Separability #

Last time: K/F separade E Ma sep. for all de K. f(x) separable (f(x), f(x)) = 1 Separabulity is one of two obstructions to heavy Galow.

Def: KIF is inseparable if KIF is not

Propi 
$$p(x) \in F(x)$$
 (read. Char(F) = p  
There is  $K_{7/D}$  and some  $p_{sep}(x) \in F(x)$   
(unique)  
 $1r(-rd. S.t. (unique))$   
 $p(x) = p_{sep}(x)^{K}$ 

Proof:  
If 
$$p$$
 is separable, we're done. Otherwise  
 $p$  not separable, so  $p(x) = p_i(x^p)$   
for some  $p_i$ . Repeat this process.  
How to stop at some point blc  
 $p(x)$  is apply so degree is finite.

p(x) = psep(x) for some psep.

If  $p_{sep}(x) = g(x)h(x)$ , p(x) = g(x)h(x)h(x)then 2) E Ø

 $deg_{sep}(g(x)) := deg(psep)$ deg: (p(x)) = pk

 $dog(p(z)) = deg(p(z)) \cdot deg(p(z))$ 

 $E_{X}: = \chi = t \text{ is irred, over } F_3(t)(x)$   $E_{X}: = \chi = t \text{ is irred, over } F_3(t)(x)$   $P_{Sep}(x) = \chi - t$ 

degsep(p(x)) = 1deg: (p(x)) = 3

Prog: KIF Char (F) = P. TFAE. 1.) KIF purely inseparable 2.1 degsep (ma(DC)) = 1 four all all K.

 $Ma(2k) = X^{2} = a$ for a EF. Rink: if K=F(a) this can prove K[F purely insep. E) deg sep (ma(x)) = 1

Froof: 1=>21 Suppose K/F Jurely Inseparable dEK. Then ma not separable.  $M_{\lambda}(x) = M_{sep}(x^{p^{\chi}})$  for Some K711.  $M_{\lambda}(\alpha) = O$  $M_{sep}(\alpha^{\beta^{\kappa}}) = 0$ Since more is Irred. apt separable. It a ge E F ble KIF purey insep.

 $a^{p^{n}} = a \in F$ a root of xr-a e FTxT.  $\longrightarrow$   $M_{a} = \chi_{bx} a go$ deg sep (ma) = 1. Z=>11 Juggore de K/F Separable. Look at mui deg(ma) = degsep(ma).deg;(ma) deg sep (mon) = 1 by assumption dey; (ma) = 7 blc & separable

deg(mon)	= 1	)	YE	F.

Structure of extensions





Fseg is called Signable closure of Fin K. Wy is K/Fsep purch Inseparable? 1.) d E K/F Sep. (HW) Separable (HW) KIF Sig. ES K/L con 2.1 LIF Sep. her K/L/F.

I) last week E hard

QEX(Fep separable =)

Fsep (x) [Fsep is Sep.

= FrepCA) [F 13 separable. I) a separable con F



a E Fsep.

If X/F finite

[K:F]seg = [Fseg:F]  $[K:F]_i = [K:Fsep]$ 

[K:F] = [K:F]seg·[K:F].

To see coly this is all Strange, will gue ex. of insep. but not purch insep. extension.

 $E_{X:}$  f(x) = X - tx + 1 $e = F_3(t)[x] = K[x].$ f(x) is used in K[x] b(c it has no root in K (check?) I claim X'-EXTI E KIXT is irred. So that KEDIK is issep. but not purely chis-root x-tx+1).  $f(x) = (x - r, \chi x - r_z) \in \mathbb{K}[x].$  $f(x^3) = (x^3 - r_1)(x^3 - r_2)$ 

 $r_1 = 6^3$   $r_2 = 7'$ 

N Contraction of the second second

 $x - r_1 = (x - 6)^3$  $x^3 - r_2 = (x - 7)^3$ 

 $=) g(z) = (x - \Theta)^{3}$ 

Suppose f(x3) factore de  $f(x^3) = g(x)h(x)$  for some  $g(x),h(x) \in KTx$ . Can he chosen Cure I: g(a), h(x)Co-prime then wiside EIXT,  $f(x') = (x-6)^{3}(x-9)^{3}$  $= g(\alpha)h(\alpha)$ 

 $N(x) = (x-x)^{3}$ cont shore any factors. blc G=r,  $\gamma=r_2 \in K$  $\Rightarrow \in$ Case II: Can't pick g(x)-h(x) Coprim  $= f(x^3) = f(x)^3 for$ TT(x) e K[x] rreg. Some all coeff. of tt(x) or Carles Qorajzr. r 72× 10  $\Pi(x)$ 

 $f(x) = a_{2} + a_{1} x - r x$  $f(\chi^3) = \chi^6 - t \chi^3 + 1$  $\exists \in$ SIC 22-6 is irred in XEXT. So t is not a color in K. =) X-tX+1 irrel over KEX)  $\longrightarrow (x(x)|k$ insep. but not purely insep. KIF is semple if Det: K = F(a)

Thus: (Primitice event theorem) KIF finile, separable. Then K = F(G) for some B.  $P_{100f}$ : (Sketch)  $K = F \alpha_1, \dots, \alpha_n) | F.$ Predece to n=2 care by induction. n=2 case is #45 on HW.  $E_{X:} \mathcal{Q}(f_2, f_3) = \mathcal{Q}(f_2 + f_3)$ |F|=q.Runk; F is fritte, Char(F) = P

Fx is cyclic of order 9-1. a 13 - generator of Ex  $F = F_{p}(A)$ SIC Hp(n) contains F the gol pours of d al alco o. Hp(~) Ħρ & To get a generator, Some The F(a,-, da)|F لى ٥٢ / ٩. よ? lineur comb. of

Def: KIF i normel if K is a splitting held of some  $f(x) \in F[x],$ 

Ex:  $Q(S_{2}, S_{3})/Q$  norm s.f. of  $(\chi^{2}-2)(\chi^{2}-3)$ 

Wormality is the other dostruction to heing Galois, as mentioned last week.

Trop: K/F normal () eung frai EFEX) irred. W root in K splits in K.

Ex: Q(352)/Q is not normal ble 23-2 has complex roots but it's sep. w/c char O.

Ex: H3(3Ft)/H3ttl is normal but not sep Note: K= F(a,-,an)/F then

Note: Karlako that K normal =>

Mailor splits in KEXT for all i.

Big theorem: XIF Galois ES K is normal + separable

Next week: (Hopefully ... ) Conquitations of Galois groups.