Dimension
Tim Smits

*Starred problems are optional problems that relate the concepts to other areas of math.

1. For each of the following vector spaces, give an example of a basis of and state the dimension of the vector space.
   (a) $\text{Span}\{(1,2,3),(4,5,6),(7,8,9)\} \subset \mathbb{R}^3$.
   (b) The subspace $S \subset \mathbb{R}^4$ of vectors (x_1,x_2,x_3,x_4) with $x_1+x_2+x_3+x_4=0$.
   (c) The solution space to the system of equations $\begin{cases} x-2y+z=0 \\ 2x-3y+z=0 \end{cases}$.
   (d) The subspace $S \subset P_3(\mathbb{R})$ of polynomials of degree at most 3 with $p''(1)=0$.
   (e) The subspace $U_n \subset \text{Mat}_n(\mathbb{R})$ of upper triangular $n \times n$ matrices.
   (f) The subspace $\text{Skew}_n(\mathbb{R}) \subset \text{Mat}_n(\mathbb{R})$ of skew-symmetic $n \times n$ matrices. (Recall a matrix is skew-symmetic if $A^t = -A$).
   (g) The quotient space $\mathbb{R}^2/W$ where $W = \text{Span}\{(1,0)\}$.
   (h) The quotient space $\mathbb{C}[x]/W$ where $W = \text{Span}\{x^a : a > 2\}$.
2. Let V be a finite dimensional vector space and $U \subset V$ a subspace. Prove that $\dim U \leq \dim V$, and $U = V$ if $\dim U = \dim V$. (Do not assume that U is finite dimensional – you must show this as part of the proof!).

3.* How many subspaces of $\mathbb{F}_p^2$ are there? (Hint: draw a picture!)

Solutions

1. (a) $\{(1,2,3),(4,5,6)\}$ is a basis and the dimension is 2.
   (b) $\{(1,-1,0,0),(1,0,-1,0),(1,0,0,-1)\}$ is a basis and the dimension is 3.
   (c) $\{(1,1,1)\}$ is a basis and the dimension is 1.
   (d) $\{1,x,x^3-3x^2\}$ is a basis and the dimension is 3.
   (e) $\{E_{ij}: 1 \leq i \leq j \leq n\}$ where $E_{ij}=1$ in the (i,j)-th entry. The dimension is $(n^2+n)/2$.
   (f) $\{E_{ij}-E_{ji}: 1 \leq i < j \leq n\}$ is a basis and the dimension is $(n^2-n)/2$.
   (g) $\{(0,1)+W\}$ is a basis and the dimension is 1.
   (h) $\{1+W, x+W, x^2+W\}$ is a basis and the dimension is 3.

2. First we show that U is finite dimensional. If $U=\{0\}$ we're done. Otherwise, pick $u_1 \in U \neq 0$. Then if $U=\text{Span}\{u_1\}$ we're done, otherwise there is $u_2 \in U$ with $w_2 \notin \text{Span}\{u_1\}$ so $\{u_1,u_2\}$ is linearly independent. Repeating this process, at the k-th step we produce $u_1,\ldots,u_k$ linearly independent with $U=\text{Span}\{u_1,\ldots,u_k\}$. This process must eventually stop, because a linearly independent set in U is linearly independent in V, and the maximal size of a linearly independent set is $\dim(V)<\infty$. Thus, there are at most $\dim(V)$ steps in this process so $\dim(U) \leq \dim(V)$ as desired. If the process stops with $\dim(V)$ vectors, we have a linearly independent subset of $\dim(V)$ size, so it's a basis of V. This says $U=V$.

3. $\mathbb{F}_p^2$ is a two dimensional vector space over $\mathbb{F}_p$. Therefore by the previous problem, the possible dimensions of a subspace are $0,1,2$. The zero subspace is the only zero dimensional subspace, and $\mathbb{F}_p^2$ is the only two dimensional subspace. Therefore, we need to count the number of one dimensional subspaces. Such a subspace looks like $\text{Span}\{v\}$ for some non-zero $v \in \mathbb{F}_p^2$. There are a total of p^2-1 such vectors, and each of the $p-1$ non-zero multiples of v span the same subspace. This gives $(p^2-1)/(p-1)=p+1$ total one dimensional subspaces, for a total of $p+3$.