Adjoint Operators

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1. Let V be an inner product space, and let $u, v \in V$ be fixed. Define $T : V \to V$ by $T(x) = \langle x, u\rangle v$. Compute $T^*(y)$ for any $y \in V$.

2. Let $V = P_2(\mathbb{R})$ with the inner product $\langle p, q\rangle = \int_{-1}^{1} p(t)q(t)\,dt$ and let $T : V \to V$ be defined by $T(p) = -2xp' + (1 - x^2)p''$. (Feel free to use WolframAlpha to avoid doing computations by hand.)
   (a) Let $\mathcal{E} = \{1, x, x^2\}$ be the standard basis of V. Perform Gram-Schmidt on $\mathcal{E}$ to get an orthonormal basis β of V.
   (b) Compute $[T^*]_\beta$
   (c) Compute $T^*(1 - x + x^2)$

3. Let V be an inner product space and $S, T : V \to V$ a linear operator with adjoints S^*, T^*. Prove the following:
   (a) $(S + T)^* = S^* + T^*$
   (b) $(aT)^* = \bar{a}T^*$ for $a \in F$
   (c) $(TS)^* = S^*T^*$
   (d) $(T^*)^* = T$

Solutions

1. For any $x, y \in V$, we have $\langle T(x), y\rangle = \langle x, T^*(y)\rangle$ by definition of T^*. Since $T(x) = \langle x, u\rangle v$, this says $\langle x, T^*(y)\rangle = \langle\langle x, u\rangle v, y\rangle = \langle x, u\rangle\langle v, y\rangle = \langle x, \langle y, v\rangle u\rangle$. This says $T^*(y) = \langle y, v\rangle u$.

2. (a) $\beta = \{\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}x, \sqrt{\frac{5}{8}}(3x^2 - 1)\}$. See the inner product handout for the computation.

   (b) $[T^*]_\beta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -6 \end{pmatrix}$

   (c) With $p(x) = 1 - x + x^2$, we have $[p]_\beta = (\frac{4\sqrt{2}}{3}, -\sqrt{\frac{2}{3}}, \frac{2}{3}\sqrt{\frac{2}{5}})$, so $[T^*(p)]_\beta = [T^*]_\beta[p]_\beta = (0, 2\sqrt{\frac{2}{3}}, -4\sqrt{\frac{2}{5}})$, so that $T^*(p) = 2\sqrt{\frac{2}{3}}\sqrt{\frac{3}{2}}x - 4\sqrt{\frac{2}{5}}\sqrt{\frac{5}{8}}(3x^2 - 1) = 2x - 2(3x^2 - 1) = 2 + 2x - 6x^2$.

3. (a) For any $x, y \in V$, we have $\langle x, (S+T)^*(y)\rangle = \langle (S+T)(x), y\rangle = \langle S(x), y\rangle + \langle T(x), y\rangle = \langle x, S^*(y)\rangle + \langle x, T^*(y)\rangle = \langle x, S^*(y) + T^*(y)\rangle$. This says that $(S+T)^* = S^* + T^*$.

   (b) For any $x, y \in V$ we have $\langle x, (aT)^*(y)\rangle = \langle (aT)(x), y\rangle = a\langle T(x), y\rangle = a\langle x, T^*(y)\rangle = \langle x, \bar{a}T^*(y)\rangle$, so that $(aT)^* = \bar{a}T^*$.

   (c) For any $x, y \in V$, we have $\langle x, (TS)^*(y)\rangle = \langle (TS)(x), y\rangle = \langle T(S(x)), y\rangle = \langle S(x), T^*(y)\rangle = \langle x, S^*(T^*(y))\rangle = \langle x, (S^*T^*)(y)\rangle$. This says that $(TS)^* = S^*T^*$.

   (d) For any $x, y \in V$, we have $\langle x, (T^*)^*(y)\rangle = \langle T^*(x), y\rangle = \overline{\langle y, T^*(x)\rangle} = \overline{\langle T(y), x\rangle} = \langle x, T(y)\rangle$, so that $(T^*)^* = T$.