Selected Solutions to Homework 3

Tim Smits

3.1.32 Let R be a ring and let $Z(R) = \{a \in R : ar = ra \text{ for all } r \in R\}$ be the center of the R. Prove that $Z(R)$ is a subring of R.

Solution: Let $a, b \in Z(R)$ and $r \in R$. Then $(a-b)r = ar - br = ra - rb = r(a-b)$, so $a - b \in R$. We also see that $(ab)r = a(br) = a(rb) = (ar)b = (ra)b = r(ab)$, so $ab \in Z(R)$. Finally, we have $1 \cdot r = r \cdot 1 = r$, so $1 \in Z(R)$. By the subring test, this says that $Z(R)$ is a subring of R. (If you don't wish to assume rings have identity for whatever reason, you can check the non-emptyness condition by noting $0 \in Z(R)$).

3.2.14 Prove the only idempotents in an integral domain R are 0 and 1.

Solution: If $e \in R$ is idempotent, then $e^2 = e$ says that $e(e-1) = 0$. Since R is an integral domain, this says either $e = 0$ or $e - 1 = 0$, i.e. $e = 1$.

3.2.40 Prove that R has no non-zero nilpotent elements if and only if the only solution to $x^2 = 0$ in R is $x = 0$.

Solution: Suppose that R has no non-zero nilpotents, then clearly $x^2 = 0$ has only $x = 0$ as a solution, because a non-zero solution would be nilpotent by definition. Conversely, suppose that the only solution to $x^2 = 0$ is $x = 0$. Suppose that there was a non-zero nilpotent element $a \in R$. Let N be the smallest positive integer such that $a^N = 0$ and $a^{N-1} \neq 0$. Then $(a^{N-1})^2 = a^{2N-2} = a^N \cdot a^{N-2} = 0$, contradicting that $x^2 = 0$ has no non-zero solutions.