A^t - 2\mathrm{tr}(A)I_2$, where $\mathrm{tr}(A)$ is the trace of A. Let $\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ be the standard basis of $M_2(\mathbb{R})$.

   (a) Find the eigenvalues of T, and bases of the corresponding eigenspaces.

   (b) Is T diagonalizable? If so, find matrices S, D such that $[T]_\beta = SDS^{-1}$.

Solutions

1. Suppose that A and B are similar, so $A = PBP^{-1}$ for some $P \in M_n(F)$. We have $p_A(x) = \det(xI_n - A) = \det(xI_n - PBP^{-1}) = \det(P(xI_n - B)P^{-1}) = \det(P)\det(xI_n - B)\det(P^{-1}) = \det(xI_n - B) = p_B(x)$ by definition of the characteristic polynomial and properties of the determinant.

2. 0 is an eigenvalue of A if and only if there is a non-trivial solution to $Av = 0$, i.e. A has a non-trivial kernel. This happens if and only if A is not invertible.

3. (a) We have $A = [T]_\beta = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$, so $p_T(x) = (x-1)^3$. This says the only eigenvalue of T is 1 (with multiplicity 3). We have $E_1 = \ker(A - I) = \operatorname{span}\{(1,0,0)\}$.

   (b) T is not diagonalizable since we cannot find a basis of eigenvectors (there is only one linearly independent eigenvector).

4. (a) We have $A = [T]_\beta = \begin{pmatrix} 3 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 3 \end{pmatrix}$, and $p_T(x) = (x-5)(x+1)(x-1)^2$, so the eigenvalues of T are $5, -1, 1$ (with multiplicity 2). We have $E_5 = \ker(A - 5I) = \operatorname{span}\{(1,0,0,1)\}$, $E_{-1} = \ker(A + I) = \operatorname{span}\{(0,-1,1,0)\}$, and $E_1 = \ker(A - I) = \operatorname{span}\{(-1,0,0,1),(0,1,1,0)\}$.

   (b) We have four linearly independent eigenvectors, so T is diagonalizable. Let $\gamma = \{(1,0,0,1),(0,-1,1,0),(-1,0,0,1),(0,1,1,0)\}$ be the eigenbasis, then change of basis says $[T]_\beta = S_\gamma^\beta [T]_\gamma (S_\gamma^\beta)^{-1}$. Take $S = S_\gamma^\beta$ the change of basis matrix. Then $S_\gamma^\beta = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$ and $D = [T]_\gamma = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.