Integration and Infinite Series

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Introduction

These notes arose from many years of being a teaching assistant, and later an instructor for, math 31B at UCLA. The content covers the core topics that one usually sees in a second semester calculus course on integration techniques and infinite series. At UCLA, such a course also contains some topics in differential calculus. The intention is to provide a large collection of examples of various levels of difficulty: not just "easy" examples like one might see in a calculus textbook. By including some "hard" examples, the hope is that the reader sharpens their problem solving skills by seeing how more complicated problems can be broken down. There may be many typos. Let me know if any are found!

Chapter 1

Infinite Series

Definition 1.0.1. A **sequence** is a function $f : \mathbb{N} \to \mathbb{R}$, where $\mathbb{N}$ is the set of non-negative integers.

We usually write a_n to denote the value $f(n)$ of the function f, because we like to think of sequences as different types of objects than functions. Often times, it's useful to think about a sequence as it's set of values $\{a_n\}$, and we typically write $\{a_n\}$ to refer to the sequence instead of f.

Example 1.0.2. We can think of the sequence $a_n = \frac{1}{n^2}$ as either being some object explicitly defined by the above formula, or as the list of values $\{1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots\}$.

Example 1.0.3. Define a sequence by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. This is an example of a **recursive sequence**, a sequence where the value at some given n depends on the previous terms. Explicitly, the first few terms of this sequence are $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$. This sequence is called the **Fibonacci sequence**.

Since sequences are functions, all the usual operations you're used to doing with functions make sense for sequences: addition/subtraction, multiplication/division, taking limits, etc. Ultimately, our goal is to understand how calculus works in the discrete world, with sequences taking the role of functions. Below are the analogies between calculus in $\mathbb{R}$ and discrete calculus that should be kept in mind to strengthen conceptual understanding.

Calculus in $\mathbb{R}$	Discrete calculus
Functions $f : \mathbb{R} \to \mathbb{R}$	Sequences $\{a_n\}$
Derivative: $\frac{d}{dx} f(x)$	Forward difference: $\Delta a_n = a_{n+1} - a_n$
Anti-derivative: $\int f(x)\, dx$	Partial sum: $\sum_{n=1}^{N} a_n$
Definite integral: $\int_a^b f(x)\, dx$	Sum: $\sum_{n=a}^{b} a_n$
Improper integral: $\int_1^\infty f(x)\, dx$	Infinite series: $\sum_{n=1}^{\infty} a_n$

1.1 Basic Definitions

Definition 1.1.1. Given a sequence $\{a_n\}$, define a new sequence $\{S_N\}$ by $S_N = a_1 + a_2 + \ldots + a_N = \sum_{n=1}^{N} a_n$. The sequence $\{S_N\}$ is called the sequence of **partial sums** of $\{a_n\}$.

An **infinite series** is an expression of the form $\sum_{n=1}^{\infty} a_n$, i.e. addition of infinitely many terms of some sequence (for convenience the starting index is 1, but it does not matter).

As is usual in calculus, to try and understand something "infinite", we have to take limits of "finite" things that we understand. Analogously to how we define $\int_1^\infty f(x)\,dx$ through limits of definite integral $\lim_{R\to\infty}\int_1^R f(x)\,dx$, we will define an infinite series by taking a limit of its partial sums.

Definition 1.1.2. We say $\sum_{n=1}^{\infty} a_n$ **converges** if $\lim_{N\to\infty} S_N$ is finite, and if $\lim_{N\to\infty} S_N = L$, we say $\sum_{n=1}^{\infty} a_n = L$. The infinite series $\sum_{n=1}^{\infty} a_n$ **diverges** if $\lim_{N\to\infty} S_N$ does not exist.

Series can be confusing at first because of the different types of objects involved. A series is a formal infinite expression of the form $a_1 + a_2 + \ldots$. A series can be assigned a value, which is obtained by taking a limit of a sequence (the sequence of partial sums). In particular, don't mix up sequences with series: series have a value, sequences are functions.

Example 1.1.3. Set $a_n = \frac{1}{n}$, so that $S_N = \sum_{n=1}^{N} \frac{1}{n}$. The first few terms of the sequence S_N are given by $1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}, \ldots$. The series $\sum_{n=1}^{\infty}\frac{1}{n}$ is called the **harmonic series**. It turns out that $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges (which is not at all obvious).

Example 1.1.4. Set $a_n = n$. Then the sequence of partial sums $\{S_N\}$ has terms given by $S_N = \sum_{n=1}^{N} n = \frac{N(N+1)}{2}$. Since $\lim_{N\to\infty} S_N = \infty$, the series $\sum_{n=1}^{\infty} n$ diverges.

Example 1.1.5. Let $\{a_n\}$ be a sequence such that the N-th partial sum is given by $S_N = 3 - \frac{1}{N^2}$. Since $\lim_{N\to\infty} S_N = 3$, this says $\sum_{n=1}^{\infty} a_n$ converges, and we have $\sum_{n=1}^{\infty} a_n = 3$. Notice we know nothing about the actual terms in the sequence $\{a_n\}$ – the definition of convergence or divergence of an infinite series depends only on the partial sums.

Example 1.1.6. With $S_N = 3 - \frac{1}{N^2}$ as above, we can recover what the general term of the sequence is. Taking a forward difference, we have $\Delta S_N = S_{N+1} - S_N = \sum_{n=1}^{N+1} a_n - \sum_{n=1}^{N} a_n = a_{N+1}$, so $a_{N+1} = \frac{1}{N^2} - \frac{1}{(N+1)^2}$. Re-indexing, we find $a_n = \frac{1}{(n-1)^2} - \frac{1}{n^2}$ for $n \geq 2$, and $S_1 = a_1 = 2$. This process is analogous to how a function can be recovered from knowledge of it's anti-derivative by differentiating.

1.2 Geometric and Telescoping Series

Definition 1.2.1. A **geometric series** is an infinite series of the form $\sum_{n=M}^{\infty} ar^n$ for some non-zero real numbers a and r, and some starting index M.

Geometric series are "simple" series in the sense that we can classify their behavior completely:

Theorem 1.2.2 (Classification of geometric series). *If $|r| < 1$, then $\sum_{n=M}^{\infty} ar^n$ converges, and $\sum_{n=M}^{\infty} ar^n = \frac{ar^M}{1-r}$. Otherwise if $|r| \geq 1$, then $\sum_{n=M}^{\infty} ar^n$ diverges.*

Example 1.2.3. The series $\sum_{n=1}^{\infty} 5(\frac{1}{2})^n$ is a geometric series with $a = 5$, $r = \frac{1}{2}$, and $M = 1$. We see $\sum_{n=1}^{\infty} 5(\frac{1}{2})^n = \frac{5/2}{1-1/2} = 5$.

Example 1.2.4. Consider the infinite series $\sum_{n=0}^{\infty} \frac{3\cdot 2^{2n-2}+(-1)^n 5^{n+1}}{6^n}$. Splitting this up, we can write this as $\sum_{n=0}^{\infty} \frac{3\cdot 2^{2n-2}}{6^n} + \sum_{n=0}^{\infty} (-1)^n \frac{5^{n+1}}{6^n}$. Using exponent rules to write each sum as a geometric series, we find $\sum_{n=0}^{\infty} \frac{3\cdot 2^{2n-2}+5^{n+1}}{6^n} = \sum_{n=0}^{\infty} \frac{3}{4}(\frac{4}{6})^n + \sum_{n=0}^{\infty} 5(-\frac{5}{6})^n = \frac{3/4}{1-2/3} + \frac{5}{1+5/6} = \frac{219}{44}$ using the above formula.

Definition 1.2.5. A **telescoping series** is an infinite series of the form $\sum_{n=1}^{\infty}(a_{n+1} - a_n)$ for some sequence $\{a_n\}$.

The name telescoping comes from writing down the summation of the terms in the series – they cancel out and "collapse" like a telescope. The N-th partial sum of a telescoping series is $S_N = \sum_{n=1}^{N}(a_{n+1} - a_n) = (a_2 - a_1) + (a_3 - a_2) + (a_4 - a_3) + \ldots + (a_N - a_{N-1}) + (a_{N+1} - a_N) = a_{N+1} - a_1$, so taking a limit gives the following:

Theorem 1.2.6 (Discrete FTC). *Suppose that* $\lim_{n\to\infty} a_n = L$. *Then* $\sum_{n=1}^{\infty}(a_{n+1} - a_n) = L - a_1$.

Writing the above statement using the forward difference operator, the theorem says $\sum_{n=1}^{\infty} \Delta a_n = L - a_1$ where $L = \lim_{n\to\infty} a_n$. The analogue is the statement that $\int_1^{\infty} f'(x)\,dx = L - f(1)$ where $L = \lim_{x\to\infty} f(x)$, which is just the fundamental theorem of calculus (applied to improper integrals).

Example 1.2.7. The series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is a telescoping series. To see this, using partial fractions we can write $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$, and we then see $\sum_{n=1}^{\infty}(\frac{1}{n} - \frac{1}{n+1})$ is a telescoping series with $a_n = -\frac{1}{n}$, and $\lim_{n\to\infty} a_n = 0$, so that $\sum_{n=1}^{\infty}(\frac{1}{n} - \frac{1}{n+1}) = 1$.

Example 1.2.8. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(\frac{(n+1)^n}{n^{(n+1)}})}{n(n+1)}$. Using log rules, we can write this as $\sum_{n=1}^{\infty} \frac{n\ln(n+1) - (n+1)\ln(n)}{n(n+1)} = \sum_{n=1}^{\infty}\left(\frac{\ln(n+1)}{n+1} - \frac{\ln(n)}{n}\right)$. This is a telescoping series with $a_n = \frac{\ln(n)}{n}$. As $\lim_{n\to\infty} \frac{\ln(n)}{n} = 0$, we find $\sum_{n=1}^{\infty} \frac{\ln(\frac{(n+1)^n}{n^{(n+1)}})}{n(n+1)} = 0$.

Unlike with integration where we have many different techniques and rules for explicitly computing anti-derivatives, finding a sequence b_n with $\Delta b_n = a_n$ is in general, very hard. Therefore, it's generally not very obvious if a series telescopes or not! Because this process is so difficult, it's not very easy to go through the definition of an infinite series to determine if it converges or diverges. We'll have to develop more theory to help us get around this issue.

1.3 The Comparison Tests

There is a useful test for quickly checking if a series diverges:

Theorem 1.3.1 (Divergence Test). *Let* $\{a_n\}$ *be a sequence. If* $\lim_{n\to\infty} a_n \neq 0$, *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Example 1.3.2. The divergence test says the series $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges, because $\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$. The series $\sum_{n=1}^{\infty}(1+\sin(n))$ also diverges, because $\lim_{n\to\infty} 1+\sin(n)$ does not exist.

Warning: the divergence test does **not** say that if $\lim_{n\to\infty} a_n = 0$, that $\sum_{n=1}^{\infty} a_n$ converges. As we will later see, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, but as mentioned before the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. In both series, the general term tends to 0, so if this happens we cannot conclude anything about convergence or divergence.

Our first two series test are going to be our most powerful ones.

Theorem 1.3.3 (Direct comparison test). *Let $\sum a_n$, $\sum b_n$ be infinite series with $a_n, b_n \geq 0$, and assume that $\sum a_n \leq \sum b_n$ eventually.*

(a) If $\sum a_n$ diverges, then $\sum b_n$ diverges.

(b) If $\sum b_n$ converges, then $\sum a_n$ converges.

Intuitively, the direct comparison test says anything smaller than a convergent series converges (i.e. anything smaller than a finite sum is finite), and anything larger than a divergent series is divergence (i.e., anything larger than an infinite sum is infinite). Notice that we only need that the inequality on series holds **eventually**. We may always rip out a finite number of terms from the sum (which doesn't change convergence) to make such an inequality explicitly true (provided $a_n \leq b_n$ eventually holds).

Theorem 1.3.4 (Limit comparison test). *Let $\sum a_n$, $\sum b_n$ be infinite series with $a_n \geq 0$ and $b_n > 0$. Set $L = \lim_{n\to\infty} \frac{a_n}{b_n}$ and assume that L exists. If $0 < L < \infty$, then $\sum a_n$ and $\sum b_n$ both converge or diverge together.*

Intuitively, if L is finite, this says eventually, that $a_n \approx L b_n$, so the terms in the series roughly differ by a constant multiple, which won't change the convergence or divergence.

Each comparison test has its own set of pros and cons. In general, the direct comparison test will be a bit harder to apply, since one needs to exhibit explicit inequalities, which might be tricky to find. The limit comparison test is typically more useful, because in the process of intuitively reasoning if a series will converge or diverge, one often gets another series to compare with for free, and computing a limit is much easier than trying determine which series is larger. The direct comparison tests is more useful in a few specific cases: when the series has terms with logarithms (which grow too slowly to find a different series with similar growth speed), or with exponentials (which grow too quickly). Another situation where the direct comparison test is useful is when trigonometric functions like sine or cosine appear, as we have explicit upper/lower bounds on these functions. When we later cover Taylor series, we will see how to come up with good approximations to these types of functions that allow the limit comparison test to more easily apply.

Before moving onto examples, we need some series whose behavior is known that we can compare to. Above we classified the convergence or divergence of geometric series. Another common family of series, known as p-series, have the following behavior:

Theorem 1.3.5 (Classification of p-series). *The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $0 < p \le 1$.*

It will be useful to know how quickly certain commonly encountered functions grow. We will use the notation "$a_n \ll b_n$" to mean the sequence a_n is eventually smaller than b_n, i.e. there is some N such that $a_n \le b_n$ for all $n \ge N$. Another way of saying this is that a_n grows slower than b_n.

Theorem 1.3.6. *The following hold for any $a > 0$ and any $b > 1$: $\ln(n) \ll n^a \ll b^n \ll n! \ll n^n$.*

Remark 1.3.7. The above theorem is actually even stronger than what is stated. As you move up the hierarchy, not only do you have eventual inequalities, but you also have eventual limit domination, meaning that the limit of the ratio tends to 0 as $n \to \infty$. For example, $\lim_{n\to\infty} \frac{\ln(n)}{n^a} = 0$ for any $a > 0$. This strengthening of the theorem can be proved by repeated applications of L'Hopital's rule (and in fact, is really how you would prove the above version anyway.)

Example 1.3.8. The series $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ converges. The general term $\frac{1}{n2^n}$ decays more quickly than $\frac{1}{2^n}$, and the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges, so we expect our series converges as well. We see that $2^n < n2^n$, so that $\frac{1}{n2^n} < \frac{1}{2^n}$ for all n. This says $\sum_{n=1}^{\infty} \frac{1}{n2^n} < \sum_{n=1}^{\infty} \frac{1}{2^n}$. The latter is a convergent geometric series, so the result follows by the direct comparison test.

Example 1.3.9. The series $\sum_{n=0}^{\infty} \frac{4}{4^n+n!}$ converges. The general term $\frac{1}{n!+4^n}$ decays more quickly than $\frac{1}{4^n}$, and the series $\sum_{n=0}^{\infty} \frac{1}{4^n}$ converges, so we expect our series converges as well. Since $n! + 4^n > 4^n$ for all $n \ge 0$, we see that $\frac{1}{4^n+n!} < \frac{1}{4^n}$, so multiplying by 4 says $\frac{4}{4^n+n!} < \frac{4}{4^n}$ for $n \ge 0$. Since $\sum_{n=0}^{\infty} \frac{4}{4^n} = 4\sum_{n=0}^{\infty} (\frac{1}{4})^n$ is a convergent geometric series, we see that $\sum_{n=0}^{\infty} \frac{4}{4^n+n!}$ converges by a direct comparison test.

Example 1.3.10. The series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n-1}$ diverges. As $n \to \infty$, $\frac{\sqrt{n}}{n-1} \approx \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$. Since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges, we think our series should diverge as well. Since $n - 1 < n$, we get $\frac{1}{n-1} > \frac{1}{n}$ so that $\frac{\sqrt{n}}{n-1} > \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$. The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent p-series with $p = 1/2$, so the result follows by the direct comparison test.

Example 1.3.11. The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}(1+\sqrt{n})}$ diverges. As $n \to \infty$, $1 + \sqrt{n} \approx \sqrt{n}$, so that $\frac{1}{\sqrt[3]{n}(1+\sqrt{n})} \approx \frac{1}{n^{5/6}}$. Since $\sum_{n=1}^{\infty} \frac{1}{n^{5/6}}$ is a divergent p-series, we expect our original series also diverges. As $1 \le \sqrt{n}$ for $n \ge 1$, we see $1 + \sqrt{n} \le \sqrt{n} + \sqrt{n} = 2\sqrt{n}$, so that $\sqrt[3]{n}(1 + \sqrt{n}) \le \sqrt[3]{n}(2\sqrt{n}) = 2n^{5/6}$. This then says $\frac{1}{\sqrt[3]{n}(1+\sqrt{n})} \ge \frac{1}{2n^{5/6}}$, and $\sum_{n=1}^{\infty} \frac{1}{2n^{5/6}}$ is a divergent p-series with $p = 5/6$. The original series diverges by a direct comparison.

Example 1.3.12. The series $\sum_{n=1}^{\infty} \frac{1}{n-\ln(n)}$ diverges. As $n \to \infty$, the only term that matters in the denominator is n, because logarithms grow slowly. So we expect $\frac{1}{n-\ln(n)} \approx \frac{1}{n}$, which would say that $\sum_{n=1}^{\infty} \frac{1}{n-\ln(n)}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ should have the same behavior. The latter is the divergent harmonic series, so we expect our original series diverges. Set $a_n = \frac{1}{n-\ln(n)}$ and $b_n = \frac{1}{n}$. Then $\frac{a_n}{b_n} = \frac{n}{n-\ln(n)}$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n-\ln(n)} = 1$ by L'Hopital's rule. Therefore by the limit comparison test, our original series diverges.

Example 1.3.13. The series $\sum_{n=1}^{\infty} \frac{n^3}{n^5+4n+1}$ converges. As $n \to \infty$, the fastest growing term in the denominator is n^5, so we expect $\frac{n^3}{n^5+4n+1} \approx \frac{n^3}{n^5} = \frac{1}{n^2}$. Using a limit comparison test with $a_n = \frac{n^3}{n^5+4n+1}$ and $b_n = \frac{1}{n^2}$, we see $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^5}{n^5+4n+1} = 1$. This says the behavior of the original series is the same as $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which is a convergent p-series. The result then follows by the limit comparison test.

Example 1.3.14. The series $\sum_{n=1}^{\infty} \frac{e^n+n}{e^{2n}-n^2}$ converges. As $n \to \infty$, the exponential terms are the only things that matter in the numerator and denominator, because they grow the fastest. Therefore, $\frac{e^n+n}{e^{2n}-n^2} \approx \frac{e^n}{e^{2n}} = (\frac{1}{e})^n$. Therefore, we expect that $\sum_{n=1}^{\infty} \frac{e^n+n}{e^{2n}-n^2}$ and $\sum_{n=1}^{\infty}(\frac{1}{e})^n$ have the same behavior. The latter series is a convergent geometric series with $r = \frac{1}{e}$, so our original series should converge as well. Set $a_n = \frac{e^n+n}{e^{2n}-n^2}$ and $b_n = \frac{1}{e^n}$. Then $\frac{a_n}{b_n} = \frac{e^{2n}+ne^n}{e^{2n}-n^2}$. After dividing both numerator and denominator by e^{2n}, we may write this as $\frac{a_n}{b_n} = \frac{1+\frac{n}{e^n}}{1-\frac{n^2}{e^{2n}}}$. Therefore, $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1+\frac{n}{e^n}}{1-\frac{n^2}{e^{2n}}} = 1$. By the limit comparison test, we get what we want.

Example 1.3.15. The series $\sum_{n=1}^{\infty} \sin(\frac{1}{n^2})$ converges. When x is close to 0, $\sin(x) \approx x$. As $n \to \infty$, $\frac{1}{n^2} \to 0$ so we expect $\sin(\frac{1}{n^2}) \approx \frac{1}{n^2}$. Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, we expect that our original series does as well. Using $a_n = \sin(\frac{1}{n^2})$ and $b_n = \frac{1}{n^2}$, we see $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin(\frac{1}{n^2})}{\frac{1}{n^2}} = \lim_{u\to 0} \frac{\sin(u)}{u} = 1$ via the substitution $u = \frac{1}{n^2}$. This says the series have the same behavior, so the result follows via the limit comparison test.

Example 1.3.16. Sometimes it's useful to chain comparison tests together. The series $\sum_{n=2}^{\infty} \frac{\ln(n)}{\sqrt{n^3-n^2}}$ is convergent. As $n \to \infty$, the fastest growing term in the denominator is n^3, so $\sqrt{n^3-n^2} \approx \sqrt{n^3} = n^{3/2}$. So we expect that $\sum_{n=2}^{\infty} \frac{\ln(n)}{\sqrt{n^3-n^2}}$ and $\sum_{n=2}^{\infty} \frac{\ln(n)}{n^{3/2}}$ have the same behavior. This can be checked using the limit comparison test: with $a_n = \frac{\ln(n)}{\sqrt{n^3-n^2}}$ and $b_n = \frac{\ln(n)}{n^{3/2}}$ we see that $\frac{a_n}{b_n} = \frac{n^{3/2}}{\sqrt{n^3-n^2}} \to 1$. Therefore, we just need to determine what $\sum_{n=2}^{\infty} \frac{\ln(n)}{n^{3/2}}$ does. In order to analyze this series, we need a fact about the growth speed of logarithms: they grow slower than any power function. Formalized mathematically, this says for any $a > 0$, there exists N such that $\ln(n) < n^a$ for $n \geq N$. Picking $a = 1/4$, this says $\ln(n) < n^{1/4}$ eventually, so that eventually $\sum \frac{\ln(n)}{n^{3/2}} < \sum \frac{n^{1/4}}{n^{3/2}} = \sum \frac{1}{n^{5/4}}$. The latter series is a convergent p-series, so by a direct comparison, $\sum \frac{\ln(n)}{n^{3/2}}$ converges and we are done.

1.4 The Integral Test

Conceptually, the integral test is the most important convergence test: it says that if the terms of an infinite series are "nice", the behavior of the series and the behavior of the corresponding improper integral should be the same. This provides the explicit link between infinite series and integration.

In practice, the integral test is often not that useful. In order for it to apply, you must know how to integrate the general term of an infinite series – this is something you either know you can do, in which case the test will work, otherwise if you don't know how to integrate the general term, the test is completely useless. It's generally best to try other convergence tests before trying the integral test, unless you are confident you can make it work.

Theorem 1.4.1 (Integral test). *Let $a_n = f(n)$ where $f(x)$ is a non-negative, continuous function that is eventually decreasing for $x \geq M$ for some M. Then $\sum a_n$ and $\int_M^\infty f(x)\,dx$ both converge or both diverge.*

Example 1.4.2. We can use the integral test to classify the behavior of p-series. Set $f(x) = \frac{1}{x^p}$. Then $f'(x) = -px^{p-1} < 0$ for $x > 0$, so that $f(x)$ is decreasing. It's clear that $f(x)$ is non-negative for $x > 0$, and also that it's continuous. Therefore by the integral test, $\sum_{n=1}^\infty \frac{1}{n^p}$ and $\int_1^\infty \frac{1}{x^p}\,dx$ have the same behavior. First we handle the case $p \neq 1$. By definition, $\int_1^\infty \frac{1}{x^p}\,dx = \lim_{R\to\infty} \int_1^R x^{-p}\,dx = \lim_{R\to\infty} \frac{x^{1-p}}{1-p}\Big|_1^R = \lim_{R\to\infty} \frac{R^{1-p}}{1-p} - \frac{1}{1-p}$. If $p > 1$, then $1 - p < 0$, so that $R^{1-p} \to 0$ as $R \to \infty$, so that $\int_1^\infty \frac{1}{x^p}\,dx = -\frac{1}{1-p} < \infty$. If $p < 1$, then $1 - p > 0$, so that $R^{1-p} \to \infty$ as $R \to \infty$, which says $\int_1^\infty \frac{1}{x^p}\,dx$ diverges. In the case $p = 1$, the integral in question that we care about is $\int_1^\infty \frac{1}{x}\,dx = \lim_{R\to\infty} \int_1^R \frac{1}{x}\,dx = \lim_{R\to\infty} \ln|x|\Big|_1^R = \lim_{R\to\infty} \ln(R) = \infty$, so that $\int_1^\infty \frac{1}{x}\,dx$ diverges. Putting this all together, we find that $\sum_{n=1}^\infty \frac{1}{n^p}$ converges for $p > 1$ and diverges for $p \leq 1$.

Example 1.4.3. One place where the integral test really shines is when there are logarithms floating around: the series $\sum_{n=2}^\infty \frac{1}{n\ln(n)}$ diverges. Set $f(x) = \frac{1}{x\ln(x)}$. Then $f'(x) = -\frac{(1+\ln(x))}{x^2\ln(x)^2} < 0$ for $x \geq 2$. The function $f(x)$ is also non-negative for $x \geq 2$, and it's clearly continuous, so by the integral test, the series $\sum_{n=2}^\infty \frac{1}{n\ln(n)}$ and the improper integral $\int_2^\infty \frac{1}{x\ln(x)}\,dx$ have the same behavior. By definition, the latter integral is $\lim_{R\to\infty} \int_2^R \frac{1}{x\ln(x)}\,dx$ $= \lim_{R\to\infty} \ln(\ln(x))\Big|_2^R = \lim_{R\to\infty} \ln(\ln(R)) - \ln(\ln(2)) = \infty$. Therefore, the integral diverges, so that the series diverges.

The right way to really think about the integral test is not as a test for convergence of infinite series, but as a test for convergence of *integrals*. The integral test is incredibly important if you think about it this way, because it gives us significantly more techniques than we had before to determine if integrals converge or diverge!

Example 1.4.4. Suppose we wanted to know if $\int_0^\infty \frac{x^3+x+1}{x^4+1}\,dx$ converges or diverges. Using what we learned before, we could split the integral up and do several direct comparison tests. Alternatively, we could use the integral test, and then a limit comparison test. The function $f(x) = \frac{x^3+x+1}{x^4+1}$ is clearly non-negative and continuous, and $f'(x) = -\frac{(x^6+3x^4+4x^3-3x^2-1)}{(x^4+1)^2}$ is negative for $x > 1$ (which is not terribly hard to see). Therefore, by the integral test, $\int_0^\infty \frac{x^3+x+1}{x^4+1}\,dx$ and $\sum_{n=0}^\infty \frac{n^3+n+1}{n^4+1}$ have the same behavior. As $n \to \infty$, $\frac{n^3+n+1}{n^4+1} \approx \frac{1}{n}$. Doing a limit comparison test on $\sum_{n=0}^\infty \frac{n^3+n+1}{n^4+1}$ with $\sum_{n=1}^\infty \frac{1}{n}$ will show it diverges, and so the integral diverges as well.

The proof of the integral test gives the following upper and lower bounds of the sum:

Theorem 1.4.5 (Integral test estimate). *Suppose that* $\sum_{n=0}^{\infty} a_n$ *is a convergent series satisfying the conditions of the integral test with* a_n *monotonically decreasing for* $n \geq N$. *Then* $\int_N^{\infty} f(x)\,dx \leq \sum_{n=N}^{\infty} a_n \leq a_N + \int_N^{\infty} f(x)\,dx$.

Example 1.4.6. Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent series satisfying the conditions of the integral test, we have $1 = \int_1^{\infty} \frac{1}{x^2}\,dx \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \leq 1 + \int_1^{\infty} \frac{1}{x^2}\,dx = 2$. The actual value of the sum was shown by Euler to be $\frac{\pi^2}{6} \approx 1.645$!

1.5 The Root and Ratio Tests

We now move on to series tests that are applicable to terms with negative terms. So far, none of our convergence tests have been "easy", in the sense that given a series, we can't just test if it converges or diverges by itself. We fix this with the root and ratio tests, which are arguably the easiest to use convergence tests. Before we do that, we need some terminology that applies to series that have negative terms.

Definition 1.5.1. An infinite series $\sum a_n$ **converges absolutely** if $\sum |a_n|$ converges. If $\sum |a_n|$ diverges and $\sum a_n$ converges, then we say $\sum a_n$ **converges conditionally**.

Absolute convergence is a "stronger" form of convergence, in the following sense:

Theorem 1.5.2 (Absolute convergence test). *If* $\sum |a_n|$ *converges, then* $\sum a_n$ *converges. That is, an absolutely convergent series converges.*

Example 1.5.3. We'll see later that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges while $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is a conditionally convergent series.

Example 1.5.4. The series $\sum_{n=1}^{\infty} \frac{\cos(n)}{2^n}$ converges absolutely. Taking absolute values, since $|\cos(n)| \leq 1$, we have $\sum_{n=1}^{\infty} \frac{|\cos(n)|}{2^n} \leq \sum_{n=1}^{\infty} \frac{1}{2^n}$ which is a convergent geometric series.

We now state the ratio and root tests:

Theorem 1.5.5 (Ratio test). *Let* $\sum a_n$ *be an infinite series. Set* $L = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$.

(a) If $0 \leq L < 1$, *then* $\sum a_n$ *converges absolutely.*

(b) If $L > 1$, *then* $\sum a_n$ *diverges.*

(c) if $L = 1$, *the ratio test says nothing.*

Theorem 1.5.6 (Root test). *Let* $\sum a_n$ *be an infinite series. Set* $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$.

(a) If $0 \leq L < 1$, *then* $\sum a_n$ *converges absolutely.*

(b) If $L > 1$, *then* $\sum a_n$ *diverges.*

(c) if $L = 1$, *the root test says nothing.*

Both the root and ratio test require only a single infinite series to perform the test, and further more, when they work they tell you explicitly whether or not the series converges (even absolutely!) or diverges. The drawback is that they don't always work. Of the two, the ratio test is more useful in practice. In fact, the root test is really only helpful when there are expressions raised to n-th powers, and *don't* involve any factorials. The ratio test is significantly more useful when factorials appear (more than any other convergence test), and handles n-th powers relatively easily as well. If you're trying to determine if an infinite series converges or not, and it's not of a special form (i.e. alternating, geometric, telescoping), and there's nothing to obviously do a limit comparison with, I recommend trying the ratio test.

One last thing that's worth pointing out: it's not a coincidence that the two tests look very similar. The root test is actually *stronger* than the ratio test, in the sense that you prove the ratio test by using the root test. If the ratio test works, you could have also done the root test. Sometimes if the ratio test doesn't work, the root test will work. If the root test *doesn't* work, don't bother with the ratio test: it won't work either!

Example 1.5.7. The series $\sum_{n=1}^{\infty} \frac{2n}{n^n}$ converges. Set $a_n = \frac{2n}{n^n}$. Then $\sqrt[n]{|a_n|} = \frac{(2n)^{1/n}}{n}$. As $n \to \infty$, we see that $(2n)^{1/n} \to 1$: this is because if $L = \lim_{n\to\infty}(2n)^{1/n}$, then $\ln(L) = \lim_{n\to\infty} \frac{\ln(2n)}{n} = 0$ by L'Hopital's rule, so $L = 1$. This says $\sqrt[n]{|a_n|} \to 0$ as $n \to \infty$. Convergence then follows from the root test.

Example 1.5.8. The series $\sum_{n=1}^{\infty}(1+\frac{1}{n})^{-n^2}$ converges. Set $a_n = (1+\frac{1}{n})^{-n^2}$, then $\sqrt[n]{|a_n|} = (1+\frac{1}{n})^{-n}$. As $n \to \infty$, we see $\lim_{n\to\infty} \frac{1}{(1+1/n)^n} = \frac{1}{e} < 1$, so the convergence follows from the root test.

Example 1.5.9. The series $\sum_{n=1}^{\infty} \frac{n!}{(2n)!}$ converges. Set $a_n = \frac{n!}{(2n)!}$. Then $|\frac{a_{n+1}}{a_n}| = \frac{(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{n!} = \frac{n+1}{(2n+2)(2n+1)} \to 0 < 1$ as $n \to \infty$. The convergence then follows by the ratio test.

Example 1.5.10. The series $\sum_{n=1}^{\infty} \frac{2^{n^2}}{n!}$ diverges. Set $a_n = \frac{2^{n^2}}{n!}$, then $|\frac{a_{n+1}}{a_n}| = \frac{2^{n^2+2n+1}}{(n+1)!} \cdot \frac{n!}{2^{n^2}} = \frac{2^{2n+1}}{n+1} \to \infty$. Therefore the series diverges by the ratio test.

Example 1.5.11. The series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges. Set $a_n = \frac{n!}{n^n}$. Then $|\frac{a_{n+1}}{a_n}| = \frac{(n+1)!n^n}{(n+1)^{n+1}n!} = \frac{(n+1)n^n}{(n+1)(n+1)^n} = (\frac{n}{n+1})^n = (1+\frac{1}{n})^{-n}$. As $n \to \infty$, we have $(1+\frac{1}{n})^{-n} \to \frac{1}{e}$, so the result follows by the ratio test.

1.6 Alternating Series

Our last type of series we study is when the negative terms are predictable, specifically, when the terms of the series alternate between positive and negative.

Definition 1.6.1. An **alternating series** is an infinite series of the form $\sum(-1)^n a_n$ where $a_n \geq 0$.

Because of the alternation between positive and negative terms, this makes it harder for the sum to diverge to infinity, so in some sense, alternating series are more "well behaved". One way of phrasing this is as follows:

Theorem 1.6.2 (Alternating series test). *Let $\sum(-1)^n a_n$ be an alternating series. If $\lim_{n\to\infty} a_n = 0$ and a_n is monotonically decreasing, then $\sum(-1)^n a_n$ converges.*

The first condition of the alternating series test is a requirement for the series to even converge in the first place (otherwise it diverges by the divergence test), so really the only condition we are imposing on the terms is that they are strictly decreasing, which is a relatively tame condition as far as "niceness" of sequences go. The alternating series test has one weakness: it cannot show an alternating series diverges. In fact, the conditions of the alternating series test say that a divergent alternating series has to be fairly complicated (provided it doesn't obviously diverge, i.e. has $\lim_{n\to\infty} a_n \neq 0$).

Example 1.6.3. The series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges conditionally: we know $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, but clearly $\frac{1}{n}$ goes to 0 and is monotonically decreasing, and so by the alternating series test, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges conditionally.

Example 1.6.4. The series $\sum_{n=1}^{\infty}(-1)^n \frac{1}{1+\frac{1}{n}}$ diverges because $\lim_{n\to\infty}(-1)^n \frac{1}{1+\frac{1}{n}}$ does not exist – not every alternating series requires the alternating series test!

Example 1.6.5. The series $\sum_{n=2}^{\infty}(-1)^n \frac{1}{n^2 \ln(n)}$ converges. With $a_n = \frac{1}{n^2\ln(n)}$, it's clear that $\lim_{n\to\infty} a_n = 0$, and we see that a_n is decreasing because the function $f(n) = \frac{1}{n^2\ln(n)}$ has derivative $f'(n) = -\frac{2\ln(n)+1}{n^3\ln(n)^2} < 0$ for $n \geq 1$. The result follows by the alternating series test. In fact, the convergence is absolute: the series $\sum_{n=2}^{\infty} \frac{1}{n^2\ln(n)}$ converges because $n^2\ln(n) > n^2$ for $n \geq 3$, so $\sum_{n=3}^{\infty} \frac{1}{n^2\ln(n)} < \sum_{n=3}^{\infty} \frac{1}{n^2}$ which is a convergent p-series.

Example 1.6.6. The series $\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n^{2/3}}$ converges conditionally. Notice that $\cos(\pi n) = (-1)^n$, so this is really just the series $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^{2/3}}$. It's clear that $\frac{1}{n^{2/3}} \to 0$ as $n \to \infty$, and the function $f(n) = n^{-2/3}$ has derivative $f'(n) = -\frac{2}{3}n^{-5/3} < 0$, so it converges by the alternating series test. However, taking an absolute value, the series $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ is a divergent p-series.

Example 1.6.7. The series $\sum_{n=1}^{\infty}(-1)^n \frac{e^{1/n}}{n}$ converges conditionally. As $n \to \infty$, $\frac{1}{n} \to 0$, and $e^x \approx 1$ for $x \approx 0$. This says as $n \to \infty$, that $\frac{e^{1/n}}{n} \approx \frac{1}{n}$. First, we show that we do not converge absolutely: the series $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n}$ should behave like the series $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges. Indeed, using the limit comparison test with $a_n = \frac{e^{1/n}}{n}$ and $b_n = \frac{1}{n}$, we have $\frac{a_n}{b_n} = e^{1/n}$ and clearly $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$. By the limit comparison test, $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n}$ diverges. The alternating series however, converges. With $a_n = \frac{e^{1/n}}{n}$, it's clear that $a_n \geq 0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{e^{1/n}}{n} = 0$. Set $f(n) = \frac{e^{1/n}}{n}$. Then $f'(n) = -\frac{e^{1/n}(n+1)}{n^3} < 0$. This says a_n is decreasing, so by the alternating series test, we are done.

Example 1.6.8. The series $\sum_{n=1}^{\infty}(-1)^{n+1} \frac{10^n}{2^{n^2}}$ converges absolutely. To see this, use the root test: $|a_n|^{1/n} = \frac{10}{2^n}$ and clearly $|a_n|^{1/n} \to 0$ as $n \to \infty$. The alternating series test isn't the only thing you should try when you see alternating series!

Example 1.6.9. We give an example of a (non-obvious) divergent alternating series. Define

$$a_n = \begin{cases} 1/n & n \text{ is even} \\ 1/n^2 & n \text{ is odd} \end{cases}$$

I claim the series $\sum_{n=1}^{\infty}(-1)^n a_n$ diverges. We show this via the definition of convergence for an infinite series. The N-th partial sum is given by $S_N = \sum_{n=1}^{N}(-1)^n a_n = -\sum_{k\leq N,\text{ k odd}} \frac{1}{k^2} + \sum_{k\leq N,\text{ k even}} \frac{1}{k}$. As $N \to \infty$, the first sum converges, because $\sum_{k \text{ odd}} \frac{1}{k^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, while the second sum diverges, because $\sum_{k \text{ even}} \frac{1}{k} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$ is the harmonic series. Since the partial sums diverge, the series $\sum_{n=1}^{\infty}(-1)^n a_n$ diverges.

Alternating series have a very nice error bound, that make it very easy to estimate these types of sums.

Theorem 1.6.10 (Alternating series error bound)**.** *Let $\sum_{n=0}^{\infty}(-1)^n a_n$ be a convergent alternating series, and let S denote the value of the sum. Then $|S - S_N| \leq a_{N+1}$.*

Example 1.6.11. We saw that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges. For any value of N, $\sum_{n=1}^{N} \frac{(-1)^n}{n}$ approximates the true value of the sum within an error of $\frac{1}{N+1}$. To guarantee two decimal places of accuracy, we can take $N = 99$, for example, and one may compute with a computer that $\sum_{n=1}^{99} \frac{(-1)^n}{n} \approx -.698$, so that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \approx -.69$. We'll see later that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -\ln(2) \approx -.693147$ – the series converges very slowly!

1.7 Summary of Tests

Test	Applicable Series	Conclusion	Additional
Divergence	$\sum a_n$	Diverges if $\lim_{n\to\infty} a_n \neq 0$	Always try this first. Inconclusive if $\lim_{n\to\infty} a_n = 0$. Can **not** show convergence!!
Geometric Series	$\sum_{n=M}^{\infty} cr^n$	Converges if $\|r\| < 1$, diverges if $\|r\| \geq 1$	Converges to value $\frac{cr^M}{1-r}$
Direct Comparison	$\sum a_n$ and $\sum b_n$ with $0 \leq a_n \leq b_n$ eventually	If $\sum b_n$ converges, then $\sum a_n$ converges If $\sum a_n$ diverges, then $\sum b_n$ diverges	
Limit Comparison	$\sum a_n$ and $\sum b_n$ with $0 < a_n, b_n$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, $0 < L < \infty$	$\sum a_n$ and $\sum b_n$ both converge or diverge	
Integral	$\sum a_n$ with $a_n = f(n)$ continuous, positive, decreasing eventually for $n \geq M$	$\sum a_n$ and $\int_M^\infty f(x)\,dx$ both converge or diverge	$\|S - S_N\| \leq \int_N^\infty f(x)\,dx$
p-Series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	Converges if $p > 1$, diverges if $p \leq 1$	
Absolute Convergence	$\sum a_n$	If $\sum \|a_n\|$ converges, $\sum a_n$ converges absolutely	If $\sum a_n$ converges but $\sum \|a_n\|$ diverges, we call this conditional convergence
Ratio	$\sum a_n$ with $a_n \neq 0$ and $\lim_{n\to\infty} \left\|\frac{a_{n+1}}{a_n}\right\| = L$	Converges (absolutely) if $L < 1$, diverges if $L > 1$	Inconclusive if $L = 1$
Root	$\sum a_n$ with $\lim_{n\to\infty} \sqrt[n]{\|a_n\|} = L$	Converges (absolutely) if $L < 1$, diverges if $L > 1$	Inconclusive if $L = 1$
Alternating Series	$\sum (-1)^n a_n$ with a_n positive, monotonically decreasing eventually, and $\lim_{n\to\infty} a_n = 0$	$\sum (-1)^n a_n$ converges	$\|S - S_N\| \leq a_{N+1}$

Chapter 2

Power Series

Our study of when infinite series converge leads to the following question: when can a function be written as an infinite series? To motivate why one would even want to do such a thing, at its heart, calculus is about *approximation*. One of the questions calculus tries to answer is how to find "good" approximations to a function $f(x)$ locally near some point x_0. You are already familiar with one such technique: "linearize" the function by finding the tangent line $L(x)$ at x_0, and then for values of x close to x_0, $L(x)$ is a "good" approximation to $f(x)$. $L(x)$ is an approximation to $f(x)$ by a polynomial of degree 1. What if we wanted an approximation to $f(x)$ by a polynomial of degree 2, or arbitrary degree n?

Since calculus is concerned with limit operations, one might ask if we could make sense of an "infinite degree" polynomial. If so, it's then natural to ask whether or not the above approximations actually become equalities, which is precisely asking when can you write a function as an infinite series! Our goal is to answer this question, and see the many powerful application that this knowledge gives us with regards to classical problems in calculus.

2.1 Basic Definitions

Definition 2.1.1. A **power series** is an infinite series of the form $F(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ for variable x, some sequence $\{a_n\}$, and some real number c, called the **center** of the power series.

Note that a power series $F(x)$ is not necessarily a well-defined function: for some values of x, the resulting series $F(x)$ may either converge or diverge.

Example 2.1.2. Let $F(x) = \sum_{n=0}^{\infty} x^n$, which is a power series centered at $c = 0$ with constant coefficients $a_n = 1$. For each fixed value of x, the resulting infinite series is a geometric series, and therefore converges if $|x| < 1$ and diverges if $|x| \geq 1$. Therefore we cannot make sense of $F(x)$ as a function defined on $\mathbb{R}$, but we *can* make sense of $F(x)$ as a function defined on $(-1, 1)$: from the formula for the sum of a geometric series, we know that for $|x| < 1$, $F(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$.

The first question we must answer then if we wish to make sense of power series, is when can we determine a domain that makes a power series a well-defined function? Since for each fixed value of x a power series is just an infinite series, we can answer this using the theory we've already developed. A bit of work will show that power series have the following behavior:

Theorem 2.1.3 (Convergence of power series). *For a power series $F(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$, exactly one of the following is true:*

- *There is a unique non-negative real number R such that $F(x)$ converges absolutely for $|x-c| < R$ and diverges for $|x-c| > R$.*
- *$F(x)$ converges absolutely for all $x \in \mathbb{R}$.*

Definition 2.1.4. The **radius of convergence** R of a power series $F(x)$ is defined as the number R in the above theorem. If $F(x)$ converges absolutely for all x, we define $R = \infty$. The **interval of convergence** is the set of all values such that $F(x)$ converges.

How can we find the radius of convergence of a power series? Assuming we can apply the ratio test to the infinite series $\sum_{n=0}^{\infty} a_n(x-c)^n$, we see that the series converges absolutely if $\lim_{n\to\infty} |\frac{a_{n+1}(x-c)^{n+1}}{a_n(x-c)^n}| = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}||x-c| < 1$, and diverges if $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}||x-c| > 1$. If $L = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$ is finite and non-zero, this says that the infinite series $F(x)$ converges absolutely if $L|x-c| < 1$ and diverges if $L|x-c| > 1$, i.e. converges absolutely for $|x-c| < 1/L$ and diverges if $|x-c| > 1/L$, so that the above theorem says $R = 1/L$. If $L = 0$, then $L|x-c| = 0$ for any value of x, so therefore $F(x)$ converges absolutely for any such choice of x, which says $R = \infty$. If L is infinite, then for any value of $x \neq c$, $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}||x-c| = \infty$, so $F(x)$ diverges, and for $x = c$ we see $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}||x-c| = 0$, i.e. $F(x)$ converges only at $x = c$, so $R = 0$. We can sum this up in the following:

Theorem 2.1.5. *Assume that $L = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$ exists. The radius of convergence of the power series $F(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ is given by $R = \frac{1}{L}$, where this is interpreted as $R = 0$ if $L = \infty$ or $R = \infty$ if $L = 0$.*

The theorem on the convergence behavior of power series tells us that if $R < \infty$, a power series must converge in the interval $(c-R, c+R)$, and diverges in $(-\infty, c-R) \cup (c+R, \infty)$. However, the theorem tells us nothing about what happens at the endpoints $x = c-R$ and $x = c+R$. To check if a power series converges for these values of x, this must be done manually using the usual convergence tests for infinite series.

Example 2.1.6. In the previous example, we determined the power series $F(x) = \sum_{n=0}^{\infty} x^n$ converges if $|x| < 1$ and diverges if $|x| \geq 1$ using properties of geometric series. In other words, the radius of convergence is $R = 1$ and the interval of convergence is $(-1, 1)$. We can also determine this using the ratio test: $F(x)$ converges absolutely if $\lim_{n\to\infty} |\frac{x^{n+1}}{x^n}| = |x| < 1$ and diverges if $|x| > 1$, so $R = 1$. If $x = 1$, then $F(1) = \sum_{n=0}^{\infty} 1$ diverges, and similarly $F(-1) = \sum_{n=0}^{\infty} (-1)^n$ also diverges, so the interval of convergence is $(-1, 1)$.

Example 2.1.7. Set $F(x) = \sum_{n=0}^{\infty} n!x^n$. What is the interval of convergence of $F(x)$? Using the ratio test, we find that $L = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty}(n+1) = \infty$. This says $R = 0$, and so $F(x)$ converges only at $x = 0$.

Example 2.1.8. Set $F(x) = \sum_{n=2}^{\infty} \frac{1}{\ln(n)}(x-1)^n$. What is the interval of convergence of $F(x)$? Using the ratio test, $F(x)$ converges absolutely if $\lim_{n\to\infty} |\frac{\frac{1}{\ln(n+1)}(x-1)^{n+1}}{\frac{1}{\ln(n)}(x-1)^n}| = \lim_{n\to\infty} \frac{\ln(n)}{\ln(n+1)}|x-1| < 1$ and diverges if $\lim_{n\to\infty} \frac{\ln(n)}{\ln(n+1)}|x-1| > 1$. Since $\lim_{n\to\infty} \frac{\ln(n)}{\ln(n+1)} = 1$, this says $F(x)$ converges absolutely if $|x-1| < 1$ and diverges if $|x-1| > 1$, i.e. $R = 1$. We then see that $F(x)$ converges in the interval $(0, 2)$. What happens at the endpoints? At $x = 0$, we have $F(0) = \sum_{n=2}^{\infty} \frac{1}{\ln(n)}(-1)^n$, which converges by the alternating series test. At $x = 2$, $F(1) = \sum_{n=2}^{\infty} \frac{1}{\ln(n)}$. Since $\ln(n) < n$ for all n, $\frac{1}{n} < \frac{1}{\ln(n)}$ so that $\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$ diverges by a direct comparison. Therefore $F(1)$ diverges, and the interval of convergence of $F(x)$ is given by $[0, 2)$.

Example 2.1.9. Set $F(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^n (n!)^2}$. In both of the two previous examples, we could have computed the radius of convergence by using the previous theorem. However here the theorem does not apply, because the power series $F(x)$ has only *even* powers of x in the sum. Therefore we need to use the ratio test to determine the radius of convergence. With $b_n = \frac{(-1)^n x^{2n}}{4^n (n!)^2}$, we have $|\frac{b_{n+1}}{b_n}| = |\frac{(-1)^{n+1} x^{2n+2}}{4^{n+1}((n+1)!)^2} \cdot \frac{4^n (n!)^2}{(-1)^n x^{2n}}| = \frac{x^2}{4(n+1)^2}$. Therefore, $\lim_{n\to\infty} |\frac{b_{n+1}}{b_n}| = \lim_{n\to\infty} \frac{x^2}{4(n+1)^2} = 0$ for any value of x. This says $R = \infty$ and $F(x)$ converges absolutely for all x.

2.2 Functions Defined by Power Series

The convergence behavior of power series says that a power series $F(x)$ determines a well-defined function on its interval of convergence. One reason why we care about power series is that doing calculus with them is extremely easy:

Theorem 2.2.1 (Integration and differentiation of power series)**.** *Let $F(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ be a power series with radius of convergence R. Then for $|x-c| < R$, we may differentiate and integrate the power series $F(x)$ term by term. That is, the following hold:*

- $F'(x) = \sum_{n=0}^{\infty} \frac{d}{dx} a_n(x-c)^n = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1}$
- $\int F(x)\,dx = \sum_{n=0}^{\infty} \int a_n(x-c)^n\,dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-c)^{n+1}$.

Furthermore, the radius of convergence remains unchanged, but the interval of convergence of these new series may differ at the endpoints.

Example 2.2.2. Let $F(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$. Then using the theorem for computing the radius of convergence, we see that $R = 1$ and $F(x)$ converges in $(-1, 1)$. If $x = 1$, $F(1) = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges, and at $x = -1$, $F(-1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is a convergent alternating series. Therefore, $F(x)$ has interval of convergence $[-1, 1)$. Taking a derivative says $F'(x) = \sum_{n=1}^{\infty} \frac{d}{dx}\frac{x^n}{n} = \sum_{n=1}^{\infty} x^{n-1} = \sum_{n=0}^{\infty} x^n$. As previously determined, this power series has radius of convergence 1 and interval of convergence $(-1, 1)$. If we integrate $F(x)$, we see $\int F(x)\,dx = \sum_{n=1}^{\infty} \int \frac{x^n}{n}\,dx = \sum_{n=1}^{\infty} \frac{1}{n}\int x^n\,dx = C + \sum_{n=1}^{\infty} \frac{1}{n(n+1)}x^{n+1}$. This series has radius of convergence $R = 1$, and so converges in the interval $(-1, 1)$. At $x = -1$, the series $C + \sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)}$ is a convergent alternating series, and at $x = 1$ the series $C + \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges by a limit comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$. This says the interval of convergence is $[-1, 1]$.

The above theorem tells us that functions defined by a power series are very special: they are not only differentiable, but are *infinitely* differentiable (the derivative of a power series is a power series so you can keep applying the theorem!), and they behave nicely with respect to the operations of differentiation and integration. Naturally then, is the following: give a function $f(x)$, how can we determine if it can be defined by a power series on some interval?

2.3 Taylor Series

Suppose we have a function $f(x)$ defined on some interval I that can be written as a power series. That is to say, $f : I \to \mathbb{R}$ is defined by $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ for some power series $\sum_{n=0}^{\infty} a_n(x-c)^n$. It turns out the coefficients a_n of the power series can very easily be determined. Plugging in $x = c$, all terms on the right hand side disappear except the $n = 0$ term, so $f(c) = a_0$. Since f is represented by a power series, it is differentiable, so we may write $f'(x) = \sum_{n=1}^{\infty} na_n(x-c)^{n-1}$. Plugging in $x = c$, all terms in the right hand side disappear except the $n = 1$ term, which says $f'(c) = a_1$. Differentiating again says $f''(x) = \sum_{n=2}^{\infty} n(n-1)a_n(x-c)^{n-2}$. Plugging in $x = c$, all terms in the right hand side disappear except the $n = 2$ term, so $f''(c) = 2a_2$ says $a_2 = \frac{f''(c)}{2}$. Continuing this process, one finds that $a_n = \frac{f^{(n)}(c)}{n!}$, so that $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$. This says that if a function $f(x)$ can be written as a power series, it necessarily has this special form.

Definition 2.3.1. Let $f(x)$ be an infinitely differentiable function. The **Taylor series** of $f(x)$ centered at c, denoted $T(x)$, is the power series $T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$. If $c = 0$, the power series is sometimes called the **MacLaurin series** of $f(x)$.

Example 2.3.2. What's the Taylor series of $f(x) = e^x$ centered at $c = 0$? By definition, this Taylor series is given by $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$, so we need to figure out what an arbitrary n-th order derivative of f looks like. Luckily, $f^{(n)}(x) = e^x$ for all n, so $f^{(n)}(0) = 1$. This says the Taylor series centered at 0 of e^x is given by $\sum_{n=0}^{\infty} \frac{1}{n!}x^n$.

Example 2.3.3. We have seen that a valid power series expansion of $f(x) = \frac{1}{1-x}$ when $|x| < 1$. By the uniqueness of a power series representation, this actually says that the Taylor series centered at $c = 0$ of $f(x)$ is given by $\sum_{n=0}^{\infty} x^n$, valid for $|x| < 1$. That is, $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$.

Example 2.3.4. What's the Taylor series centered at $c = 0$ for $f(x) = \sin(x)$? Derivatives of $\sin(x)$ have a simple pattern: they cycle $\cos(x), -\sin(x), -\cos(x), \sin(x)$. If we plug in $x = 0$, the pattern goes $1, 0, -1, 0$, i.e. the even order derivatives at 0 are all 0 and the odd order derivatives at 0 alternate between 1 and -1. Then $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n \text{ odd}} \frac{f^{(n)}(0)}{n!}x^n$. We can loop the sum over all odd integers by writing $n = 2k+1$ and then letting k vary from 0 to ∞, i.e. $\sum_{n \text{ odd}} \frac{f^{(n)}(0)}{n!}x^n = \sum_{k=0}^{\infty} \frac{f^{(2k+1)}(0)}{(2k+1)!}x^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}x^{2k+1}$.

What we determined at the beginning of the section is that *if* a function can be written as a power series centered at some point c, that power series *must* be its Taylor series. We have **not** said that a function is equal to its Taylor series. Indeed, this is false:

Example 2.3.5. Consider the function f defined by $f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$. Through quite a lot of (difficult) work, one can show that this function has the following unusual properties: f is infinitely differentiable, and $f^{(n)}(0) = 0$ for all $n \geq 0$. The Taylor series of $f(x)$ centered at $c = 0$ is then given by $T(x) = 0$, which obviously is not the same as $f(x)$.

To finish off the section, we give some examples of how one goes about computing Taylor series. The take away from these examples should all be the same: to compute a Taylor series, perform operations on *known* power series to arrive at an answer. Do not try and work with the definition!

Example 2.3.6. Let's compute the Taylor series of $f(x) = \frac{x^2}{(1-x)^3}$ centered at $c = 0$. We start with the known Taylor series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, valid for $|x| < 1$. If we differentiate once, we find $\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$, and if we differentiate again we see $\frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1)x^{n-2}$. This says $\frac{1}{(1-x)^3} = \frac{1}{2}\sum_{n=2}^{\infty} n(n-1)x^{n-2}$, so multiplying by x^2 says $\frac{x^2}{(1-x)^3} = \frac{x^2}{2}\sum_{n=2}^{\infty} n(n-1)x^{n-2} = \frac{1}{2}\sum_{n=2}^{\infty} n(n-1)x^n$. This expansion is still valid for $|x| < 1$ (differentiating doesn't change the radius of convergence!). Since we have found a power series representing $f(x)$ in some interval, this *forces* it to be the Taylor series of $f(x)$ by uniqueness.

Example 2.3.7. Similarly, by integrating the Taylor series of $\frac{1}{1-x}$, we can find the Taylor series of $\ln(1-x)$. We have $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, valid for $|x| < 1$. Integrating says $\ln(1-x) = C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$. To figure out C, plug in $x = 0$: we then have $\ln(1) = C$, so $C = 0$. It's easy to see we pick up convergence at the end point $x = -1$, so we have $\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$ for $x \in [-1, 1)$. Since we have found a power series representation for our function, it must be its Taylor series.

Example 2.3.8. Let's compute the Taylor series of $f(x) = \frac{1}{1-x}$ centered at $c = 4$. We know that $\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n$ is a valid power series expansion when $|u| < 1$. Instead of directly computing derivatives, we can do a clever trick to find the Taylor series. We know that the Taylor series of $f(x)$ centered at 4 is of the form $\sum_{n=0}^{\infty} a_n(x-4)^n$ for some coefficients a_n, so if we can find such a power series, uniqueness forces it to be the Taylor series of $f(x)$. To do so, we will perform a substitution and use the above formula to make an $(x-4)^n$ term appear in the sum. Write $\frac{1}{1-x} = \frac{1}{1-(x-4+4)} = \frac{1}{-3-(x-4)} = -\frac{1}{3}\frac{1}{1-(-\frac{x-4}{3})}$. Set $u = -\frac{x-4}{3}$. Then the above says $\frac{1}{1-x} = -\frac{1}{3}\sum_{n=0}^{\infty}(-\frac{x-4}{3})^n = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{(x-4)^n}{3^{n+1}}$, which is valid for $|\frac{x-4}{3}| < 1$, i.e. $|x-4| < 3$. Since we have found a power series of f centered at 4, it must be its Taylor series.

Example 2.3.9. Let's find the Taylor series of $f(x) = \frac{2}{1-2x} - \frac{1}{1-x}$ centered at $c = 0$. Similarly to above, we use the expansion $\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n$ for $|u| < 1$. Writing each term as a power series, we have $\frac{2}{1-2x} - \frac{1}{1-x} = 2\sum_{n=0}^{\infty}(2x)^n - \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} 2^{n+1}x^n - \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty}(2^{n+1} - 1)x^n$, which is only valid when *both* series converge. Since the first series convergences only for $|x| < 1/2$, we see this power series expansion is only valid for $|x| < 1/2$. Since we have found a power series expansion of f that's valid in some interval around 0, this says it must be the Taylor series of f.

Example 2.3.10. Let's find the Taylor series of $f(x) = \int_0^x \frac{e^{t^2}-1}{t}\,dt$ centered at $c = 0$. Note that it is not at all possible to compute an anti-derivative of the integrand – the only method here is to integrate its Taylor series. The Taylor series of e^t centered at 0 is given by $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$ which is valid for all t, so $e^{t^2} = \sum_{n=0}^{\infty} \frac{(t^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{t^{2n}}{n!}$. This series looks like $1 + t^2 + \frac{t^4}{2} + \ldots$, so $e^{t^2} - 1 = t^2 + \frac{t^4}{2} + \ldots = \sum_{n=1}^{\infty} \frac{t^{2n}}{n!}$, which is a valid expansion for all t. Dividing through by t then says $\frac{e^{t^2}-1}{t} = \frac{1}{t}\sum_{n=1}^{\infty} \frac{t^{2n}}{n!} = \sum_{n=1}^{\infty} \frac{t^{2n-1}}{n!}$. We then have $\int_0^x \frac{e^{t^2}-1}{t}\,dt = \int_0^x \sum_{n=1}^{\infty} \frac{t^{2n-1}}{n!}\,dt = \sum_{n=1}^{\infty} \int_0^x \frac{t^{2n-1}}{n!}\,dt = \sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)n!}$, and further this expression is valid for all x. Since we have found a power series representation of $f(x)$, this must be its Taylor series.

Example 2.3.11. Let's find the Taylor series of $f(x) = \tan^{-1}(x)$ centered at $c = 0$. We know that $\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$, so let's start by finding the Taylor series of this function instead, which is much easier. Starting with $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, we have $\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$ by replacing x with $-x^2$. Integrating then says $\tan^{-1}(x) = C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$. Since $\tan^{-1}(0) = 0$, we find $C = 0$, and so $\tan^{-1}(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$. The radius of convergence is 1 because we did not do any operations to change it from our starting series. Testing the endpoints, the series converges at both $x = 1, -1$ by the alternating series test, and so the power series representation is valid on $[-1, 1]$. Since we have found a power series representation of $f(x)$, this must be its Taylor series.

2.4 Polynomial Approximations

Determining when a function is equal to its Taylor series is quite a subtle question. In fact, it's also quite hard: in general, there is not much we can say. At the very minimum however, we can say the following. For each fixed value of x, consider the n-th partial sum $T_n(x)$ of the Taylor series $T(x)$, that is $T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$. Saying that $f(x) = T(x)$ is the same as saying that $T_n(x) \to f(x)$ as $n \to \infty$. If we set $R_n(x) = f(x) - T_n(x)$, this says that if $f(x) = T(x)$, if and only if $R_n(x) \to 0$ as $n \to \infty$.

Theorem 2.4.1 (Representation by Taylor series)**.** *An infinitely differentiable function $f(x)$ can be written as a power series centered at c if and only if the n-th order remainder term $R_n(x) = f(x) - T_n(x)$ satisfies $\lim_{n\to\infty} R_n(x) = 0$ for all $x \in I$.*

To reiterate for emphasis, this theorem is saying very little: we merely translated the statement that $f(x) = T(x)$ into a statement about its partial sums via the definition of convergence of an infinite series. Explicitly computing the remainder term $R_n(x)$ is generally a hopeless task. Therefore if one wants to check using the above criterion that $f(x)$ can be represented by a power series, we need to come up with a way of estimating the remainder term $R_n(x)$ if we want to see if it tends to 0.

Definition 2.4.2. The n-th order **Taylor polynomial** of $f(x)$ centered at c is the n-th partial sum of its Taylor series, $T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$.

Example 2.4.3. Let's compute the 4-th order Taylor polynomial of $f(x) = xe^{x^2}$ centered at $c = 0$. By definition, this is given by $f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \frac{f^{(4)}(0)}{24}x^4$. One such approach is to just calculate all the relevant derivatives and plug in $x = 0$. It's an easy computation to check that $f'(x) = (2x^2+1)e^{x^2}$, $f''(x) = (4x^3+6x)e^{x^2}$, $f'''(x) = (8x^4+24x^2+6)e^{x^2}$ and $f^{(4)}(x) = (16x^5 + 80x^3 + 60x)e^{x^2}$. Plugging in 0 then gives $T_4(x) = x + x^3$. Another way we could have done this computation is as follows. $T_4(x)$ is the 4-th degree polynomial that comes from the Taylor series of $f(x)$, so if we compute it's Taylor series, we can chop off terms to get the Taylor polynomial. Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, we have $xe^{x^2} = x\sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!} = x + x^3 + \frac{x^5}{2} + \ldots$. We recover $T_4(x)$ by chopping off the sum at the degree 4 term (of which we see there is none), so $T_4(x) = x + x^3$.

This example illustrates several things: using Taylor series to compute Taylor polynomials is significantly faster, that the n-th degree Taylor polynomial doesn't even need to have degree n, and that two Taylor polynomials could be equal (here we have $T_3(x) = T_4(x)$).

Example 2.4.4. Let $f(x) = x^4 + \frac{1}{2}x^2 - 1$. Then the 4-th order Taylor polynomial of $f(x)$ centered at $c = 0$ is just $x^4 + \frac{1}{2}x^2 - 1$. This is because $f(x)$ is *already* a degree 4 polynomial.

The remainder term $R_n(x)$ tells you how far off from $f(x)$ the approximation $T_n(x)$ is. Determining how "good" of an approximation Taylor polynomials are is one of the major theorems of calculus!

Theorem 2.4.5 (Taylor's Theorem)**.** *Let f be a function such that $f^{(n+1)}(x)$ exists and is continuous. Suppose there is a constant K_{n+1} such that $|f^{(n+1)}(z)| \leq K_{n+1}$ for all z between x and c. Then $|R_n(x)| = |f(x) - T_n(x)| \leq \frac{K_{n+1}}{(n+1)!}|x - c|^{n+1}$.*

Taylor's inequality tells us that the size of the remainder term $R_n(x)$ depends on the size of the $(n+1)$-st derivative of f. This makes it more explicit *why* it's hard to show a function can be written as a power series: computing arbitrary order derivatives is generally not easy.

Example 2.4.6. Consider the Taylor expansion of e^x centered at $c = 0$. Since $\frac{d^n}{dx^n}e^x = e^x$ for all $n \geq 0$, Taylor's theorem says that for any value of x and any $n \geq 0$, we have $|R_n(x)| \leq \frac{e^{|x|}}{(n+1)!}|x|^{n+1}$ because the maximum value of e^x on the interval $[-x, x]$ happens at whichever endpoint is positive. Taking a limit as $n \to \infty$ then shows that $R_n(x) \to 0$, and so this means the Taylor series of e^x converges to e^x, so we get an equality $e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$. Some calculus books give this as the *definition* of the exponential function. In particular, we get the numerical identity $e = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \ldots$.

As a first application, we can handle one simple case of when a function can be written as a power series: let $T(x)$ be the Taylor series of $f(x)$ centered at c, and suppose that $T(x)$ has radius of convergence R. If there is some number K such that $|f^{(n)}(x)| \leq K$ for all n and all x such that $|x - c| < R$, then applying Taylor's inequality says that $|R_n(x)| \leq \frac{K}{(n+1)!}|x - c|^{n+1} \leq \frac{K}{(n+1)!}R^{n+1}$. Taking $n \to \infty$ says that $|R_n(x)| \to 0$, so that we have proved the following:

Theorem 2.4.7. *Let f be an infinitely differentiable function. Let $T(x)$ be the Taylor series of $f(x)$ centered at c with radius of convergence R. Suppose there is some constant K such that $|f^{(n)}(x)| \leq K$ for all n and all x such that $|x-c| < R$. Then $f(x) = T(x)$ for $|x-c| < R$. That is to say, such an f has a power series representation.*

Example 2.4.8. With $f(x) = \sin(x)$, we see that $|f^{(n)}(x)| \leq 1$ for all $n \geq 0$. The above theorem says that the Taylor series of $\sin(x)$ converges to $\sin(x)$, and so from our prior example we have an actual equality $\sin(x) = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!}$, valid for all x because the series has infinite radius of convergence. We can compute the Taylor series of $\cos(x)$ by taking the derivative of the Taylor series of $\sin(x)$. Doing so, we find $\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}x^{2k}$, which is again valid for all x, so we have the equality $\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}x^{2k}$.

We now give some examples of how the error bound inequality can be used to quantify how "good" a polynomial approximation is.

Example 2.4.9. Suppose $f(x) = e^{-x}$ and we have $T_3(x)$ centered at $c = 1$. How good of an approximation to $f(1.1)$ is $T_3(1.1)$? The error bound formula says $|f(1.1) - T_3(1.1)| \leq \frac{K_4}{4!}|.1|^4$, where K_4 is an upper bound of $|f^{(4)}(x)|$ on the interval $[1, 1.1]$. Since $|f^{(4)}(x)| = f^{(4)}(x) = e^{-x}$, K_4 is just an upper bound of e^{-x} on the interval $[1, 1.1]$. The function e^{-x} is strictly decreasing on this interval, so it attains it's maximal value on the interval at the left endpoint $x = 1$. This says the maximal value is given by $\frac{1}{e}$, so we can take $K_4 = \frac{1}{e}$. Although this is a perfectly valid choice of K_4, if the point is to do an approximation *by hand*, it's completely useless to choose a value of K_4 that would involve e, since that's another thing we have to approximate. Since $e \approx 2.718$, in particular we have $e \geq \frac{5}{2}$ so $\frac{1}{e} \leq \frac{2}{5}$. We will then instead take $K_4 = \frac{2}{5}$, the trade off being the error estimate will be a little bit worse, but computable by hand. Plugging into the error bound formula, this says $|f(1.1) - T_3(1.1)| \leq \frac{2/5}{24}(.1)^4 = \frac{1}{60 \cdot 10^4} \approx .000001$. This says if we want to estimate $e^{-1.1}$, that $T_3(1.1)$ is a very good estimate. Indeed, we can compute that $T_3(x)$ centered at $c = 1$ is given by $T_3(x) = \frac{1}{e} - \frac{1}{e}(x-1) + \frac{1}{2e}(x-1)^2 - \frac{1}{6e}(x-1)^3$, so that $T_3(1.1) = \frac{1}{e} - \frac{1}{10e} + \frac{1}{200e} - \frac{1}{6000e} = \frac{5429}{6000e}$. If we use the approximation $e \approx 2.718$, then $T_3(1.1) \approx .3329$, while $f(1.1) \approx .33287$ (we obviously lost a bit more precision by having to approximate e).

Example 2.4.10. Suppose $f(x) = xe^{x^2}$. Earlier, we computed that $T_3(x) = x + x^3$. How good of an approximation is this to $f(x)$? For an arbitrary value $x > 0$, the error bound formula says $|f(x) - T_3(x)| \leq \frac{K_4}{4!}x^4$, where K_4 is an upper bound of $|f^{(4)}(z)|$ on the interval $[0, x]$. We also computed that $g(z) = |f^{(4)}(z)| = f^{(4)}(z) = (16z^5 + 80z^3 + 60z)e^{z^2}$. By definition, K_4 is an upper bound of this function on the interval $[0, x]$. The function $g(z)$ is strictly increasing, because $g'(z) = (32z^6 + 240z^4 + 360z^2 + 60)e^{z^2} \geq 0$ when z is in the interval $[0, x]$. In particular, this says $g(z)$ attains it's maximal value at the right endpoint of this interval, i.e. at $z = x$. Therefore, we may choose $K_4 = (16x^5 + 80x^3 + 60x)e^{x^2}$. If we plug this in, this says $|f(x) - T_3(x)| \leq \frac{(16x^5+80x^3+60x)e^{x^2}}{24}x^4 = \frac{(16x^9+80x^7+60x^5)e^{x^2}}{24}$.

This says at worst, the error grows at the same rate as the function $\frac{(16x^9+80x^7+60x^5)e^{x^2}}{24}$. Since $\lim_{x\to 0} \frac{(16x^9+80x^7+60x^5)e^{x^2}}{24} = 0$ (and it goes to 0 quite quickly), for values of x close to 0, the approximation will be quite good. For example, using a calculator we find $|f(.1) -$

$T_3(.1)| \leq .00002559$, so $T_3(.1) = .101$ approximates $f(.1)$ to within 4 decimal places. Indeed, we see $f(.1) \approx .101005$. However, as $x \to \infty$, we have $\frac{(16x^9+80x^7+60x^5)e^{x^2}}{24} = \infty$, and moreover, this function is growing extremely quickly (faster than an exponential function!). This says for values of x far from 0, the error bound will be terrible. For example, at $x = 1$, we see $|f(1) - T_3(1)| \leq \frac{13e}{2} \approx 17.668$. This an absolutely useless estimate, because we knew $f(1) = e$ and $T_3(1) = 2$, so in actuality $|f(1) - T_3(1)| \approx .718$!

Example 2.4.11. Suppose we want to compute $\ln(1.1)$ to within 4 decimal places of accuracy. How can we do this? One such approach using what we have done so far is to figure out how many terms in the Taylor series of $\ln(1.1)$ are necessary in order for the N-th Taylor polynomial $T_N(x)$ to approximate to within that level of error by using the error bound formula. In order for the error bound formula to remain useful, we must make sure that we center $T_N(x)$ somewhere close to 1.1. One such approach is to center it at $c = 1$, so that phrased mathematically, we want to find N such that $|f(1.1) - T_N(1.1)| \leq \frac{1}{10^4}$, where $f(x) = \ln(x)$ and $T_N(x)$ is centered at $c = 1$.

The error bound formula says $|f(1.1) - T_N(1.1)| \leq \frac{K_{N+1}}{(N+1)!}(.1)^{N+1}$, so if we make this smaller than $\frac{1}{10^4}$, then we're good. In order to do this, we need to figure out how to pick K_{N+1}. By definition, K_{N+1} is an upper bound of $|f^{(N+1)}(x)|$ on the interval $[1, 1.1]$. First, we compute an arbitrary order derivative of $f(x)$. We have $f(x) = \ln(x)$, $f'(x) = x^{-1}$, $f''(x) = -x^{-2}$, $f'''(x) = 2x^{-3}$, $f^{(4)}(x) = -6x^{-4}$, and so on. Continuing the pattern, we see that $f^{(N)}(x) = (-1)^{N+1}(N-1)!x^{-N}$, so that $|f^{(N+1)}(x)| = N!x^{-(N+1)}$. In particular, this function is decreasing (because the derivative is always negative), so that it's maximal value happens at the left endpoint $x = 1$. Plugging this in, the maximum value of $N!x^{-(N+1)}$ is just $N!$, so we may take $K_{N+1} = N!$. We then need to solve the inequality $\frac{K_{N+1}}{(N+1)!}\frac{1}{10^{N+1}} \leq \frac{1}{10^4}$. Plugging in our choice of K_{N+1}, this is the same thing as solving $\frac{1}{(N+1)10^{N+1}} \leq \frac{1}{10^4}$, i.e. $10^4 \leq (N+1)10^{N+1}$. We see that $N = 3$ is the smallest such choice of N that works, so we only need to take 3 terms in the Taylor series to get the desired level of accuracy.

If we wanted to then approximate $\ln(1.1)$, we can then easily compute $T_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$, so that $T_3(1.1) = \frac{1}{10} - \frac{1}{200} + \frac{1}{3000} = \frac{143}{1500} \approx .09533$, and indeed, this is a good approximation to $\ln(1.1) \approx .09531$.

2.5 Applications of Taylor Series

Taylor series are the ultimate tool of calculus – they can be used to answer almost all classical calculus problems you might be interested in solving. In particular, we will see how Taylor series can be used to do the following:

- Compute limits.
- Compute derivatives at a point.
- Compute the value of an infinite series.

- Approximate the value of a definite integral when the integrand does not have an anti-derivative we can write down.

- Analyze the growth rate of functions, making it easier to apply the limit comparison test.

Example 2.5.1. Suppose we want to compute $\lim_{x\to 0} \frac{\sin(x^4)-x^4}{(x^4-\frac{1}{6}x^3)^4}$. If you try and use L'Hopital's rule, you'll very quickly convince yourself that it will be extremely difficult. How else can we compute this limit? One approach is replace the numerator with it's Taylor series, and do the resulting limit computation. We have $\sin(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$, so $\sin(x^4) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{8n+4}}{(2n+1)!} = x^4 - \frac{1}{6}x^{12} + \ldots$, thus $\sin(x^4) - x^4 = \sum_{n=1}^{\infty}(-1)^n \frac{x^{8n+4}}{(2n+1)!} = -\frac{1}{6}x^{12} + \frac{1}{120}x^{20} + \ldots$, so that $\lim_{x\to 0} \frac{\sin(x^4)-x^4}{(x-\frac{1}{6}x^3)^4} = \lim_{x\to 0} \frac{-\frac{1}{6}x^{12}+\frac{1}{120}x^{20}+\ldots}{(x^4-\frac{1}{6}x^3)^4}$. Dividing the numerator and denominator through by x^{12}, we find $\lim_{x\to 0} \frac{-\frac{1}{6}x^{12}+\frac{1}{120}x^{20}+\ldots}{(x^4-\frac{1}{6}x^3)^4} = \lim_{x\to 0} \frac{-\frac{1}{6}+\frac{1}{120}x^8+\ldots}{(x-\frac{1}{6})^4} = -6^3 = -216$.

Example 2.5.2. Let $f(x) = e^{x^2}$. Suppose we wanted to calculate the 1000-th derivative of f at 0, $f^{(1000)}(0)$. It's obviously impossible to calculate 1000 derivatives by hand, and no computer will be able to calculate the derivative explicitly. How can we do this? The easiest way is to compute the Taylor series of $f(x)$ centered at 0, which encodes information about all derivatives of f at 0. We have $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, so that $e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$. In particular, the definition of the Taylor series says $e^{x^2} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$, so to recover the value of $f^{(1000)}(0)$, we need to look at the coefficient of x^{1000} in the Taylor series of $f(x)$. We see that the coefficient is just $\frac{1}{500!}$, so by comparing coefficients in these two series we find $\frac{1}{500!} = \frac{f^{(1000)}(0)}{1000!}$, which says $f^{(1000)}(0) = \frac{1000!}{500!}$.

Example 2.5.3. The series $\sum_{n=0}^{\infty} \frac{2n+1}{4^n}$ converges by doing a direct comparison test with a geometric series. As it turns out, we can actually compute the value of this sum. The way we do this is as follows: write $\sum_{n=0}^{\infty} \frac{2n+1}{4^n} = \sum_{n=0}^{\infty}(2n+1)(\frac{1}{4})^n$. Then the value of this sum is $f(1/4)$, where $f(x) = \sum_{n=0}^{\infty}(2n+1)x^n$. If we can find a function whose Taylor series centered at 0 is equal to $f(x)$, then we can find the exact value of the series. It's a quick check to see that $f(x)$ converges for $|x| < 1$, so we can write $f(x) = 2\sum_{n=0}^{\infty} nx^n + \sum_{n=0}^{\infty} x^n$ for such values. We know that the second series is the Taylor series of $\frac{1}{1-x}$. What about the first series? The key observation is that this series is "almost" what you would get if you differentiated the second series. In fact, the only difference is a factor of x, so we have $\sum_{n=0}^{\infty} nx^n = x\frac{d}{dx}\sum_{n=0}^{\infty} x^n = x\frac{d}{dx}\frac{1}{1-x} = \frac{x}{(1-x)^2}$. This tells us that $f(x) = \frac{2x}{(1-x)^2} + \frac{1}{1-x}$, for $|x| < 1$. Plugging in $x = \frac{1}{4}$ then says $f(1/4) = \sum_{n=0}^{\infty} \frac{2n+1}{4^n} = \frac{20}{9}$.

Example 2.5.4. At some point in your life, someone has probably mentioned that the function $f(x) = e^{-x^2}$ does not have an anti-derivative that you can write down. However, functions that look like this are of fundamental importance in fields relating to mathematics, for example, $f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ is what is known as the "standard normal distribution" in statistics. If you have taken such a course before, then you know that computing definite integrals of this function is extremely important. How can we do it? As an example, we'll approximate $\int_{-1}^{1} \frac{1}{\sqrt{2\pi}}e^{-x^2/2}\,dx$. A statistical interpretation of this integral is that if you have a population that is normally distributed, this integral computes the proportion of the population that lies within 1 standard deviation from the mean.

First, we'll find a Taylor series of the integrand centered at 0. This is easy to do: we know $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, so $e^{-x^2/2} = \sum_{n=0}^{\infty} \frac{(-x^2/2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^n n!}$. Integrating this series then says $\int_{-1}^{1} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^n n!}\, dx = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \int_{-1}^{1} (-1)^n \frac{x^{2n}}{2^n n!}\, dx = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{2}{(2n+1)2^n n!}$. In order to approximate the sum, we can use the remainder estimate from the alternating series test. First, let's pick a desired level of accuracy, say, 4 decimal places. If we let S denote the value of the series, then the alternating series test says $|S - S_N| \leq a_{N+1}$. Then we would like to find N such that $a_{N+1} \leq \frac{1}{10^4}$, that is, solve $\frac{2}{(2N+3)2^{N+1}(N+1)!} \leq \frac{1}{10^4}$, which is equivalent to solving $20000 \leq (2N+3)2^{N+1}(N+1)!$. We see that $N = 4$ is the smallest value of N that works, so we arrive at the desired accuracy by computing S_4. Using a calculator to do so, we then conclude that $\int_{-1}^{1} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx \approx .6827$.

Example 2.5.5. Suppose we wanted to know if $\sum_{n=1}^{\infty} (1/n - \tan^{-1}(1/n))$ converges or diverges. In order to come to a conclusion, we need to understand how the summand grows as $n \to \infty$. The 3rd order Taylor polynomial of $\tan^{-1}(x)$ centered at $x = 0$ is $\tan^{-1}(x) \approx x - \frac{1}{3}x^3$. Since $1/n \to 0$ as $n \to \infty$, this means that $\tan^{-1}(1/n) \approx \frac{1}{n} + \frac{1}{3n^3}$, so as $n \to \infty$ we see that $1/n - \tan^{-1}(1/n) \approx \frac{1}{3n^3}$. Therefore, our original sum should converge.

To formally show this, we'll examine $\sum_{n=1}^{\infty} |1/n - \tan^{-1}(1/n)|$ instead (because showing the terms in our original sum are *positive* so we can use a comparison test is rather tricky!). By Taylor's theorem, we have $\tan^{-1}(x) = x - \frac{1}{3}x^3 + R_3(x)$, where $|R_3(x)| \leq Cx^4$ for some constant C. Therefore, $\tan^{-1}(1/n) = \frac{1}{n} - \frac{1}{3n^3} + R_3(1/n)$, with $|R_3(1/n)| \leq \frac{C}{n^4}$. We have $\lim_{n\to\infty} \frac{|1/n - \tan^{-1}(1/n)|}{1/(3n^3)} = \lim_{n\to\infty} \frac{|1/(6n^3) - R_3(1/n)|}{1/(6n^3)} \leq 1 + \lim_{n\to\infty} \frac{|R_3(1/n)|}{1/(6n^3)} \leq 1 + \lim_{n\to\infty} \frac{6C}{n} = 1$. Similarly, we also see that $\lim_{n\to\infty} \frac{|1/n - \tan^{-1}(1/n)|}{1/(3n^3)} \geq \lim_{n\to\infty} \frac{1/n - \tan^{-1}(1/n)}{1/(3n^3)} = 1$, and so we conclude that $\lim_{n\to\infty} \frac{|1/n - \tan^{-1}(1/n)|}{1/(3n^3)} = 1$. Therefore by the limit comparsion test, $\sum_{n=1}^{\infty} |1/n - \tan^{-1}(1/n)|$ converges, and so the original series does too.