Selected Solutions to Homework 4

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3.3.8 Prove the map $f : \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{2})$ defined by $f(a + b\sqrt{2}) = a - b\sqrt{2}$ is an isomorphism of rings.

Solution: Let $x = a + b\sqrt{2}$ and $y = c + d\sqrt{2}$ for some $a, b, c, d \in \mathbb{Q}$. Then $x + y = (a+c)+(b+d)\sqrt{2}$ and $xy = (ac+2bd)+(ad+bc)\sqrt{2}$ by how addition and multiplication in $\mathbb{Q}(\sqrt{2})$ are defined. We see that $f(x+y) = (a+c) - (b+d)\sqrt{2} = (a - b\sqrt{2}) + (c - d\sqrt{2}) = f(x) + f(y)$ and $f(xy) = (ac + 2bd) - (ad + bc)\sqrt{2} = (a - b\sqrt{2})(c - d\sqrt{2}) = f(x)f(y)$. Clearly $f(1) = 1$, so f is a ring homomorphism.

Now we check that f is bijective. The map f is clearly surjective – for any $a+b\sqrt{2} \in \mathbb{Q}(\sqrt{2})$ the element $a - b\sqrt{2}$ maps to it under f. If $x = a + b\sqrt{2} \in \ker(f)$, then $a - b\sqrt{2} = 0$ says that $a = b\sqrt{2}$. The right hand side is irrational, while the left hand side is rational, so the only way this is possible is if $a = 0$ and $b = 0$, i.e. that $x = 0$. This then gives that f is injective, so that f is an isomorphism.

3.3.28

(a) Give an example of a ring homomorphism $f : R \to S$ such that R is a ring and S is a ring without identity.

(b) Give an example of a ring homomorphism $f : R \to S$ such that S is a ring and R is a ring without identity.

Solution:

(a) Let $R = \{0\}$ and S be your favorite ring without identity. I like none of them, but I'll take $S = 2\mathbb{Z}$. There is only one map $f : R \to S$ we can even pick, which is the zero map. This obviously satisfies Hungerford's definition of a ring homomorphism.

(b) The exact same example works with the letters R and S swapped above.

3.3.38 Let F be a field, R be a ring, and $f : F \to R$ a ring homomorphism.

(a) If there is $c \neq 0 \in F$ with $f(c) = 0$, prove that $f = 0$.

(b) Prove that f is either the zero map or injective.

Solution:

(a) Suppose that $f(c) = 0$ for some $c \neq 0 \in F$. Since F is a field, c has a multiplicative inverse. Then $1 = c \cdot \frac{1}{c}$, so for any $x \in F$ we have $f(x) = f(x \cdot 1) = f(x \cdot c \cdot \frac{1}{c}) = f(x)f(c)f(\frac{1}{c}) = 0$ since f is a ring homomorphism.

(b) Part a) is the statement that if f has a non-trivial kernel, then $f = 0$. The contrapositive of this statement is if $f \neq 0$, then f must have a trivial kernel, i.e. f is injective.

General comments

- If you want to show that a ring homomorphism is injective, it is much easier to check that $\ker(f) = \{0\}$.
- In 3.3.8, Many of you wrote something along the lines of "$a - b\sqrt{2} = c - d\sqrt{2} \implies a = c$ and $b = d$". This is a non-trivial statement that *uses* the assumption that a, b, c, d are rational! If $a, b, c, d \in \mathbb{R}$ for example, you could take $a = \sqrt{2}$ and $b = 1$, and $c = 2\sqrt{2}$ and $d = 2$. You must be justifying all of your claims!
- In 3.3.28, it does not make sense to speak of a ring homomorphism without specifying a map. Make sure the map you write down actually makes sense, (e.g. has domain and co-domain what you claim), that your choices of R and S actually are rings, and the map you write down actually is a homomorphism!
- In 3.3.28 giving an example means specifying choices of R, S and a ring homomorphism $f : R \to S$.
- In 3.3.38 you should make it clear how part (a) is being used in part (b).