Selected Solutions to Homework 6

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4.4.18 Let $\varphi : \mathbb{C} \to \mathbb{C}$ be an isomorphism of rings such that $\varphi(a) = a$ for all $a \in \mathbb{Q}$. Suppose that $r \in \mathbb{C}$ is a root of $f(T) \in \mathbb{Q}[T]$. Prove that $\varphi(r)$ is also a root of $f(T)$.

Solution: Let $f(T) = a_0 + \ldots + a_n T^n$. Since r is a root of $f(T)$, we have $a_0 + \ldots + a_n r^n = 0$. Applying φ to both sides, we have $0 = \varphi(0) = \varphi(a_0 + \ldots + a_n r^n) = \varphi(a_0) + \ldots + \varphi(a_n)\varphi(r)^n = a_0 + \ldots + a_n\varphi(r)^n$. This says that $\varphi(r)$ is also a root of $f(T)$.

4.5.2 Prove that $\sqrt{p}$ is irrational for all prime p.

Solution: If $\sqrt{p}$ was rational, it would be a root of $f(T) = T^2 - p \in \mathbb{Q}[T]$. By the rational root theorem, the only possible rational roots of $f(T)$ are $\pm p$, and it's obvious neither of these are roots, a contradiction. Therefore, $\sqrt{p}$ is irrational.

4.5.18 Which of the following polynomials $f(T)$ are irreducible in $\mathbb{Q}[T]$?.

(a) $T^4 - T^2 + 1$

(b) $T^4 + T + 1$

(c) $T^5 + 4T^4 + 2T^3 + 3T^2 - T + 5$

(d) $T^5 + 5T^2 + 4T + 7$.

Solution:

(a) $T^4 - T^2 + 1$ clearly has no roots, so if it factors, it must factor as a product of irreducible quadratics. By Gauss's lemma, we can write $T^4 - T^2 + 1 = (T^2 + aT + b)(T^2 + cT + d)$ for some $a, b, c, d \in \mathbb{Z}$. Expanding, this says $T^4 - T^2 + 1 = T^4 + (a+c)T^3 + (ac+b+d)T^2 + (ad+bc)T + bd$. Comparing coefficients, we have $a + c = 0, ac + b + d = -1, ad + bc = 0, bd = 1$. The first equation says $a = -c$. Plugging in says $c^2 - 1 = b + d$, $c(b - d) = 0$ and $bd = 1$. Note that $b = d = -1$ or $b = d = 1$. The former says $c^2 = -1$ and the latter says $c^2 = 3$, neither of which are possible. Therefore, $T^4 - T^2 + 1$ is irreducible in $\mathbb{Q}[T]$.

(b) In $(\mathbb{Z}/2\mathbb{Z})[T]$, the only degree 2 monic irreducible is $T^2 + T + 1$, and $(T^2 + T + 1)^2 = T^4 + T^2 + 1$. This says $T^4 + T + 1$ is irreducible mod 2 (it clearly has no roots, so no linear factors) and therefore is irreducible in $\mathbb{Q}[T]$.

(c) In $(\mathbb{Z}/3\mathbb{Z})[T]$, we have $\bar{f}(T) = T^5 + T^4 + 2T^3 + 2T + 2$. This polynomial has no root in $\mathbb{Z}/3\mathbb{Z}$, so if it factors, it must have a degree 2 factor. The degree 2 monic irreducible polynomials in $(\mathbb{Z}/3\mathbb{Z})[T]$ are $T^2 + 1, T^2 + T + 1$, and $T^2 + 2T + 2$. One can check that none of these divide $T^5 + T^4 + 2T^3 + 2T + 2$ in $(\mathbb{Z}/3\mathbb{Z})[T]$, so $f(T)$ is irreducible mod 3 and therefore is irreducible in $\mathbb{Q}[T]$.

(d) In $(\mathbb{Z}/2\mathbb{Z})[T]$, we have $\bar{f}(T) = T^5 + T^2 + 1$. This has no root in $\mathbb{Z}/2\mathbb{Z}$, so if it factors, it must be divisible by the monic irreducible quadratic polynomial in $(\mathbb{Z}/2\mathbb{Z})[T]$, $T^2 + T + 1$. One can check that this doesn't divide $T^5 + T^2 + 1$, so that $f(T)$ is irreducible mod 2. Therefore, $f(T)$ is irreducible in $\mathbb{Q}[T]$.