Selected Solutions to Homework 5

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4.1.12 If $f, g \in R[T]$ and $f + g \neq 0$, prove that $\deg(f + g) \leq \max\{\deg(f), \deg(g)\}$.

Solution: Write $f(T) = a_n T^n + \ldots + a_0$ and $g(T) = b_m T^m + \ldots + b_0$, with $a_n, b_m \neq 0$. If $n \neq m$, then it's clear that $\deg(f + g) = \max\{\deg(f), \deg(g)\}$. If $n = m$, the leading coefficient of $f + g$ is $a_n + b_n$. If this is non-zero, $\deg(f + g) = n = \max\{\deg(f), \deg(g)\}$. Otherwise if it is zero, $\deg(f + g) < n = \max\{\deg(f), \deg(g)\}$. Putting this all together says $\deg(f + g) \leq \max\{\deg(f), \deg(g)\}$.

4.2.14 Let $f, g \in F[T]$ with $(f, g) = 1$. Prove that if $f \mid h$ and $g \mid h$ for some $h \in F[T]$, that $fg \mid h$.

Solution: Since $(f, g) = 1$, we can write $fp + gq = 1$ for some $p, q \in F[T]$ by Bezout's lemma. Multiplying by h says $fhp + ghq = h$. Since $g \mid h$, the first term is divisible by fg. Since $f \mid h$, the second term is divisible by fg. Therefore, $fg \mid h$.

4.3.14 Show that $T^2 + T \in (\mathbb{Z}/6\mathbb{Z})[T]$ can be factored in two ways in $(\mathbb{Z}/6\mathbb{Z})[T]$ as a product of non-constant polynomials that are not units, and not associates of T or $T + 1$.

Solution: We want to write $T^2 + T = (aT + b)(cT + d)$ for some $a, b, c, d \in (\mathbb{Z}/6\mathbb{Z})$, i.e. $T^2 + T = acT^2 + (ad + bc)T + bd$. Comparing coefficients, this says $ac = 1$, $ad + bc = 1$, and $bd = 0$. The units of $\mathbb{Z}/6\mathbb{Z}$ are 1 and 5, so either $a = c = 1$ or $a = c = 5$. In the first case, this says $b + d = 1$ and $bd = 0$, so $d = 1 - b$ gives $b = b^2$. The idempotents in $\mathbb{Z}/6\mathbb{Z}$ are $0, 1, 3, 4$. Since we *don't* want to allow $b = 0, 1$ we get that $b = 3, 4$. Therefore, one such possible factorization is $T^2 + T = (T + 3)(T + 4)$. If $a = c = 5$, we get $5(b + d) = 1$ and $bd = 0$. This says $d = b - 5$, so $b^2 = 5b$. This has solutions $b = 0, 2, 3, 5$. Since we don't want any associates of $T, T + 1$ we can't have $b = 0, 5$ so $b = 2, 3$ are possible. This gives the factorization $T^2 + T = (5T + 2)(5T + 3)$. Note that none of the linear polynomials we have found are units or associates of $T, T + 1$ because none of the constant terms are units.

General comments

- In 4.3.14, you must be careful with your factorizations. It was very commonly written that the two factorizations were $(T+3)(T+4)$ and $(T-3)(T-2)$. However, in $(\mathbb{Z}/6\mathbb{Z})$, $3 = -3$ and $-2 = 4$ so these factorizations are the same!
- In 4.3.14, you have to actually check that your factorizations you wrote satisfy the conditions listed in the problem, i.e. that the polynomials are not units and not associates of $T, T+1$. This is not obvious, because we're not working in a domain!