Selected Solutions to Homework 8

Tim Smits

6.2.10 Let $f : R \to S$ be a surjective ring homomorphism, and let $I \subset R$ be an ideal.

(a) Prove that $f(I)$ is an ideal of S.

(b) Show that part (a) is false if f is not surjective.

Solution:

(a) Let $x, y \in f(I)$. Then $x = f(a)$ and $y = f(b)$ for some $a, b \in R$. This says $x + y = f(a) + f(b) = f(a + b)$, so $x + y \in f(I)$. For any $s \in S$, we have $s = f(r)$ for some r because f is surjective. We have $rx = f(r)f(a) = f(ra)$ and $xr = f(a)f(r) = f(ar)$, so that $rx, xr \in f(I)$. This means that $f(I)$ is an ideal as desired.

(b) Consider the inclusion map $i : \mathbb{Z} \to \mathbb{Q}$, and take $I = 2\mathbb{Z}$. then $f(I) = 2\mathbb{Z}$, which is not an ideal of $\mathbb{Q}$. ($\mathbb{Q}$ is a field, so it's only ideals are (0) and $\mathbb{Q}$!).

6.2.18 Let R be a commutative ring such that every ideal is principal (this is called a *principal ideal ring*, or PIR for short). Prove that if $f : R \to S$ is a ring homomorphism, then $f(R)$ is a PIR.

Solution: By shrinking the co-domain, we can consider f as a map $f : R \to f(R)$ instead. Let J be an ideal of $f(R)$. By the problem below, we have $I = f^{-1}(J)$ is an ideal of R, and because R is a PIR we can write $I = (r)$ for some $r \in R$. Since f surjects onto $f(R)$, we have $f(f^{-1}(J)) = J$ for set theoretic reasons, so $f(I) = J$. If we show that $f(I)$ is principal, we're done. We have $f(I) = f((r)) = \{f(ar) : a \in R\} = \{f(a)f(r) : a \in R\} = \{bf(r) : b \in f(R)\} = (f(r))$, because varying over all elements of R makes f hit all elements of $f(R)$. This says that $J = (f(r))$, so we're done.

6.2.22 Let $f : R \to S$ be a ring homomorphism. If J is an ideal of S, prove that $I = f^{-1}(J) = \{r \in R : f(r) \in J\}$ is an ideal of R with $\ker(f) \subset I$.

Solution: Let $x, y \in f^{-1}(J)$. We want to show that $x + y \in f^{-1}(J)$, i.e. that $f(x + y) \in J$. We have $f(x + y) = f(x) + f(y) \in J$ since $f(x), f(y) \in J$ and J is an ideal of S, so it's closed under addition. Next, let $r \in R$. We want to show that rx and $xr \in f^{-1}(J)$, i.e. that $f(rx)$ and $f(xr) \in J$. Once more, we have $f(rx) = f(r)f(x)$ and $f(xr) = f(x)f(r)$ are both in J, because $f(x) \in J$ and J absorbs multiplication by elements of J. This says that $f^{-1}(J)$ is an ideal. Note that if $x \in \ker(f)$, then $f(x) = 0 \in J$, so $\ker(f) \subset f^{-1}(J)$.