Change of Basis
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1. The majority of our example computations have been concerned with changing to or from a "nice" basis of some vector space. What do you do when you want to change between to bases that are not "nice"?

   (a) Let $V$ be a vector space, and let $\beta, \gamma, \mathcal{E}$ be bases of $V$. Prove that $S_\beta^\gamma = S_\mathcal{E}^\gamma S_\beta^\mathcal{E}$.

   Use this idea to compute the change of basis matrix $S_\beta^\gamma$ for each of the following:

   (b) $V = \mathbb{R}^2, \beta = \{(1,1),(1,3)\}, \gamma = \{(2,1),(-1,2)\}$

   (c) $V = P_2(\mathbb{R}), \beta = \{x^2 - x, x^2 + 1, x - 1\}$, $\gamma = \{5x^2 - 2x - 3, -2x^2 + 5x + 5, 2x^2 - x - 3\}$

2. Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be given by $T(p) = x^2 p'' + xp' + p$. Let $\beta = \{1, x, x^2\}$ be the standard basis and $\gamma = \{x^2 + x + 1, x^2 + x, x^2\}$.

   (a) Compute $[T]_\gamma$.

   (b) Compute the change of basis matrix $S_\beta^\gamma$, and compute $[p(x)]_\gamma$ for $p(x) = 5x^2 - 2x + 3$. Use this to write $p(x)$ as a linear combination of vectors in $\gamma$.

   (c) Use the change of basis formula to compute $[T]_\beta$.

   (d) Is $T$ an isomorphism? Justify your answer.

3. The change of basis proceedure still works for arbitrary linear maps $T : V \to W$. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $T(x, y, z) = (y + 2z, 3x + 4y + 5z)$ and let $\beta = \{e_1, e_2, e_3\}$ and $\gamma = \{f_1, f_2\}$ be the standard bases of $\mathbb{R}^3$ and $\mathbb{R}^2$. Let $\beta' = \{(1,0,0),(1,1,0),(1,1,1)\}$ and $\gamma' = \{(1,0),(1,1)\}$ be other bases of $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively.

   (a) Write down the change of basis matrix $S_\beta^{\beta'}$.

   (b) Write down the change of basis matrix $S_\gamma^{\gamma'}$.

   (c) Write down the change of basis diagram relating $[T]_\beta^\gamma$ and $[T]_{\beta'}^{\gamma'}$. (This should only require a slight modification of the one we've been drawing!)

   (d) Use your picture to compute $[T]_{\beta'}^{\gamma'}$.

## Solutions

1. (a) Let $\beta = \{v_1, \dots, v_n\}$. Then $[v_i]_\gamma = S^\gamma_\beta [v_i]_\beta$ is the $i$-th column of $S^\gamma_\beta$. On the other hand, the $i$-th column of $S^\gamma_\mathcal{E} S^\mathcal{E}_\beta$ is given by $(S^\gamma_\mathcal{E} S^\mathcal{E}_\beta)[v_i]_\beta = S^\gamma_\mathcal{E}[v_i]_\mathcal{E} = [v_i]_\gamma$, so these matrices are equal.

   (b) $S^\gamma_\beta = \dfrac{1}{5}\begin{pmatrix} 3 & 5 \\ 1 & 5 \end{pmatrix}$.

   (c) $S^\gamma_\beta = \dfrac{1}{36}\begin{pmatrix} 12 & 16 & -8 \\ 0 & 9 & 0 \\ -12 & -13 & 20 \end{pmatrix}$.

   For both of these, take $\mathcal{E}$ to be the standard basis.

2. (a) $[T]_\gamma = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 3 & 3 & 5 \end{pmatrix}$.

   (b) $S^\gamma_\beta = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$ and $[p(x)]_\gamma = S^\gamma_\beta [p(x)]_\beta = (3, -5, 7)$. This says $p(x) = 3(x^2 + x + 1) - 5(x^2 + x) + 7x^2$.

   (c) $S^\beta_\gamma = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$, so $[T]_\beta = S^\beta_\gamma [T]_\gamma S^\gamma_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$.

   (d) Many ways to see this; for example, $\det([T]_\gamma) \neq 0$, so $T$ is an isomorphism.

3. (a) $S^{\beta'}_\beta = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$.

   (b) $S^{\gamma'}_\gamma = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$.

   (c)
$$\begin{array}{ccc} [x]_\beta & \xrightarrow{[T]^\gamma_\beta} & [T(x)]_\gamma \\ \big\downarrow S^{\beta'}_\beta & & \big\downarrow S^{\gamma'}_\gamma \\ [x]_{\beta'} & \xrightarrow{[T]^{\gamma'}_{\beta'}} & [T(x)]_{\gamma'} \end{array}$$

   (d) From the diagram, $[T]^{\gamma'}_{\beta'} = S^{\gamma'}_\gamma [T]^\gamma_\beta S^\beta_{\beta'}$. We have $[T]^\gamma_\beta = \begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{pmatrix}$, so $[T]^{\gamma'}_{\beta'} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -3 & -6 & -9 \\ 3 & 7 & 12 \end{pmatrix}$.