Klt varieties of general type with small volume

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For projective varieties of general type, the *volume* measures the asymptotic growth of the plurigenera: $\mathrm{vol}(X) = \lim_{m\to\infty} h^0(X, mK_X)/(m^n/n!)$. This is equal to the intersection number K_X^n if the canonical class K_X is ample. A central fact about the classification of algebraic varieties is the theorem of Hacon-M$^{\mathrm{c}}$Kernan-Xu, which says in particular: for mildly singular (klt) complex projective varieties X with ample canonical class, there is a constant u_n depending only on the dimension n of X such that the pluricanonical linear system $|mK_X|$ gives a birational embedding of X into projective space for all $m \geq u_n$ [10, Theorem 1.3]. It follows that there is a positive lower bound v_n for the volume of all klt n-folds with ample canonical class: namely, $1/(u_n)^n$ is a lower bound. It is a fundamental problem to find the optimal values of these constants.

We focus here on constructing klt varieties of general type with small volume. (We also construct klt Fano varieties with similar exotic behavior.) It is also interesting to look for small volumes in the narrower setting of varieties with canonical singularities and ample canonical class, since these arise as canonical models of smooth projective varieties of general type. In that direction, Ballico, Pignatelli, and Tasin constructed smooth projective n-folds of general type with volume about $1/n^n$ [2, Theorems 1 and 2]. After several advances, Esser and the authors constructed smooth projective n-folds of general type with volume about $1/2^{2^{n/2}}$ [7, Theorem 1.1]. That paper also gives comparably extreme examples of Fano and Calabi-Yau varieties. Returning to the klt setting, our examples here have volume roughly $1/2^{2^n}$. These examples should actually be close to optimal, by the following discussion.

In the more general context of klt pairs, Kollár proposed what may be the klt pair (Y, Δ) of general type with standard coefficients that has minimum volume [9, Introduction]. (Here "general type" means that $K_Y + \Delta$ is big, and "standard coefficients" means that all coefficients of the $\mathbf{Q}$-divisor Δ are of the form $1-1/m$ for $m \in \mathbf{Z}^+$.) There is some positive lower bound for such volumes, and the minimum is attained, by Hacon-M$^{\mathrm{c}}$Kernan-Xu's theorem that these volumes satisfy DCC [10, Theorem 1.3]. The example is

$$(Y, \Delta) = \left(\mathbf{P}^n, \frac{1}{2}H_0 + \frac{2}{3}H_1 + \frac{6}{7}H_2 + \cdots + \frac{c_{n+1}-1}{c_{n+1}}H_{n+1}\right),$$

where $H_0, H_1, \ldots, H_{n+1}$ are $n+2$ general hyperplanes and $c_0, c_1, c_2, \ldots$ is Sylvester's sequence,

$$c_0 = 2 \text{ and } c_{m+1} = c_m(c_m - 1) + 1.$$

In this case, the volume of $K_Y + \Delta$ is $1/(c_{n+2}-1)^n$, which is really small, less than $1/2^{2^n}$. The optimality of Kollár's example is known only in dimension 1, where it is the "Hurwitz orbifold" of volume 1/42 [1, section 10].

How small can the volume be for a klt variety with ample canonical class, as opposed to a klt pair? In dimension 2, Alexeev and Liu gave an example with volume 1/48983 [1, Theorem 1.4]. In high dimensions, we give examples as follows (Theorems 2.1 and 4.1). Following a long tradition in algebraic geometry [11, 12, 2, 4], our examples are weighted projective hypersurfaces. These exhibit a huge range of behavior, and finding good examples is not easy.

Theorem 0.1. *For every integer $n \geq 2$, there is a complex klt n-fold X with ample canonical class such that $\mathrm{vol}(K_X) < 1/2^{2^n}$. More precisely, $\log(\mathrm{vol}(K_X))$ is asymptotic to $\log(\mathrm{vol}(K_Y + \Delta))$ as the dimension goes to infinity, where (Y, Δ) is Kollár's klt pair above.*

Since Kollár's example is conjecturally optimal in the broader setting of klt pairs with standard coefficients, Theorem 0.1 means that our klt varieties with ample canonical class should be close to optimal in high dimensions. The details of the construction are intricate, combining Sylvester's sequence with several sequences of polynomials defined by recurrence relations.

Finally, we construct a klt Fano variety X in every dimension n such that the linear system $|-mK_X|$ is empty for all $1 \leq m < b$, with b doubly exponential in n (Theorem 5.1). More precisely, b is roughly 2^{2^n}. (In the narrower setting of terminal Fano varieties, Esser and the authors gave examples with b roughly $2^{2^{n/2}}$ [7, Theorem 3.9].) Birkar's theorem on the boundedness of complements implies that there is an upper bound on the number of vanishing spaces of sections $H^0(X, -mK_X)$, for all klt Fano varieties of a given dimension [3, Theorem 1.1]. Our examples show that the bound must grow extremely fast as the dimension increases.

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1 Background on weighted projective spaces

Some introductions to the singularities of the minimal model program, such as terminal, canonical, or Kawamata log terminal (klt), are [15, 13]. We work over $\mathbf{C}$, although much of the following would work in any characteristic.

For positive integers $a_0, \ldots, a_n$, weighted projective space $Y = \mathbf{P}(a_0, \ldots, a_n)$ means the quotient variety $(A^{n+1} - 0)/G_m$, where the multiplicative group G_m acts by $t(x_0, \ldots, x_n) = (t^{a_0}x_0, \ldots, t^{a_n}x_n)$. Here Y is said to be *well-formed* if $\gcd(a_0, \ldots, \widehat{a_j}, \ldots, a_n) = 1$ for each j. We always assume this. (In other words, the analogous quotient stack $[(A^{n+1} - 0)/G_m]$ has trivial stabilizer group in codimension 1.) For well-formed Y, the canonical class of Y is given by $K_Y = O(-a_0 - \cdots - a_n)$ [6, Theorem 3.3.4]. Here $O(m)$ is the reflexive sheaf associated to a Weil divisor for any integer m; it is a line bundle if and only if m is a multiple of every weight a_i. The intersection number $\int_Y c_1(O(1))^n$ is equal to $1/a_0 \cdots a_n$. (To check this, think of the intersection number as $\mathrm{vol}(O(1))$, and use that the coordinate ring of $O(1)$ is the graded polynomial ring with generators in degrees $a_0, \ldots, a_n$.)

Since weighted projective spaces have quotient singularities, they are klt. A closed subvariety X of a weighted projective space $\mathbf{P}(a_0, \ldots, a_n)$ is called *quasi-smooth* if its affine cone in A^{n+1} is smooth outside the origin. It follows that X

has only cyclic quotient singularities. A subvariety X in $Y = \mathbf{P}(a_0, \ldots, a_n)$ is said to be *well-formed* if Y is well-formed and the codimension of $X \cap Y^{\text{sing}}$ in X is at least 2. Notably, the adjunction formula holds for a well-formed quasi-smooth hypersurface X of degree d in Y, meaning that $K_X = O_X(d - \sum a_i)$ [11, section 6.14]. A general hypersurface of degree d is well-formed if and only if d is an $\mathbf{N}$-linear combination of $a_0, \ldots, \widehat{a_i}, \ldots, \widehat{a_j}, \ldots, a_n$ for all $i < j$; that holds for all examples in this paper. Indeed, assuming that d is not equal to any a_i (as will be true in our examples), a quasi-smooth hypersurface of dimension at least 3 is always well-formed [11, Theorem 6.17].

Iano-Fletcher proved the following criterion for quasi-smoothness, using that we are in characteristic zero [11, Theorem 8.1]. Here $\mathbf{N}$ denotes the natural numbers, $\{0, 1, \ldots\}$.

Lemma 1.1. *A general hypersurface of degree d in $\mathbf{P}(a_0, \ldots, a_n)$ is quasi-smooth if and only if*

either (1) $a_i = d$ for some i,

or (2) for every nonempty subset I of $\{0, \ldots, n\}$, either (a) d is an $\mathbf{N}$-linear combination of the numbers a_i with $i \in I$, or (b) there are at least $|I|$ numbers $j \notin I$ such that $d - a_j$ is an $\mathbf{N}$-linear combination of the numbers a_i with $i \in I$.

2 Klt varieties with ample canonical class

As in the introduction, let $c_0, c_1, c_2, \ldots$ be Sylvester's sequence [14],

$$c_0 = 2 \text{ and } c_{n+1} = c_n(c_n - 1) + 1.$$

The first few terms are $c_0 = 2$, $c_1 = 3$, $c_2 = 7$, $c_3 = 43$, $c_4 = 1807$. We give the following examples of klt varieties with ample canonical class. We will generalize the construction as Theorem 4.1, giving better but more complicated examples.

Theorem 2.1. *Let n be an integer at least 2, and define integers $a_0, \ldots, a_{n+1}$ as follows. Let $y = c_{n-1} - 1$ and*

$$\begin{aligned} a_2 &= y^3 + y + 1 \\ a_1 &= y(y+1)(1 + a_2) - a_2 \\ a_0 &= y(1 + a_2 + a_1) - a_1. \end{aligned}$$

Let $x = 1 + a_0 + a_1 + a_2$, $d = yx = c_0 \cdots c_{n-2}x = y^7 + y^6 + y^5 + 4y^4 + 2y^3 + 2y^2 + 2y$, and $a_{i+3} = c_0 \cdots \widehat{c_i} \cdots c_{n-2}x$ for $0 \leq i \leq n-2$. Let X be a general hypersurface of degree d in the complex weighted projective space $\mathbf{P}(a_0, \ldots, a_{n+1})$. Then X is a klt projective variety of dimension n with ample canonical class, and

$$\text{vol}(K_X) = \frac{1}{y^{n-3}x^{n-2}a_0a_1a_2}.$$

It follows that $\text{vol}(K_X) < \frac{1}{(c_{n-1}-1)^{7n-1}}$ *and hence* $\text{vol}(K_X) < \frac{1}{2^{2^n}}$.

This example is not optimal, but it should be fairly close to optimal, given the fast-growing functions involved. Indeed, Kollár's conjecturally optimal klt pair (Y, Δ) from the introduction has $\text{vol}(K_Y + \Delta) = 1/(c_{n+2} - 1)^n \doteq 1/(c_{n-1} - 1)^{8n}$, while the klt variety X in Theorem 2.1 has $\mathrm{vol}(K_X) \doteq 1/(c_{n-1}-1)^{7n-1}$, thus about the 7/8th power of the volume of Kollár's klt pair. See Theorem 4.1 for a generalization, producing better examples.

Some cases of Theorem 2.1 in low dimensions, klt varieties with ample canonical class, are:

$X_{316} \subset \mathbf{P}^3(158, 85, 61, 11)$ of dimension 2, with volume $2/57035 \doteq 3.5 \times 10^{-5}$.

$X_{340068} \subset \mathbf{P}^4(170034, 113356, 47269, 9185, 223)$ of dimension 3, with volume

$$1/5487505331993410 \doteq 1.8 \times 10^{-16}.$$

The klt 4-fold with ample canonical class given by Theorem 2.1 has volume about 1.4×10^{-44}. For comparison, the smallest known volume for a klt 4-fold with ample canonical class is about 1.4×10^{-47} [5, ID 538926].

Proof. Sylvester's sequence satisfies $c_m = c_0 \cdots c_{m-1} + 1$. It follows that any two terms in the sequence are relatively prime. Another notable feature is that

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \cdots + \frac{1}{c_m} = 1 - \frac{1}{c_{m+1} - 1},$$

which converges very quickly to 1 as m increases.

We first show that the weighted projective space $Y = \mathbf{P}(a_0, \ldots, a_{n+1})$ is well-formed. That is, we have to show that $\gcd(a_0, \ldots, \widehat{a_m}, \ldots, a_{n+1}) = 1$ for each $0 \le m \le n+1$. It suffices to show that a_0, a_1, a_2 are pairwise relatively prime.

Indeed, 1 is a $\mathbf{Z}[y]$-linear combination of any two of a_0, a_1, a_2. For clarity, however, let us check by hand that a_0, a_1, a_2 are pairwise relatively prime. To show that $\gcd(a_1, a_2) = 1$, let p be a prime number dividing a_1 and a_2. By the formula for a_1, p divides $y(y+1)$. But as a polynomial in y, $a_2(y) = y^3 + y + 1$ satisfies $a_2(0) = 1$ and $a_2(-1) = -1$; so $a_2 \equiv 1 \pmod{y}$ and $a_2 \equiv -1 \pmod{y+1}$. This contradicts that p divides a_2. So $\gcd(a_1, a_2) = 1$.

Next, let p be a prime number that divides a_0 and a_1. By the formula for a_0, either p divides $1 + a_2$ or p divides y. In both cases, the formula for a_1 gives that p divides a_2, contradicting that $\gcd(a_1, a_2) = 1$. So $\gcd(a_0, a_1) = 1$.

Finally, let p be a prime number that divides a_0 and a_2. Then the formulas for a_0 and a_1 imply that $a_1 \equiv y(a_1 + 1) \pmod{p}$, that is, $(y-1)a_1 + y \equiv 0 \pmod{p}$, and $a_1 \equiv y(y+1) \pmod{p}$. Combining these shows that $(y-1)y(y+1) + y = y^3 \equiv 0 \pmod{p}$. So p divides y. But then $a_1 \equiv y(y+1) \equiv 0 \pmod{p}$, contradicting that $\gcd(a_1, a_2) = 1$. It follows that the weighted projective space Y is well-formed.

Next, let us show that the general hypersurface X of degree d in Y is quasi-smooth. We use the following sufficient condition in terms of a cycle of congruences.

Lemma 2.2. *For positive integers d and $a_0, \ldots, a_{n+1}$, a general hypersurface of degree d in $\mathbf{P}(a_0, \ldots, a_{n+1})$ is quasi-smooth if $d \ge a_i$ for every i and there is a positive integer r such that:*

(1) $a_i | d$ if $i \ge r$ (that is, all but the first r weights divide d),

and (2) $d - a_{r-1} \equiv 0 \pmod{a_{r-2}}$, ..., $d - a_1 \equiv 0 \pmod{a_0}$, and $d - a_0 \equiv 0 \pmod{a_{r-1}}$.

Proof. Use Lemma 1.1. We have something to prove for each nonempty subset I of $\{0, \ldots, n+1\}$. If I contains a number $i \ge r$, then a_i divides d and we are done.

Otherwise, I is contained in the set $S = \{0, \ldots, r-1\}$. Consider S as the vertices of a directed graph, with arrows from $r-1$ to $r-2$ to ... to 0 to $r-1$; then S is a directed cycle of length r. If I contains two vertices j, i with an edge from j to i, then the congruence $d - a_j \equiv 0 \pmod{a_i}$ implies that d is an $\mathbf{N}$-linear combination of a_i and a_j, using that $d \geq a_j$, and we are done.

Otherwise, I contains no edge in the graph. Let J be the set of vertices that point to some element of I. We have $|J| = |I|$, and J is disjoint from I because I contains no edge. For each element $j \in J$ pointing to a vertex $i \in I$, the congruence $d - a_j \equiv 0 \pmod{a_i}$ implies that $d - a_j$ is an $\mathbf{N}$-linear combination of the numbers a_m with $m \in I$. This checks the condition of Lemma 1.1 for quasi-smoothness. $\square$

Returning to the proof of Theorem 2.1, let us use Lemma 2.2 to prove that the general hypersurface X of degree d in Y is quasi-smooth. We know that a_i divides d for each $i \geq 3$; also, d is greater than every a_i. Given that, it suffices to prove the cycle of 3 congruences: $d - a_2 \equiv 0 \pmod{a_1}$, $d - a_1 \equiv 0 \pmod{a_0}$, and $d - a_0 \equiv 0 \pmod{a_2}$. Using that $x = 1 + a_0 + a_1 + a_2$ and $d = yx$, we compute that

$$\begin{aligned} d - a_2 &= (y^2 + 1)a_1 \\ d - a_1 &= (y + 1)a_0 \\ d - a_0 &= (y^4 + 3y - 1)a_2, \end{aligned}$$

proving the desired congruences. That completes the proof that X is quasi-smooth. In particular, X has only cyclic quotient singularities, and so X is klt.

Since Y is well-formed and X is quasi-smooth, $K_X = O_X(d - \sum a_i)$. Here $d - \sum_{i=3}^{n+1} a_i = c_0 \cdots c_{n-2}(1 - \sum_{i=0}^{n-2} 1/c_i)x = x = 1 + a_0 + a_1 + a_2$, and so $K_X = O_X(1)$. As a result,

$$\begin{aligned} \operatorname{vol}(K_X) &= \frac{d}{a_0 \cdots a_{n+1}} \\ &= \frac{(c_0 \cdots c_{n-2})x}{(c_0 \cdots c_{n-2})^{n-2} x^{n-1} a_0 a_1 a_2} \\ &= \frac{1}{y^{n-3} x^{n-2} a_0 a_1 a_2}. \end{aligned}$$

In terms of $y = c_{n-1} - 1$, we have $a_2 = y^3 + y + 1 > y^3$, $a_1 = y^5 + y^4 + 3y^2 + y - 1 > y^5$, $a_0 = y^6 + 3y^3 - y^2 + 1 > y^6$, and $x = y^6 + y^5 + y^4 + 4y^3 + 2y^2 + 2y + 2 > y^6$. Therefore, $\operatorname{vol}(K_X) < 1/y^{7n-1} = 1/(c_{n-1} - 1)^{7n-1}$.

There is a constant $c \doteq 1.264$ such that c_i is the closest integer to $c^{2^{i+1}}$ for all $i \geq 0$ [8, equations 2.87 and 2.89]. This implies the crude statement that $\operatorname{vol}(K_X) < 1/2^{2^n}$ for all $n \geq 2$. $\square$

3 Some polynomial sequences defined by recurrence relations

Here we define five sequences of polynomials in $\mathbf{Z}[y]$ by recurrence relations, f_i, e_i, b_i, z_i and d_i. These will be used for defining our examples of klt varieties with ample canonical class in Theorem 4.1, generalizing Theorem 2.1. It would be interesting to know if these polynomials (such as f_i, below) have been encountered before.

We found these polynomials by trying to choose weights for our weighted projective space such that the largest weights are of the form $d/2, d/3, \ldots, d/c_j$, while the other weights satisfy a cycle of congruences as in Lemma 2.2 (so that we get a quasi-smooth hypersurface). For a cycle of length 3, we were led to the polynomials in Theorem 2.1. Cycles of even length seem not to lead to good examples: we want large weights which also yield a well-formed weighted projective space. We found similar polynomials to produce a cycle of length 5, and generalizing to a cycle of any odd length led to the polynomials in this section. These polynomials seem related to the sequence of iterates of the polynomial $y^2 - y + 1$, which comes up because the Sylvester numbers satisfy $c_{i+1} = c_i^2 - c_i + 1$.

Definition 3.1. For each $i \geq 0$, define a polynomial f_i in $\mathbf{Z}[y]$ by: $f_0 = y + 1$, $f_1 = y^2 + 1$, and $f_i = f_{i-1}f_{i-2} + (f_{i-1} - 1)(f_{i-1} - 2)$ for $i \geq 2$.

For example, $f_0 = y + 1$, $f_1 = y^2 + 1$, and $f_2 = y^4 + y^3 + y + 1$. Clearly the polynomial f_i has degree 2^i for each $i \geq 0$. The following description of f_i may seem more natural.

Lemma 3.2. *For all $i \geq 0$,*

$$f_i = 1 + y(f_0 \cdots f_{i-1} - f_0 \cdots f_{i-2} + \cdots + (-1)^i).$$

Proof. Temporarily define a sequence of polynomials h_i in $\mathbf{Z}[y]$ by

$$h_i = 1 + y(h_0 \cdots h_{i-1} - h_0 \cdots h_{i-2} + \cdots + (-1)^i).$$

We want to show that $h_i = f_i$ for all $i \geq 0$. We have $h_0 = y + 1 = f_0$ and $h_1 = y^2 + 1 = f_1$. It remains to show that h_i satisfies the recurrence relation that defines f_i for $i \geq 2$. We clearly have $h_i + h_{i-1} - 2 = yh_0 \cdots h_{i-1}$. Likewise, $h_{i-1} + h_{i-2} - 2 = yh_0 \cdots h_{i-2}$. Therefore, $h_i + h_{i-1} - 2 = h_{i-1}(h_{i-1} + h_{i-2} - 2)$, which is equivalent to the desired relation $h_i = h_{i-1}h_{i-2} + (h_{i-1} - 1)(h_{i-1} - 2)$. So $h_i = f_i$ for all $i \geq 0$. ☐

The next polynomial sequence we will need is:

Definition 3.3. For each $i \geq 0$, define a polynomial e_i in $\mathbf{Z}[y]$ by

$$e_i = yf_0 \cdots f_{i-1}.$$

For example, $e_0 = y$, $e_1 = y(y + 1) = y^2 + y$, and $e_2 = y(y + 1)(y^2 + 1) = y^4 + y^3 + y^2 + y$. By Lemma 3.2, we have

$$e_i = f_i + f_{i-1} - 2$$

for all $i \geq 1$, which can be viewed as an alternative definition of e_i. We can also say that $e_i = f_{i-1}e_{i-1}$ for all $i \geq 1$. The polynomial e_i has degree 2^i for each $i \geq 0$.

Definition 3.4. For each $i \geq 0$, define a polynomial b_i in $\mathbf{Z}[y]$ by $b_0 = 1$ and

$$b_i = (-1)^i + f_{i-1}b_{i-1}$$

for $i \geq 1$.

It follows by induction that

$$b_i = f_0 \cdots f_{i-1} - f_1 \cdots f_{i-1} + \cdots + (-1)^i$$

for all $i \geq 0$. For example, $b_0 = 1$, $b_1 = y$, and $b_2 = y^3 + y + 1$ (which was the smallest weight of the weighted projective space in Theorem 2.1). The polynomial b_i has degree $2^i - 1$ for each $i \geq 0$.

Definition 3.5. For each $i \geq 0$, define a polynomial z_i in $\mathbf{Z}[y]$ by $z_0 = y - 1$, $z_1 = y^2 - y + 1$, and

$$z_i = e_{i-1} z_{i-1} + z_{i-2}$$

for all $i \geq 2$.

For example, $z_2 = y^4 + 2y - 1$. The polynomial z_i has degree 2^i for each $i \geq 0$. The following identity, needed for Theorem 4.1, relates the polynomial z_i to b_i and f_i, which may be considered simpler.

Lemma 3.6. *For every $i \geq 0$,*

$$f_0 \cdots f_{i-1} z_i = (-1)^{i+1} + b_i(f_i - 1).$$

Proof. The lemma holds for $i = 0$ (since $y - 1 = -1 + 1(y)$) and for $i = 1$ (since $(y+1)(y^2 - y + 1) = 1 + y(y^2)$). Now let $i \geq 2$ and assume the lemma for smaller values of i. Then the definition of z_i gives that:

$$\begin{aligned}
f_0 \cdots f_{i-1} z_i &= f_0 \cdots f_{i-1}(e_{i-1} z_{i-1} + z_{i-2}) \\
&= (f_0 \cdots f_{i-2} z_{i-1})(f_{i-1} e_{i-1}) + (f_0 \cdots f_{i-3} z_{i-2})(f_{i-2} f_{i-1}) \\
&= (f_0 \cdots f_{i-2} z_{i-1}) e_i + (f_0 \cdots f_{i-3} z_{i-2})(f_{i-2} f_{i-1}) \\
&= \big[(-1)^i + b_{i-1}(f_{i-1} - 1)\big] e_i \\
&\quad + \big[(-1)^{i-1} + b_{i-2}(f_{i-2} - 1)\big] f_{i-2} f_{i-1},
\end{aligned}$$

using that the lemma holds for smaller values of i. So the lemma holds for i if 0 is equal to

$$\begin{aligned}
&\big[(-1)^i + b_{i-1}(f_{i-1} - 1)\big](-e_i) + \big[(-1)^{i-1} + b_{i-2}(f_{i-2} - 1)\big](-f_{i-2} f_{i-1}) \\
&\quad + (-1)^{i+1} + b_i(f_i - 1) \\
&= \big[(-1)^i + b_{i-1} f_{i-1}\big](-e_i) + \big[(-1)^{i-1} + b_{i-2} f_{i-2}\big](-f_{i-2} f_{i-1}) \\
&\quad + (-1)^{i+1} + b_i(f_i - 1) + b_{i-1} e_i + b_{i-2} f_{i-2} f_{i-1} \\
&= \big[(-1)^i + b_{i-1} f_{i-1}\big](-e_i + e_{i-1} - f_{i-2} + 1) + \big[(-1)^{i-1} + b_{i-2} f_{i-2}\big](f_{i-1} - f_{i-2} f_{i-1}) \\
&\quad + b_i(f_i - 1) + b_{i-1}(e_i - e_{i-1} f_{i-1} + f_{i-1} f_{i-2} - f_{i-1}) + (-1)^i(-e_{i-1} + f_{i-1} + f_{i-2} - 2).
\end{aligned}$$

By definition of b_i, we have $b_{i-1} = (-1)^{i-1} + b_{i-2} f_{i-2}$ and likewise $b_i = (-1)^i + b_{i-1} f_{i-1}$. Also, $e_i = e_{i-1} f_{i-1}$. So we need to show that 0 is equal to

$$\begin{aligned}
&b_i(-e_i + e_{i-1} - f_{i-2} + 1) + b_{i-1}(f_{i-1} - f_{i-1} f_{i-2}) \\
&\quad + b_i(f_i - 1) + b_{i-1}(f_{i-1} f_{i-2} - f_{i-1}) + (-1)^i(f_{i-1} + f_{i-2} - e_{i-1} - 2) \\
&= b_i(-e_i + e_{i-1} - f_{i-2} + f_i) + (-1)^i(f_{i-1} + f_{i-2} - e_{i-1} - 2).
\end{aligned}$$

This is zero by the identities $e_i = f_i + f_{i-1} - 2$ and $e_{i-1} = f_{i-1} + f_{i-2} - 2$. Lemma 3.6 is proved. $\square$

The last sequence of polynomials we define is:

Definition 3.7. For each $i \geq 0$, define a polynomial d_i in $\mathbf{Z}[y]$ by

$$d_i = e_i + b_i(f_i - 1).$$

For example, $d_0 = 2y$, $d_1 = y^3+y^2+y$, and $d_2 = y^7+y^6+y^5+4y^4+2y^3+2y^2+2y$ (which was the degree of the hypersurface in Theorem 2.1). The polynomial d_i has degree $2^{i+1} - 1$ for each $i \geq 0$. By Lemma 3.6, another formula for d_i is that $d_i = (-1)^i + f_0 \cdots f_{i-1}(z_i + y)$.

4 Better klt varieties with ample canonical class

We now construct klt varieties X with ample canonical class and with smaller volume than in Theorem 2.1. These should be close to optimal in high dimensions. Indeed, we give examples with $\log(\mathrm{vol}(K_X))$ asymptotic to $\log(\mathrm{vol}(K_Y + \Delta))$ as the dimension goes to infinity, where (Y, Δ) is Kollár's conjecturally optimal klt pair from the introduction.

For any odd number $r \geq 3$ and any dimension $n \geq r - 1$, we give an example with weights chosen to satisfy a cycle of r congruences. For $r = 3$, this is the example in Theorem 2.1. For each odd $r \geq 3$, our klt variety X compares to Kollár's conjecturally optimal klt pair by

$$\frac{\log(\mathrm{vol}(K_X))}{\log(\mathrm{vol}(K_Y + \Delta))} \to \frac{2^r - 1}{2^r}$$

as n goes to infinity. Thus, by increasing r as n increases, we can make this ratio converge to 1.

The example given by Theorem 4.1 in dimension 4, with $r = 5$, is a general hypersurface of degree 147565206676 in

$$\mathbf{P}^5(73782603338, 39714616165, 28421358181, 5458415771, 187980859, 232361).$$

Here X has volume $\doteq 7.4 \times 10^{-45}$. This is better than the klt 4-fold given by Theorem 2.1, although the smallest known volume for a klt 4-fold with ample canonical class is about 1.4×10^{-47} [5, ID 538926].

Let $c_0, c_1, \ldots$ be Sylvester's sequence; see section 2 for the properties of that sequence. We also use the five sequences of polynomials f_i, e_i, b_i, z_i and d_i in $\mathbf{Z}[y]$ from section 3.

Theorem 4.1. *Let r be an odd integer at least 3 and let n be an integer at least $r - 1$. Define integers $a_0, \ldots, a_{n+1}$ as follows. Let $y = c_{n-r+2} - 1$ and*

$$\begin{aligned}
a_{r-1} &= b_{r-1} \\
a_0 &= d_{r-1} - (z_{r-1} + y)a_{r-1} \\
a_1 &= d_{r-1} - f_0 a_0 \\
&\cdots \\
a_{r-2} &= d_{r-1} - f_{r-3} a_{r-3}
\end{aligned}$$

These are positive integers. Let $x = 1 + a_0 + \cdots + a_{r-1}$*; then* $d_{r-1} = yx = c_0 \cdots c_{n-r+1}x$*. Let* $a_{r+i} = c_0 \cdots \widehat{c_i} \cdots c_{n-r+1}x$ *for* $0 \leq i \leq n - r + 1$*. Let* X *be a general hypersurface of degree* d_{r-1} *in the complex weighted projective space* $\mathbf{P}(a_0, \ldots, a_{n+1})$*. Then* X *is a klt projective variety of dimension* n *with ample canonical class, and*

$$\mathrm{vol}(K_X) = \frac{1}{y^{n-r}x^{n-r+1}a_0 \cdots a_{r-1}}.$$

It follows that $\mathrm{vol}(K_X) < \frac{1}{(c_{n-r+2}-1)^{(2^r-1)n-1}}$ *and hence* $\mathrm{vol}(K_X) < \frac{1}{2^{2^n}}$*.*

Remark 4.2. As mentioned in the introduction, Hacon-McKernan-Xu showed that for each positive integer n, there is a constant u_n such that for every klt projective variety X of dimension n with K_X ample, the linear system $|u_n K_X|$ gives a birational embedding of X into projective space [10, Theorem 1.3]. Although we have emphasized the role of volume, Theorem 4.1 shows that u_n must also grow at least doubly exponentially with n. Indeed, the variety X has

$$H^0(X, mK_X) = 0$$

for all $1 \leq m < b_{r-1}$, where $b_{r-1} \geq (c_{n-r+2} - 1)^{2^{r-1}-1}$. Taking $r = n + 1$ if n is even and $r = n$ if n is odd, we deduce that the bottom weight b_{r-1} is at least 2^{2^n-1} if n is even and at least $6^{2^{n-1}-1}$ if n is odd.

Proof. For any positive integer r and $n \geq r-1$, define $a_0, \ldots, a_{r-1}$ as in the theorem. We start by proving various identities that we need, leading up to the proof that $d_{r-1} = yx$. We only introduce the assumption that r is odd and at least 3 when we prove that Y is well-formed.

A first step is to show that $a_{r-1} = d_{r-1} - f_{r-2}a_{r-2}$ if $r \geq 2$. By section 3, we have $d_{r-1} = (-1)^{r-1} + f_0 \cdots f_{r-2}(z_{r-1} + y)$. Multiplying by $(-1)^{r-1}a_{r-1}$ gives that

$$\begin{aligned}
a_{r-1} &= (-1)^{r-1}d_{r-1}a_{r-1} + (-1)^r f_0 \cdots f_{r-2}(z_{r-1} + y)a_{r-1} \\
&= d_{r-1}\big[1 - f_{r-2} + f_{r-3}f_{r-2} - \cdots + (-1)^{r-1}f_0 \cdots f_{r-2}\big] \\
&\quad + (-1)^r f_0 \cdots f_{r-2}(z_{r-1} + y)a_{r-1},
\end{aligned}$$

using a formula for $a_{r-1} = b_{r-1}$ from section 3. By definition of a_0, this gives that

$$\begin{aligned}
a_{r-1} = d_{r-1}\big[1 - f_{r-2} + f_{r-3}f_{r-2} - \cdots + (-1)^{r-2}f_1 \cdots f_{r-2}\big] \\
+ (-1)^{r-1}f_0 \cdots f_{r-2}a_0.
\end{aligned}$$

Now successively apply the definitions of a_1, a_2, and so on, giving

$$\begin{aligned}
&\cdots \\
a_{r-1} &= d_{r-1}(1 - f_{r-2} + f_{r-3}f_{r-2}) - f_{r-4}f_{r-3}f_{r-2}a_{r-4} \\
&= d_{r-1}(1 - f_{r-2}) + f_{r-3}f_{r-2}a_{r-3} \\
&= d_{r-1} - f_{r-2}a_{r-2}.
\end{aligned}$$

That is what we wanted.

Lemma 4.3. *The following r equations hold.*

$$\begin{aligned}
a_{r-1} &= b_{r-1} \\
a_{r-2} &= e_{r-2}(1 + a_{r-1}) - a_{r-1} \\
a_{r-3} &= e_{r-3}(1 + a_{r-1} + a_{r-2}) - a_{r-2} \\
&\cdots \\
a_0 &= e_0(1 + a_{r-1} + \cdots + a_1) - a_1.
\end{aligned}$$

The integer y is at least 2. Given that, Lemma 4.3 shows that the a_i's are positive integers. We will also use it to prove the identity $d_{r-1} = yx$, which is important for Theorem 4.1.

Proof. (Lemma 4.3) The first equation, $a_{r-1} = b_{r-1}$, holds by definition of a_{r-1}. Next, if $r \geq 2$, we want to show that $a_{r-2} = e_{r-2}(1 + a_{r-1}) - a_{r-1}$. It suffices to prove this after multiplying by f_{r-2}, which we do in order to use the result above that $a_{r-1} = d_{r-1} - f_{r-2}a_{r-2}$. So we want to show that 0 is equal to

$$\begin{aligned}
&f_{r-2}e_{r-2}(1 + a_{r-1}) - f_{r-2}a_{r-1} - (d_{r-1} - a_{r-1}) \\
&= a_{r-1}(f_{r-2}e_{r-2} - f_{r-2} + 1) + (f_{r-2}e_{r-2} - d_{r-1}). \\
&= b_{r-1}(f_{r-1} - 1) + (e_{r-1} - d_{r-1}),
\end{aligned}$$

where we used the identities that $e_{r-1} = f_{r-1} + f_{r-2} - 2$ and $e_{r-1} = f_{r-2}e_{r-2}$ from section 3. By definition, $d_{r-1} = e_{r-1} + b_{r-1}(f_{r-1} - 1)$, and so the desired equation holds. So we have $a_{r-2} = e_{r-2}(1 + a_{r-1}) - a_{r-1}$.

Now suppose we have proved the equation in Lemma 4.3 for a_{i+1}, with $0 \leq i \leq r - 3$; let us prove it for a_i. That is, we want to show that $a_i = e_i(1 + a_{r-1} + \cdots + a_{i+1}) - a_{i+1}$. By definition, $a_{i+1} = d_{r-1} - f_i a_i$, and so $f_i a_i = d_{r-1} - a_{i+1}$. It suffices to prove the desired identity after multiplying by f_i; so we want to show that 0 is equal to

$$\begin{aligned}
&f_i e_i(1 + a_{r-1} + \cdots + a_{i+1}) - f_i a_{i+1} - (d_{r-1} - a_{i+1}) \\
&= a_{i+1}(f_i e_i - f_i + 1) + f_i e_i(1 + a_{r-1} + \cdots + a_{i+2}) - d_{r-1} \\
&= a_{i+1}(f_{i+1} - 1) + e_{i+1}(1 + a_{r-1} + \cdots + a_{i+2}) - d_{r-1},
\end{aligned}$$

using the identities $e_{i+1} = f_{i+1} + f_i - 2$ and $e_{i+1} = f_i e_i$ from section 3.

By induction, we know that $a_{i+1} = e_{i+1}(1 + a_{r-1} + \cdots + a_{i+2}) - a_{i+2}$. So we want to show that 0 is equal to

$$a_{i+1}f_{i+1} + a_{i+2} - d_{r-1},$$

which is true by definition of a_{i+2} (or, in the case $i = r - 3$, by the equality $a_{r-1} = d_{r-1} - a_{r-2}f_{r-2}$ which we proved). That completes the proof of Lemma 4.3. □

We return to the proof of Theorem 4.1. From Lemma 4.3, it is clear that $a_0, \ldots, a_{r-1}$ are positive integers. Writing $x = 1 + a_0 + \cdots + a_{r-1}$, let us show that $d_{r-1} = yx$, part of the statement of the theorem. If $r = 1$, then $a_0 = b_0 = 1$ and $d_{r-1} = 2y$, so $x = 1 + a_0 = 2$ and we see that $d_{r-1} = yx$. If $r \geq 2$, then Lemma 4.3 says that $a_0 = e_0(1 + a_{r-1} + \cdots + a_1) - a_1$, where $e_0 = y$. Equivalently,

$x = 1 + a_0 + \cdots + a_r$ satisfies $a_1 = yx - (y+1)a_0$. Here $a_1 = d_{r-1} - (y+1)a_0$, by definition if $r \geq 3$ and by the formula $a_{r-1} = d_{r-1} - f_{r-2}a_{r-2}$ shown above if $r = 2$. We conclude that $d_{r-1} = yx$. Since $y = c_{n-r+2} - 1 = c_0 \cdots c_{n-r+1}$ by the properties of Sylvester's sequence, we can also say that $d_{r-1} = c_0 \cdots c_{n-r+1}x$.

As in the statement of the theorem, define $a_{r+i} = c_0 \cdots \widehat{c_i} \cdots c_{n-r+1}x$ for $0 \leq i \leq n-r+1$. We now show that the weighted projective space $Y = \mathbf{P}(a_0, \ldots, a_{n+1})$ is well-formed, when r is odd and at least 3. That is, we have to show that $\gcd(a_0, \ldots, \widehat{a_m}, \ldots, a_{n+1}) = 1$ for each $0 \leq m \leq n+1$. It suffices to show that $\gcd(a_0, \ldots, \widehat{a_m}, \ldots, a_{r-1}) = 1$ for each $0 \leq m \leq r-1$.

We first compute some of section 3's polynomial sequences modulo y. By induction on i, we have $f_i \equiv 1 \pmod{y}$ for all $i \geq 0$. It follows that

$$b_i \equiv \begin{cases} 1 \pmod{y} & \text{if } i \text{ is even} \\ 0 \pmod{y} & \text{if } i \text{ is odd.} \end{cases}$$

Also by induction, we find that

$$f_i \equiv \begin{cases} 0 \pmod{y+1} & \text{if } i \text{ is even} \\ 2 \pmod{y+1} & \text{if } i \text{ is odd,} \end{cases}$$

and hence

$$b_i \equiv \begin{cases} 1 \pmod{y+1} & \text{if } i = 0 \\ -1 \pmod{y+1} & \text{if } i > 0. \end{cases}$$

From there, we can show that $\gcd(a_0, \ldots, a_{r-1}) = 1$, a step towards our goal. Namely, if a prime number p divides a_j for all $0 \leq j \leq r-1$, then the formula for a_0 from Lemma 4.3 shows that p divides $e_0 = y$. But $a_{r-1} = b_{r-1} \equiv 1 \pmod{y}$ (using that r is odd), contradicting that p divides a_{r-1}. So we have shown that $\gcd(a_0, \ldots, a_{r-1}) = 1$.

Using that, let us show that $\gcd(a_0, \ldots, \widehat{a_m}, \ldots, a_{r-1}) = 1$ for each $2 \leq m \leq r-1$. (We handle the cases where m is 0 or 1 afterward.) Let p be a prime number that divides a_j for all $0 \leq j \leq r-1$ with $j \neq m$. The formula for a_0 from Lemma 4.3 gives that $0 \equiv y(1 + a_m) \pmod{p}$. As a result, the formula for a_{m-1} from the lemma gives that $0 \equiv e_{m-1}(1 + a_m) - a_m \equiv -a_m \pmod{p}$, using that e_{m-1} is a multiple of y. This contradicts the fact that $\gcd(a_0, \ldots, a_{r-1}) = 1$.

Next, we show that $\gcd(a_0, a_2, \ldots, a_{r-1}) = 1$. Let p be a prime number that divides a_0 as well as a_j for all $2 \leq j \leq r-1$. The formula for a_0 from Lemma 4.3 gives that $0 \equiv y(1 + a_1) - a_1 \equiv y + (y-1)a_1 \pmod{p}$. The formula for a_1 from the lemma gives that $a_1 \equiv e_1 = y(y+1) \pmod{p}$. Combining these, we have $0 \equiv y + (y-1)y(y+1) = y^3 \pmod{p}$. So p divides y. But then $a_1 \equiv 0 \pmod{p}$, contradicting that $\gcd(a_0, \ldots, a_{r-1}) = 1$.

Finally, we show that $\gcd(a_1, \ldots, a_{r-1}) = 1$. Let p be a prime number that divides a_j for all $1 \leq j \leq r-1$. By the formula for a_1 from Lemma 4.3, p divides $e_1 = y(y+1)$. If p divides y $(= e_0)$, then the formula for a_0 from the lemma gives that p divides a_0, contradicting that $\gcd(a_0, \ldots, a_{r-1}) = 1$. So p divides $y+1$. But $a_{r-1} = b_{r-1} \equiv -1 \pmod{y+1}$ since r is at least 3, contradicting that p divides a_{r-1}. This completes the proof that $\gcd(a_0, \ldots, \widehat{a_m}, \ldots, a_{r-1}) = 1$ for each $0 \leq m \leq r-1$. So Y is well-formed.

Next, let us show that the general hypersurface X of degree d_{r-1} in Y is quasi-smooth. For each $i > r-1$, we know that a_i divides d_{r-1}; also, d_{r-1} is greater than each a_i. (For $a_0, \ldots, a_{r-1}$, that follows from the fact that $d_{r-1} = y(1 + a_0 + \cdots + a_{r-1})$.) Given this, Lemma 2.2 shows that quasi-smoothness follows from a cycle of r congruences, namely that $d_{r-1} - a_{r-1} \equiv 0 \pmod{a_{r-2}}$, $d_{r-1} - a_{r-2} \equiv 0 \pmod{a_{r-3}}$, …, $d_{r-1} - a_1 \equiv 0 \pmod{a_0}$, and $d_{r-1} - a_0 \equiv 0 \pmod{a_{r-1}}$. These are immediate from the definitions of a_i, together with the identity $a_{r-1} = d_{r-1} - f_{r-2}a_{r-2}$ which we proved. So X is quasi-smooth. In particular, X has only cyclic quotient singularities, and so X is klt.

Therefore, $K_X = O_X(d_{r-1} - \sum a_i)$. Here

$$\begin{aligned}
d_{r-1} - \sum_{i=r}^{n+1} a_i &= c_0 \cdots c_{n-r+1}x - \sum_{i=r}^{n+1} a_i \\
&= c_0 \cdots c_{n-r+1}\left(1 - \sum_{i=0}^{n-r+1} 1/c_i\right)x \\
&= x \\
&= 1 + a_0 + \cdots + a_{r-1},
\end{aligned}$$

and so $K_X = O_X(1)$. As a result,

$$\begin{aligned}
\mathrm{vol}(K_X) &= \frac{d_{r-1}}{a_0 \cdots a_{n+1}} \\
&= \frac{(c_0 \cdots c_{n-r+1})x}{(c_0 \cdots c_{n-r+1})^{n-r+1}x^{n-r+2}a_0 \cdots a_{r-1}} \\
&= \frac{1}{y^{n-r}x^{n-r+1}a_0 \cdots a_{r-1}}.
\end{aligned}$$

In terms of $y = c_{n-r+2} - 1$, we have $a_{r-1} > y^{2^{r-1}-1}$. Use the r equations from Lemma 4.3 to estimate the other a_i's. By descending induction on i, using that $e_j \geq y^{2^j}$ for each j, it follows that $a_i > y^{2^r - 2^i - 1}$ for $0 \leq i \leq r-1$. Therefore, $x \geq a_0 > y^{2^r - 2}$. It follows that $\mathrm{vol}(K_X) < 1/y^{(2^r-1)n-1} = 1/(c_{n-r+2} - 1)^{(2^r-1)n-1}$.

There is a constant $c \doteq 1.264$ such that c_i is the closest integer to $c^{2^{i+1}}$ for all $i \geq 0$ [8, equations 2.87 and 2.89]. This implies the crude statement that $\mathrm{vol}(K_X) < 1/2^{2^n}$ for all $n \geq r-1$. $\square$

5 Klt Fano varieties with $H^0(X, -mK_X) = 0$ for a large range of positive integers m

We now construct klt Fano varieties such that $H^0(X, -mK_X) = 0$ for a large range of positive integers m (Theorem 5.1). This is of interest in connection with Birkar's theorem on the boundedness of complements. Namely, for each positive integer n, there is a positive integer $e = e_n$ such that for every klt Fano variety X of dimension n, the linear system $|-eK_X|$ is not empty, and in fact it contains a divisor M with mild singularities in the sense that the pair $(X, \frac{1}{e}M)$ is log canonical [3, Theorem 1.1]. Our examples show that e_n must grow at least doubly exponentially, roughly like 2^{2^n}.

In low dimensions, our examples are good but not optimal. In dimension 2, Theorem 5.1 gives the klt Fano surface of degree 256 in $\mathbf{P}^3(128, 69, 49, 11)$. The optimal bottom weight (for quasi-smooth hypersurfaces of dimension 2 with $K_X = O_X(-1)$) is 13, which occurs in the examples $X_{256} \subset \mathbf{P}^3(128, 81, 35, 13)$ and $X_{127} \subset \mathbf{P}^3(57, 35, 23, 13)$. In dimension 3, Theorem 5.1 gives the klt Fano 3-fold

$$X_{336960} \subset \mathbf{P}^4(168480, 112320, 46837, 9101, 223),$$

which has $K_X = O_X(-1)$. The optimal bottom weight here is 407, from Johnson and Kollár's klt Fano 3-fold [12, Remark 3]:

$$X_{37584} \subset \mathbf{P}^4(18792, 12528, 5311, 547, 407).$$

So Theorem 5.1 has excellent asymptotics in high dimensions, but it is not optimal.

Our klt Fano varieties also have fairly small volume of $-K_X$; but that has no particular significance, because the volume of klt Fano varieties in a given dimension can be arbitrarily small. (For example, for any positive integer a, the weighted projective plane $Y = \mathbf{P}^2(2a+1, 2a, 2a-1)$ is a klt Fano surface with $\mathrm{vol}(-K_Y) = 18a/(4a^2-1)$.)

The definition of our klt Fano varieties is much like that of the klt varieties of general type in Theorem 4.1. Again, let $c_0, c_1, \ldots$ be Sylvester's sequence; see section 2 for the properties of that sequence. We use the five sequences of polynomials f_i, e_i, b_i, z_i and d_i in $\mathbf{Z}[y]$ from section 3. The one slightly different polynomial we need here is $\widetilde{d_i} := -e_i + b_i(f_i - 1)$, in place of $d_i = e_i + b_i(f_i - 1)$.

Theorem 5.1. *Let r be an odd integer at least 3 and let n be an integer at least $r-1$. Define integers $a_0, \ldots, a_{n+1}$ as follows. Let $y = c_{n-r+2} - 1$ and*

$$\begin{aligned}
a_{r-1} &= b_{r-1} \\
a_0 &= \widetilde{d}_{r-1} - (z_{r-1} - y)a_{r-1} \\
a_1 &= \widetilde{d}_{r-1} - f_0 a_0 \\
&\cdots \\
a_{r-2} &= \widetilde{d}_{r-1} - f_{r-3}a_{r-3}
\end{aligned}$$

These are positive integers. Let $x = -1 + a_0 + \cdots + a_{r-1}$; then $\widetilde{d}_{r-1} = yx = c_0 \cdots c_{n-r+1}x$. Let $a_{r+i} = c_0 \cdots \widehat{c_i} \cdots c_{n-r+1}x$ for $0 \leq i \leq n-r+1$. Let X be a general hypersurface of degree $\widetilde{d}_{r-1}$ in the complex weighted projective space $\mathbf{P}(a_0, \ldots, a_{n+1})$. Then X is a klt Fano variety of dimension n, and

$$H^0(X, -mK_X) = 0$$

for all $1 \leq m < b_{r-1}$. Here $b_{r-1} \geq (c_{n-r+2} - 1)^{2^{r-1}-1}$.

Taking $r = n+1$ if n is even and $r = n$ if n is odd, we deduce that the bottom weight b_{r-1} is at least 2^{2^n-1} if n is even and at least $6^{2^{n-1}-1}$ if n is odd.

Proof. The proof is identical to that of Theorem 4.1, with sign changes where needed. For example, in place of the identity $d_i = (-1)^i + f_0 \cdots f_{i-1}(z_i + y)$, use that $\widetilde{d}_i = (-1)^i + f_0 \cdots f_{i-1}(z_i - y)$. As in the proof of Theorem 4.1, start by showing that $a_{r-1} = \widetilde{d}_{r-1} - f_{r-2}a_{r-2}$. The analog of Lemma 4.3 says that

$$\begin{aligned}
a_{r-1} &= b_{r-1} \\
a_{r-2} &= e_{r-2}(-1 + a_{r-1}) - a_{r-1} \\
a_{r-3} &= e_{r-3}(-1 + a_{r-1} + a_{r-2}) - a_{r-2} \\
&\cdots \\
a_0 &= e_0(-1 + a_{r-1} + \cdots + a_1) - a_1.
\end{aligned}$$

That makes it clear that the a_i's are positive integers. The rest of the proof shows that X is a well-formed quasi-smooth hypersurface with $K_X = O_X(-1)$, and its bottom weight is b_{r-1}. □

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