Bases & Induction

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*Starred problems are optional problems that relate the concepts to other areas of math.

1. Prove using induction that $\sum_{k=1}^{n}(2k-1) = n^2$.

2. Let $P_3(\mathbb{R})$ be the vector space of polynomials of degree at most 3. Show that $\{x^3 - 2x^2 + 1, 4x^2 - x + 3, 3x - 2, 1\}$ is a basis of $P_3(\mathbb{R})$.

3. Consider the subspace $U = \{(x, y, z) \in \mathbb{R}^3 : x - 2y + z = 0\}$ of $\mathbb{R}^3$.

   (a) Find (with proof!) a basis B of U.

   (b) Find (with proof!) a basis S of $\mathbb{R}^3$ with $B \subset S$.

   (c) Find a vector space W such that $\mathbb{R}^3 = U \oplus W$. Interpret this geometrically.

4.* Recall from the problems on subspaces $C^\infty([a, b])$, the vector space of smooth functions $f : [a, b] \to \mathbb{R}$. Let $S = \{f \in C^\infty([a, b]) : f'' - f' = 0\}$, which is a subspace of $C^\infty([a, b])$. In this problem, you will find a basis for S.

   (a) Let $g : [a, b] \to \mathbb{R}$ be a function with $g'(x) = g(x)$. Using calculus, show that $g(x) = ce^x$ for some $c \in \mathbb{R}$.

   (b) Suppose that $f''(x) - f'(x) = 0$, and set $g(x) = f'(x)$. Use the previous part to show that $f(x) = ce^x + b$ for some constants $c, b \in \mathbb{R}$.

   (c) Prove that $\{1, e^x\}$ is a basis of S.

Solutions

1. For $n = 1$, the left hand side is $2 \cdot 1 - 1 = 1$ while the right hand side is $1^2 = 1$, so the base case holds. Suppose that for some n we know $\sum_{k=1}^{n}(2k-1) = n^2$. Then we wish to show that $\sum_{k=1}^{n+1}(2k-1) = (n+1)^2$. We have $\sum_{k=1}^{n+1}(2k-1) = \sum_{k=1}^{n}(2k-1) + 2(n+1) - 1 = n^2 + 2n + 1 = (n+1)^2$, where we used the induction hypothesis in the second equality. Therefore it's true for $n+1$ as well, so by induction, we're done.

2. We need to show that $\{x^3 - 2x^2 + 1, 4x^2 - x + 3, 3x - 2, 1\}$ is a linearly independent set that spans $P_3(\mathbb{R})$. First, we show it's linearly independent. Suppose that $a(x^3 - 2x^2 + 1) + b(4x^2 - x + 3) + c(3x - 2) + d = 0$ for some $a, b, c, d \in \mathbb{R}$. Expanding this out, we find $ax^3 + (4b - 2a)x^2 + (3c - b)x + (a + 3b - 2c + d) = 0$. Since $\{1, x, x^2, x^3\}$ is a basis of $P_3(\mathbb{R})$, this says all the coefficients must equal 0. This gives a system of equations $\begin{cases} a = 0 \\ 4b - 2a = 0 \\ 3c - b = 0 \\ a + 3b - 2c + d = 0 \end{cases}$, from which it's evident that $a = b = c = d = 0$ is the only solution. To see that this is a spanning set, we want to show that $\mathrm{Span}(S) = P_3(\mathbb{R})$, i.e. any arbitrary element $c_0 + c_1 x + c_2 x^2 + c_3 x^3$ can be written as a linear combination of the polynomials in S. The same set-up will show we want to solve $ax^3 + (4b-2a)x^2 + (3c-b)x + (a+3b-2c+d) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$, which by comparing coefficients, gives the system $\begin{cases} a = c_3 \\ 4b - 2a = c_2 \\ 3c - b = c_1 \\ a + 3b - 2c + d = c_0 \end{cases}$. If you solve the system of equations, you'll find $a = c_3, b = \dfrac{c_2 + 2c_3}{4}, c = \dfrac{4c_1 + c_2 + 2c_3}{12}, d = \dfrac{12c_0 + 8c_1 - 7c_2 - 24c_3}{12}$, so that S spans V as desired. (Alternatively, use row reduction to show that the system of equations is consistent).

3. (a) I claim that $B = \{(2, 1, 0), (-1, 0, 1)\}$ is a basis for U. If $(x, y, z) \in U$, then $x - 2y + z = 0$, so $x = 2y - z$. Thus, $(x, y, z) = (2y - z, y, z) = y(2, 1, 0) + z(-1, 0, 1)$ is a linear combination of those two vectors, so that they span U. They're clearly linearly independent, because they're not multiples of each other, so they're a basis.

   (b) I claim that $S = \{(2, 1, 0), (-1, 0, 1), (1, -2, 1)\}$ is a basis for $\mathbb{R}^3$. This is linearly independent, because we may write $S = B \cup \{(1, -2, 1)\}$ and $(1, -2, 1) \notin \mathrm{Span}(B) = U$ because it doesn't satisfy the required conditions on the coordinates. To show that S is a spanning set for $\mathbb{R}^3$, we need to show the system of equations $\begin{cases} 2a - b + c = x \\ a - 2c = y \\ b + c = z \end{cases}$ for an arbitrary vector $(x, y, z) \in \mathbb{R}^3$ has a solution. Solving this you'll find $a = \dfrac{x + y + z}{3}, b = \dfrac{-x + 2y + 5z}{6}, c = \dfrac{x - 2y + z}{6}$, so this is a spanning and therefore a basis. (Alternatively, use row reduction to show the system is consistent.)

   (c) Take U as above and $W = \mathrm{Span}\{(1, -2, 1)\}$. Then part (b) above immediately tells us that $\mathbb{R}^3 = U \oplus W$. Geometrically, this says every point in $\mathbb{R}^3$ can be written as a vector in the plane U and a vector on the line W.

4. (a) Suppose that $g'(x) = g(x)$. Then $(g(x)e^{-x})' = g'(x)e^{-x} - g(x)e^{-x} = 0$ says that $g(x)e^{-x} = c$ for some $c \in \mathbb{R}$. This then says $g(x) = ce^x$ for some $c \in \mathbb{R}$ as desired.

(b) Setting $g(x) = f'(x)$, we have $g'(x) - g(x) = 0$. This says $g(x) = f'(x) = ce^x$ for some c, so integrating says $f(x) = b + ce^x$ for some b, c.

(c) The above part says $\{1, e^x\}$ is a spanning set for S, so we just need to show that S is linearly independent. Suppose that $c + be^x = 0$ for some $c, b \in \mathbb{R}$. Taking a derivative says $be^x = 0$, i.e. $b = 0$. This then says $c = 0$, so we are done.