Linear Transformations
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1. For each of the following maps, determine if they are linear transformations or not. Additionally, check if each map is injective/surjective.

   (a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x, y) = (x + 1, 2 - y)$.
   (b) $T : \mathbb{R}^3 \to \mathbb{R}^2$ given by $T(x, y, z) = (2x - 4y + 3z, 6x - y)$.
   (c) $T : \mathbb{R}^3 \to \mathbb{R}$ given by $T(x, y, z) = x + y + z$.
   (d) $T : \mathbb{R}^2 \to \mathbb{R}$ given by $T(x, y) = xy$.
   (e) $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(\vec{x}) = \vec{v} \times \vec{x}$ where $\vec{v} = (1, 1, 1)$.
   (f) $I : C(\mathbb{R}) \to C(\mathbb{R})$ given by $I(f) = \displaystyle\int_0^x f(t)\, dt$.
   (g) $T : \mathbb{C}_n[x] \to \mathbb{C}$ given by $T(p) = p(1) - 2p(0)$.
   (h) $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ given by $T(A) = A - A^t$.
   (i) $T : F[x] \to F[x]$ given by $T(p) = xp'' + x$.
   (j) $R : F^\infty \to F^\infty$ given by $R(x_0, x_1, x_2, \ldots) = (x_1, x_2, x_3, \ldots)$. Here, $F^\infty = \{(x_0, x_1, x_2, \ldots) : x_i \in F\}$ is the so-called *sequence space* of infinite tuples of elements of F.

2. Below, you will show that neither the additivity or homogeneity conditions alone are enough to imply a function is a linear transformation.

   (a) Give an example of a function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ such that $\varphi(ax) = a\varphi(x)$ for all $a \in \mathbb{R}$ but φ is not $\mathbb{R}$-linear.
   (b) Give an example of a function $\varphi : \mathbb{C} \to \mathbb{C}$ such that $\varphi(x + y) = \varphi(x) + \varphi(y)$ but φ is not $\mathbb{C}$-linear.

Solutions

1. For each of these maps, you should be able to verify below that what I've claimed is correct. If you're unsure how to fill in the details, please let me know!!

   (a) Not linear; injective; surjective.

   (b) Linear; not injective; surjective.

   (c) Linear; not injective; surjective.

   (d) Not linear; not injective; surjective.

   (e) Linear; not injective; not surjective.

   (f) Linear; injective; not surjective.

   (g) Linear; not injective; surjective.

   (h) Linear; not injective; not surjective.

   (i) Not linear; not injective; not surjective.

   (j) Linear; not injective; surjective.

2. (a) Set $\vec{x} = (x, y)$ and take $\varphi(\vec{x}) = \varphi(x, y) = \sqrt[3]{x^3 + y^3}$. Then φ is homogeneous, because for any $a \in \mathbb{R}$, $\varphi(a\vec{x}) = \varphi(ax, ay) = \sqrt[3]{a^3x^3 + a^3y^3} = a\sqrt[3]{x^3 + y^3} = a\varphi(\vec{x})$. However, φ is not additive: $\varphi(1, 0) = 1$ and $\varphi(0, 1) = 1$ but $\varphi(1, 1) = \sqrt[3]{2} \neq 2 = \varphi(1, 0) + \varphi(0, 1)$.

   (b) Take $\varphi(z) = \bar{z}$, where $\overline{a + bi} = a - bi$ is complex conjugation. Then φ is additive: writing $z = a + bi$ for some $a, b \in \mathbb{R}$ we have $\varphi(z + z') = \varphi((a + bi) + (a' + b'i)) = \varphi((a + a') + (b + b')i) = (a + a') - (b + b')i = (a - bi) + (a' - b'i) = \varphi(z) + \varphi(z')$. However, φ is not homogeneous because $\varphi(i \cdot i) = \varphi(-1) = -1$, while $i \cdot \varphi(i) = i \cdot -i = 1$.