Differential Galois Theory

Classical Galois Theory : Structure in polynomials

Differential Galois Theory : Structure in differential equations

Def: A <u>differential field</u> is a char 0 field F equipped w/ a derivation D.

$D: F \to F$ F-linear

$D(ab) = D(a)b + a\,D(b)$ $a, b \in F$.

$\operatorname{Ker}(D) = C_F$ Constants of F

Note: C_F is a subfield of F.

Ex: $F = \mathbb{C}(t)$ $D = \frac{d}{dt}$

Prototypical example

what are field extensions in this setting?

$$\begin{array}{c} E \\ | \\ F \end{array}$$ diff field ext^n s.t. $D_E|_F = D_F$.

<u>Ex:</u> $F = \mathbb{C}(t)$ $E = \mathbb{C}(t, \log(t))$

$E = F(\log(t))$

Note that $\log(t)$ satisfies

$$\frac{d}{dt}\log(t) = \frac{1}{t} \in F.$$

In order to talk about "Galois Theory" need analogues of Galois and Galois gps.

"Picard-Vessiot" extensions play the role of Galois extensions.

$$L = L(Y) = Y^{(n)} + a_{n-1} Y^{(n-1)} + \cdots + a_1 Y'$$
$$+ a_0 Y^{(0)} \qquad a_i \in F$$

homogenous differential operators over F.

If E/F is a differential field ext^n we can apply L to elements of E.

$$V_L = \{ y \in E : L(y) = 0 \}$$

note that V_L is closed under add/scalar by constants of E.

Def: L homo. diff. operator over F
E/F differential field extⁿ is
called Picard-Vessiot of:

1.) $C_E = C_F$

2.) E has a full set V_L of
solutions to $L = 0$.
(i.e. has n l.i. solⁿs over C_E)

3.) E is generated over F by solⁿ
to $L = 0$

2) is some analogue of separability
3) is some analogue of normality

Def: A differential automorphism of a differential field F

$\varphi : F \to F$ field auto.

$\varphi(Df) = D\varphi(f)$

Def: E/F P.V. $Gal(E/F)$ the gp of differential field auto. that fix F.

Ex: $F = \mathbb{C}(t)$ $E = F(\log(t))$

$D_F = D_E = \frac{d}{dt}$

Check E/F is P.V. ext^n:

- $C_E = C_F$ is clear

Although E is generated over F by a sol^n to $Y' - \frac{1}{t} = 0$

This isn't homogenous

$Y'' + \frac{1}{t} Y' = 0$ has two l.i.

Sol^{n}s over $C_E = \mathbb{C}$

$\{1, \log(t)\}$

and these generate E over F.

So E/F P.V. So we

Can talk about $Gal(E/F)$.

What is it?

Any $\sigma \in Gal(E/F)$

is determined by where $\log(t)$

maps.

$$\sigma((\log t)') = \sigma(\log t)'$$

$$\shortparallel$$

$$\sigma\left(\frac{1}{t}\right)$$

$$\shortparallel$$

$$\frac{1}{t} = \sigma(\log t)'$$

σ sends $\log t$ to another

Sol^n to $y' - \frac{1}{t} = 0$.

$$\Rightarrow \left(\sigma(\log t) - \log t\right)' = 0$$

$$\Rightarrow \sigma(\log t) - \log t = c$$

for some $c \in \mathbb{C}$.

$$\varphi : \mathrm{Gal}(E/F) \longrightarrow \mathbb{C}$$

$$\sigma \longrightarrow \sigma(f) - f$$

$f = \log t$

$$\varphi(\sigma\tau) = (\sigma\tau)(f) - f$$

$$= \sigma(f + \tau(f) - f) - f$$

$$= \sigma(f) + \tau(f) - f - f$$

$$= (\sigma(f) - f) + (\tau(f) - f)$$

$$= \varphi(\sigma) + \varphi(\tau)$$

Can check that this map is indeed an iso.

$$\mathrm{Gal}(E/F) \cong \mathbb{C}.$$

There are analogues of all the big things from usual Galois theory.

- Subgps of $Gal(f)$ correspond to intermediate ext^{n}s btwn base field and s.f.

$$\Big\updownarrow$$

- (closed) Subgps of $Gal(L)$ diff. correspond to intermediate ext^n of diff. splitting field of L and base field

- Gal(f) is transitive subgp of S_n

$$\updownarrow$$

- Gal(L) is an algebraic subgp of $GL(V_L)$

($\mathrm{Gal}(L) = \mathrm{Gal}(E_L/F)$ $V_L \subset E_L$
$E_L =$ diff. s.f. of L over F).

Galois Theory: Studied poly. equations through solvability

Diff. Galois Theory: integrability of certain functions

Def: L/F is a diff'n field ext'n

L/F is called <u>elementary</u> if

$F = L_0 \subset L_1 \subset \cdots \subset L_n = \cancel{L}$ s.t.

- $L_i \subset L_{i+1}$ is a finite ext'n
- L_{i+1} is gotten by adjoining a log of f or ~~and~~ exponential of f for some $f \in L_i$.

$$y' - \frac{f'}{f} = 0 \qquad \log$$

$$y' - f'y = 0 \qquad \exp$$

Ex: $\mathbb{C}(t)(e^{it})$ is elementary.

Point: f is contained in an elementary ext^n of $\mathbb{C}(t)$

$\iff$ f is elementary function.

(in the usual sense)

Thm: (Liouville) $F = \mathbb{C}(t)$

$\alpha \in F$. Then α has an anti-derivative in some elementary ext^n of F $\iff$ $\exists\, c_1, \ldots, c_n \in \mathbb{C}$

$\beta_1, \ldots, \beta_n, \gamma \in F$ s.t.

$$\alpha = \sum_i c_j \frac{\beta_j'}{\beta_j} + \gamma'$$

<u>Cor:</u> $g \in F$ s.t. e^g is transcendental over F, $f \in F$.

Then $fe^g \in F(e^g)$ has an elementary anti-derivative $\iff$ $\exists h \in F$ s.t. $f = h' + hg'$

<u>Ex:</u> e^{-x^2} has no elementary anti-derivative.

$F = \mathbb{C}(x)$ $\quad$ e^{-x^2} has elem. anti-der

$g = -x^2$ $\quad$ $\Longrightarrow$ $\exists h \in \mathbb{C}(x)$

$f = 1$ $\quad$ s.t.

$$1 = h' - 2xh.$$

$$h = \frac{p}{q} \quad (p,q) = 1 \quad p, q \in \mathbb{C}[x].$$

$$1 = \frac{p'q - pq'}{q^2} - \frac{2xpq}{q^2}$$

$$\Longrightarrow \quad q^2 = (p' - 2xp)\,q - pq'$$

$$\Longrightarrow \quad q \mid pq'$$

$$\Longrightarrow \quad q \mid q' \quad \text{b/c } (q,p) = 1.$$

$\Rightarrow$ q is constant.

So $h \in \mathbb{C}[x]$. $\Rightarrow$ $\not\in$

by looking at degrees $\square$