Chow groups with twisted coefficients

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It is natural to ask whether we can define the Chow group of algebraic cycles with twisted coefficients, as we can do for cohomology. Rost gave such a definition. Namely, he defined Chow groups with coefficients in a locally constant étale sheaf E, assuming that E is torsion of exponent invertible in the base field k [34, Remarks 1.11 and 2.5]. We generalize the definition so that E need not be torsion (and, in characteristic $p > 0$, p need not act invertibly on E). It was not obvious how to make such a definition, because Chow groups are defined in terms of the Zariski topology, not the étale topology. For the constant sheaf $\mathbf{Z}_X$, the twisted Chow group $CH_i(X, \mathbf{Z}_X)$ is the usual Chow group CH_iX.

Chow groups with twisted coefficients have hardly been studied at all, although Rost's more general notion of cycle modules has been influential. The main goal of this paper is to present some methods for computing twisted Chow groups, in the hope of making these groups more useful in practice. This continues the grand theme of importing ideas from homotopy theory into algebraic geometry, in order to understand torsion phenomena.

Twisted Chow groups are directly related to Serre's notion of "negligible cohomology" for finite groups [15, section 26]. This paper was prompted by a remarkable computation of negligible cohomology by Merkurjev and Scavia [31], which we generalize in terms of twisted Chow groups (Theorem 8.1). In short, twisted Chow groups are always generated by the Chow groups of suitable covering spaces. (For now, we are not giving new calculations of the usual Chow groups.) Twisted Chow groups tensored with the rationals can easily be computed in terms of the Chow groups of suitable covering spaces (Lemma 2.2), and so their main novelty is integral or modulo a prime number.

Guillot, Di Lorenzo, and Pirisi defined equivariant Chow groups with coefficients in any cycle module [21, 11]. In particular, that gives a definition of equivariant Chow groups with coefficients, $CH^*_G(X, M)$ for a $\mathbf{Z}G$-module M (section 5). An interesting special case is when X is a point; then we get a definition of twisted Chow groups of the classifying space of G, $CH^*(BG_k, M)$. (These groups map to the cohomology of G with coefficients in M.) We prove some general bounds for generators of $CH^*(BG_k, M)$ (Theorems 9.2 and 12.1). We give complete calculations of these invariants (for all G-modules) when G is cyclic, quaternion, or the group $\mathbf{Z}/2 \times \mathbf{Z}/2$ (Theorems 9.1, 11.1, 13.1). The results are related to group cohomology, but with some striking differences (Remarks 12.2 and 13.2).

Heller, Voineagu, and Østvær have defined twisted motivic cohomology [22]. We reformulate the definition and provide some computational tools, such as relating twisted motivic cohomology with twisted higher Chow groups (section 4). Surprisingly, there is a surjection from $H^{2i}_{\mathrm{M}}(X, E(i))$ to the twisted Chow group $CH^i(X, E)$, but it is not always an isomorphism (Theorems 4.5 and 14.1). In the example, the monodromy group of E is $\mathbf{Z}/2 \times \mathbf{Z}/2$. The example depends on relating twisted Chow groups with the theory of algebraic tori, such as the notion of coflasque resolutions.

I believe that twisted Chow groups and twisted motivic cohomology are both worth investigating. Each theory has its own advantages (Remark 14.2). More generally, we conjecture a definition of Chow groups twisted by any birational sheaf with transfers, in the sense of Kahn–Sujatha (Conjecture C.1).

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1 Definition

We start by recalling Rost's definition of Chow groups with twisted coefficients in the torsion case. The definition mixes the Zariski and étale topologies in a nontrivial way.

Definition 1.1. Let X be a separated scheme of finite type over a field k. Let E be a locally constant étale sheaf on X which is killed by a positive integer r that is invertible in k. Then $CH_i(X, E)$ is defined to be the cokernel of the residue map on Galois cohomology groups:

$$\oplus_{x \in X_{(i+1)}} H^1(k(x), E(1)) \to \oplus_{x \in X_{(i)}} H^0(k(x), E).$$

Here $E(1)$ means the Galois module $E \otimes_{\mathbf{Z}/r} \mu_r$, where μ_r denotes the rth roots of unity in a separable closure of k. Also, $X_{(i)}$ means the set of points of the scheme X whose closure has dimension i.

For $E = (\mathbf{Z}/r)_X$, we have $H^1(k(x), E(1)) \cong H^1(k(x), \mu_r) \cong k(x)^*/(k(x)^*)^r$, and so $CH_i(X, \mathbf{Z}/r)$ is the usual Chow group modulo r, $CH_i(X)/r$. *Throughout the paper, cohomology of fields with no topology specified will denote Galois cohomology (or equivalently, étale cohomology).* Schemes of finite type over a field will be assumed to be separated.

Even though the definition involves étale cohomology of fields, the twisted Chow groups of X cannot be viewed as étale cohomology of X, regardless of the choice of coefficients. In particular, for X smooth over k, there is a natural homomorphism from $CH^i(X, E)$ to étale motivic cohomology $H^{2i}_{\text{et}}(X, E(i))$ (Theorem 6.1), and this is an isomorphism rationally, but not integrally. For example, when k is the complex numbers $\mathbf{C}$, étale motivic cohomology $H^{2i}_{\text{et}}(X, \mathbf{Z}/r(i))$ can be identified with ordinary cohomology, $H^{2i}(X(\mathbf{C}), \mathbf{Z}/r)$. That is usually quite different from $CH^i(X, \mathbf{Z}/r) = CH^i(X)/r$.

In this paper, we generalize the definition of twisted Chow groups to any locally constant étale sheaf E of abelian groups, not necessarily torsion, as follows.

Definition 1.2. Let X be a separated scheme of finite type over a field k. Let E be a locally constant étale sheaf on X. Then $CH_i(X, E)$ is defined to be the cokernel of the residue map on Galois cohomology groups:

$$\oplus_{x \in X_{(i+1)}} H^1(k(x), E(1)) \to \oplus_{x \in X_{(i)}} H^0(k(x), E).$$

Here we interpret $\mathbf{Z}(1)$ as $G_m[-1]$, a shift of the multiplicative group in the derived category of étale sheaves over a field, as in Voevodsky's theory of motivic cohomology [30, Theorem 4.1]. Define $E(1)$ as the derived tensor product $E \otimes^L_{\mathbf{Z}} G_m[-1]$.

This agrees with Rost's definition (Definition 1.1) when E is torsion of exponent invertible in k. Also, when E is the constant étale sheaf $\mathbf{Z}_X$, $CH_i(X, \mathbf{Z}_X)$ is the usual Chow group CH_iX. That makes it quite natural to consider non-torsion coefficients. See also Conjecture C.1 for a more general attempt to define Chow groups twisted by any birational motivic sheaf.

Remark 1.3. Define an étale sheaf E of abelian groups on a scheme X to be *naively locally constant* if there is an étale covering $\{X_\alpha \to X\}$ on which E is constant. For convenience, define an étale sheaf to be *locally constant* if it is a direct limit of naively locally constant sheaves. For X connected with a choice of a geometric base point, locally constant sheaves in this sense correspond to discrete abelian groups M with a continuous action of $\pi_1^{\text{et}}X$, whereas naively locally constant sheaves correspond to the special case where M is fixed by some open subgroup of $\pi_1^{\text{et}}X$.

Rost's arguments imply essentially all the formal properties of Chow groups with twisted coefficients, although we need an extra argument (Corollary A.3) to avoid inverting the exponential characteristic e of k. (By definition, $e = 1$ if k has characteristic zero, and $e = p$ if k has characteristic $p > 0$.) Let us see how this works. The first step is to observe that a locally constant étale sheaf E on a scheme X of finite type over k determines a cycle module $H^*[E]$ on X. To describe what this means, first define a *field over* X to be a field F with a morphism $\operatorname{Spec} F \to X$ such that F is finitely generated over k. Then a *cycle module* M on X is a $\mathbf{Z}$-graded abelian group $M(F)$ associated to every field over X, along with various operations on these groups. In our case, given a locally constant étale sheaf E on X, we define $H^*[E](F)$ as the étale cohomology groups:

$$H^*[E](F) = \oplus_{j \geq 0} H^j(F, E(j)).$$

Here $\mathbf{Z}(j)$ denotes the étale sheafification of Voevodsky's motivic cohomology complex, and $E(j) = E \otimes^L_{\mathbf{Z}} \mathbf{Z}(j)$. In particular, if E is torsion of exponent r invertible in k, then we have the more elementary description $E(j) \cong E \otimes_{\mathbf{Z}/r} \mu_r^{\otimes j}$ in $D(X_{\text{et}})$, giving the description of the cycle module $H^*[E]$ from Rost's paper [34, Remark 1.11].

For clarity, we recall the operations required of a cycle module. First, for each inclusion $\varphi\colon F_1 \to F_2$ of fields over X, we are given a homomorphism $\varphi_*\colon M(F_1) \to M(F_2)$ of degree 0 (a "pullback" homomorphism, in geometric language). For each finite extension $\varphi\colon F_1 \to F_2$ of fields over X, we have a "transfer" or "pushforward" homomorphism $\varphi^*\colon M(F_2) \to M(F_1)$ of degree 0. For each field F over X, the group $M(F)$ is a graded left module over the Milnor K-theory ring $K_*^M F$. Finally, suppose that a field F has a "valuation over X", meaning a discrete valuation v together with a morphism $\operatorname{Spec} O_v \to X$ such that O_v is the local ring of a normal variety over X at a point of codimension 1. Then we are given a "residue" homomorphism $\partial_v\colon M(F) \to M(k(v))$ of degree -1, where $k(v)$ is the residue field of v. We omit the relations that these operations are required to satisfy, for M to be called a cycle module [34, definitions 1.2 and 2.1].

We need the étale sheaf E on X to be locally constant in order to define the residue homomorphisms on $H^*[E]$. Namely, for a discrete valuation v over X on a field F with residue field $k(v)$, E pulls back to a locally constant étale sheaf on $\operatorname{Spec} O_v$, and that is the situation in which we have a residue homomorphism on étale motivic cohomology:

$$\partial_v \colon H^b(F, E(b)) \to H^{b-1}(k(v), E(b-1)).$$

This is easier to construct if E is a $\mathbf{Z}[1/e]$-module; then it comes from an isomorphism in the derived category of étale sheaves on $\operatorname{Spec} k(v)$:

$$\mathbf{Z}[1/e](b-1)_{k(v)}[-2] \cong i^! \mathbf{Z}[1/e](b)_{O_v}$$

for any $b \geq 1$, where $i\colon \operatorname{Spec} k(v) \to \operatorname{Spec} O_v$ is the inclusion and $i^!$ is the exceptional inverse image functor [6, proof of Proposition 7.1.10]. This fits into the localization sequence for étale motivic cohomology due to Cisinski and Déglise: for a regular closed subscheme Y of codimension r in a regular excellent scheme X, we have a long exact sequence [6, Theorem 5.6.2 and Proposition 7.1.6]:

$$\begin{aligned} H^{i-2r}_{\text{et}}(Y, \mathbf{Z}[1/e](j-r)) \to H^i_{\text{et}}(X, \mathbf{Z}[1/e](j)) \to H^i_{\text{et}}(X-Y, \mathbf{Z}[1/e](j)) \\ \to H^{i-2r+1}_{\text{et}}(Y, \mathbf{Z}[1/e](j-r)). \end{aligned}$$

Here $\mathbf{Z}[1/e](i)$ is the étale sheafification of the usual motivic cohomology complex (with e inverted) for $i \geq 0$, whereas it is torsion for $i < 0$: $\mathbf{Z}[1/e](i) \cong \oplus_{l \neq e} \mathbf{Q}_l/\mathbf{Z}_l(i)[-1]$ for $i < 0$.

In characteristic $p > 0$, the residue map in the derived category of étale sheaves does not exist without inverting $e = p$. Nonetheless, we construct a residue homomorphism

$$\partial_v \colon H^b(F, E(b)) \to H^{b-1}(k(v), E(b-1))$$

for a locally constant étale sheaf E on $\operatorname{Spec} O_v$ (without inverting p) in Corollary A.3. As a result, $H^*[E]$ is a cycle module, and so Chow groups with twisted coefficients satisfy the desired properties without having to invert the exponential characteristic.

Once we know that $H^*[E]$ is a cycle module, Rost's theory implies essentially all the formal properties one would want for Chow groups with twisted coefficients. We have:

- Proper pushforward. For a proper morphism $f\colon X \to Y$ of schemes over k and a locally constant étale sheaf E on Y, we have a homomorphism

  $$f_* \colon CH_i(X, f^*E) \to CH_i(Y, E)$$

  [34, section 5].

- Flat pullback. For a flat morphism $f\colon X \to Y$ of relative dimension n and a locally constant étale sheaf E on Y, we have a homomorphism

  $$f^* \colon CH_i(Y, E) \to CH_{i+n}(X, f^*E)$$

  [34, section 5].

- Localization sequence. For a closed subscheme $Z \subset X$ and a locally constant étale sheaf E on X, we have an exact sequence

$$CH_i(Z,E) \to CH_i(X,E) \to CH_i(X-Z,E) \to 0.$$

  This sequence can be extended to the left, using the cycle module $H^*[E]$. Writing $C_i(X,E)_j$ for Rost's $C_i(X;H^*[E],j)$, we define

$$C_i(X,E)_j = \oplus_{x \in X_{(i)}} H^{i+j}(k(x), E(i+j)).$$

  Let $A_i(X,E)_j$ be the homology of the boundary maps on these groups,

$$C_{i+1}(X,E)_j \to C_i(X,E)_j \to C_{i-1}(X,E)_j.$$

  Then the localization sequence extends to a long exact sequence [34, section 5]:

$$\begin{aligned} \cdots \to A_{i+1}(X-Z,E)_{-i} & \\ & \to A_i(Z,E)_{-i} \to A_i(X,E)_{-i} \to A_i(X-Z,E)_{-i} \to 0. \end{aligned}$$

- Homotopy invariance. For an affine bundle $\pi\colon V \to X$ of relative dimension n and a locally constant étale sheaf E on X, the pullback

$$\pi^*\colon CH_i(X,E) \to CH_{i+n}(V, \pi^*E)$$

  is an isomorphism [34, Proposition 8.6].

- Products. For a smooth scheme X over k, write $CH^i(X,E)$ for the codimension-i Chow group with coefficients. (Thus, if X has dimension n everywhere, we have $CH^i(X,E) = CH_{n-i}(X,E)$.) Then $CH^*(X,E)$ is a module over the usual Chow ring CH^*X [34, section 14].

- Pullback for smooth schemes. For any morphism $f\colon X \to Y$ of smooth schemes over k and a locally constant étale sheaf E on Y, we have a homomorphism

$$f^*\colon CH^i(Y,E) \to CH^i(X, f^*E)$$

  [34, section 12].

Rost also proves the expected compatibilities among these operations. For the constant étale sheaf A_X associated to an abelian group A, the operations above are the usual operations on $CH_i(X,A_X) \cong CH_i(X) \otimes_{\mathbf{Z}} A$.

2 Basic calculations

We now give some basic calculations of Chow groups with twisted coefficients, emphasizing cases in which they reduce to the usual Chow groups.

Lemma 2.1. *Let $f\colon Y \to X$ be a finite étale morphism of schemes of finite type over a field k. For a locally constant étale sheaf E on Y,*

$$CH_i(X, f_*E) \cong CH_i(Y,E).$$

Equivalently, Chow groups of X with coefficients in an induced representation E of the fundamental group reduce to Chow groups with coefficients for a covering space of X. When E is a permutation representation of the étale fundamental group $\pi_1 X$ (over some commutative ring R), $CH_i(X, E)$ is the usual Chow group $CH_i(Y) \otimes_{\mathbf{Z}} R$ of a covering space Y of X (possibly with several connected components).

Proof. (Lemma 2.1) One can prove this by hand, but an efficient approach is to use Rost's results about an arbitrary morphism $f: Y \to X$ [34, Corollary 8.2]. Namely, for any cycle module M on Y, there is a convergent "Leray-Serre" spectral sequence

$$E^2_{pq} = A_p(X, A_q[f; M]) \Rightarrow A_{p+q}(Y; M),$$

for some cycle modules $A_q[f; M]$ on X. Namely, for each field F over X, let $Y_F = Y \times_X \operatorname{Spec} F$. Then we define

$$A_q[f; M](F) = A_q(Y_F; M).$$

Let $f: Y \to X$ be a finite étale morphism, and let E be a locally constant étale sheaf on Y. In this case, Y_F has dimension 0 for each field F over X. So we read off that

$$A_q[f; H^*[E]] \cong \begin{cases} 0 & \text{if } q \neq 0 \\ H^*[f_* E] & \text{if } q = 0. \end{cases}$$

Therefore, the spectral sequence reduces to an isomorphism $A_i(X, H^*[f_* E]) \cong A_i(Y, H^*[E])$ of graded abelian groups. In degree $-i$, this gives that $CH_i(X, f_* E) \cong CH_i(Y, E)$, as we want. $\square$

Next, we consider the relation between Chow groups with twisted coefficients and the usual Chow groups. We get a complete answer with rational coefficients. Namely, let G be a finite group, $f: Y \to X$ an étale G-torsor, and E a $\mathbf{Z}G$-module. Then we can view E as a locally constant étale sheaf on X, and every sheaf associated to a representation of $\pi_1 X$ with finite image arises this way. In this situation, we have the flat pullback homomorphism

$$CH_i(X, E) \to CH_i(Y, f^* E) = CH_i(Y) \otimes_{\mathbf{Z}} E.$$

Since $f^* E$ is a G-equivariant sheaf on Y, this homomorphism lands in the G-fixed subgroup:

$$CH_i(X, E) \to (CH_i(Y) \otimes_{\mathbf{Z}} E)^G.$$

One may ask how close this is to an isomorphism; it is always an isomorphism tensor $\mathbf{Q}$, as we will see.

Since f is finite, we also have the pushforward homomorphism

$$CH_i(Y) \otimes_{\mathbf{Z}} E = CH_i(Y, f^* E) \to CH_i(X, E).$$

Using that $f^* E$ is a G-equivariant sheaf on Y, this homomorphism factors through the coinvariants of G:

$$(CH_i(Y) \otimes_{\mathbf{Z}} E)_G \to CH_i(X, E).$$

Again, one may ask how close this is to an isomorphism.

Lemma 2.2. *Both homomorphisms above become isomorphisms tensor* $\mathbf{Q}$.

Proof. Consider the composition

$$CH_i(X,E) \to (CH_i(Y)\otimes E)^G \to (CH_i(Y)\otimes E)_G \to CH_i(X,E).$$

This is f_*f^* on twisted Chow groups, which is multiplication by $|G|$, as one can check on generators. Next, consider the composition

$$(CH_i(Y)\otimes E)^G \to (CH_i(Y)\otimes E)_G \to CH_i(X,E) \to (CH_i(Y)\otimes E)^G.$$

This is f^*f_*, which is the trace $\sum_{g\in G} g^*$. (This follows from the fact that proper pushforward commutes with flat pullback [34, Proposition 4.1(3)], applied to the Cartesian diagram

$$\begin{array}{ccc} Y\times_X Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X, \end{array}$$

where $Y\times_X Y \cong Y\times G$.)

Since we are considering f^*f_* on the G-invariant subgroup of $CH_i(Y)\otimes E$, it follows that f^*f_* is multiplication by $|G|$ on this subgroup. Thus, tensoring with $\mathbf{Q}$, f^* gives an isomorphism

$$CH_i(X,E)\otimes\mathbf{Q} \cong (CH_i(Y)\otimes E)^G\otimes\mathbf{Q},$$

as we want. Also, for any abelian group A with an action of the finite group G, the natural homomorphism $A^G \to A_G$ becomes an isomorphism tensor $\mathbf{Q}$. It follows that f_* also gives an isomorphism tensor $\mathbf{Q}$:

$$(CH_i(Y)\otimes E)_G\otimes\mathbf{Q} \cong CH_i(X,E)\otimes\mathbf{Q}.$$

□

Finally, we observe that Chow groups with twisted coefficients do not always have the exactness properties one might wish for, by analogy with cohomology. For example, given an exact sequence $0\to A\to B\to C\to 0$ of locally constant étale sheaves on X, we have a long exact sequence of étale cohomology groups,

$$\cdots \to H^i_{\text{et}}(X,A) \to H^i_{\text{et}}(X,B) \to H^i_{\text{et}}(X,C) \to H^{i+1}_{\text{et}}(X,A) \to \cdots.$$

(When $k=\mathbf{C}$, we also have the analogous sequence of ordinary cohomology groups with twisted coefficients.) For twisted Chow groups, we will give a partial substitute in Theorem 3.1.

Lemma 2.3. *Let* $0\to A\to B\to C\to 0$ *be a short exact sequence of locally constant étale sheaves on a scheme* X *of finite type over* k. *Then the complex* $0\to CH_i(X,A)\to CH_i(X,B)\to CH_i(X,C)\to 0$ *need not be exact at any of the three terms. There are counterexamples with* X *smooth over* k.

Proof. Let G be the group $\mathbf{Z}/2$. The regular representation of G over $\mathbf{F}_2$, $B=\mathbf{F}_2G$, fits into a short exact sequence $0\to A\to B\to C\to 0$ of G-modules, with both

A and C isomorphic to the trivial representation $\mathbf{F}_2$. We will give the desired counterexamples for this coefficient sequence.

Let $f\colon Y \to X$ be a finite étale morphism of degree 2. Let $B = f_*(\mathbf{F}_2)_Y$. Then the exact sequence above gives an exact sequence $0 \to A \to B \to C \to 0$ of locally constant étale sheaves on X, with both A and C isomorphic to $(\mathbf{F}_2)_X$. By Lemma 2.1, the resulting complex of Chow groups with twisted coefficients has the form:

$$0 \to CH_i(X)/2 \to CH_i(Y)/2 \to CH_i(X)/2 \to 0,$$

where the first homomorphism is pullback and the second is pushforward.

For example, take $k = \mathbf{C}$, $X = A^1 - 0$, $Y = A^1 - 0$, and define $f\colon Y \to X$ by $f(y) = y^2$. Then exactness fails on the right for $i = 1$. (The generator of $CH_1 Y = \mathbf{Z}$ maps to 2 times the generator of $CH_1 X = \mathbf{Z}$, hence to zero modulo 2.) Next, take $k = \mathbf{C}$, $X = (A^2 - 0)/G$ (where $G = \mathbf{Z}/2$ acts by ± 1), and $Y = A^2 - 0$. Then $CH_1 X \cong CH^1 BG \cong \mathbf{Z}/2$, whereas $CH_1 Y = 0$, and so the sequence is not exact on the left.

For the middle, let $k = \mathbf{Q}$ and let E be an elliptic curve over $\mathbf{Q}$ such that the Mordell-Weil group $E(\mathbf{Q})$ contains $(\mathbf{Z}/2)^2$ and has rank at least 1. (For example, E could be the curve [29, Elliptic Curve 117.a3].) Let Y be E minus the 2-torsion subgroup $E[2]$; so Y is E minus 4 rational points. Let $G = \mathbf{Z}/2$ act on Y by ± 1; then G acts freely on Y, and $X := Y/G$ is isomorphic to $\mathbf{P}^1_{\mathbf{Q}}$ minus 4 rational points. So $CH_0 X = 0$ and hence $CH_0(X)/2 = 0$. On the other hand, $CH_0 Y \cong E(\mathbf{Q})/E[2]$, and so $CH_0(Y)/2 \neq 0$. Thus the sequence $CH_0(X)/2 \to CH_0(Y)/2 \to CH_0(X)/2$ is not exact. $\square$

Remark 2.4. We can also give examples over $\mathbf{C}$ for which exactness fails in the middle, in Lemma 2.3. Let M be the K3 surface which is the double cover of $\mathbf{P}^2_{\mathbf{C}}$ ramified along the smooth sextic curve $C = \{0 = x^6 + y^6 + z^6 - 10(x^3y^3 + y^3z^3 + z^3x^3)\}$. Mukai observed that the automorphism group of M contains the Mathieu group $M_9 = 3^2 Q_8$ of order 72, one of the largest finite groups of symplectic automorphisms of a K3 surface [33, Theorem 0.3]. His arguments imply that M has Picard group $\mathbf{Z}^{20}$. Let $X = \mathbf{P}^2 - C$, which has the étale double cover $Y := M - C$. Then $CH^1 X \cong \mathbf{Z}/6$, and so $CH^1(X)/2 \cong \mathbf{Z}/2$. On the other hand, C has genus 10, and so $C^2 = 2g - 2 = 18$ on the K3 surface M. It follows that C is not divisible by 2 in $CH^1 M$: if $C \sim 2D$, then $C^2 = 4D^2$, but C^2 is not zero modulo 4. So $CH^1(Y)/2 = (CH^1(M)/2)/\langle C\rangle$ is isomorphic to $(\mathbf{Z}/2)^{19}$. It follows that the sequence

$$CH^1(X)/2 \to CH^1(Y)/2 \to CH^1(X)/2$$

is not exact.

3 Chow groups and coflasque resolutions

We now give a sufficient condition for an exact sequence of coefficient modules to give an exact sequence of twisted Chow groups. The statement uses the notion of a coflasque resolution from the theory of algebraic tori.

Let G be a finite group and M a $\mathbf{Z}G$-lattice, meaning a finitely generated $\mathbf{Z}G$-module that is $\mathbf{Z}$-torsion free. Following Colliot-Thélène and Sansuc, M is called *invertible* if it is a summand of a permutation module [8, section 0.5]. Next, M is *coflasque* if $H^1(H, M) = 0$ for every subgroup H of G. Finally, M is *flasque* if the dual lattice M^* is coflasque. An invertible $\mathbf{Z}G$-lattice is flasque and coflasque. By Endo and Miyata, every coflasque $\mathbf{Z}G$-lattice is invertible if and only if every Sylow subgroup of G is cyclic [7, Proposition 2].

More generally, let R be a Dedekind domain that is $\mathbf{Z}$-torsion free. We can then make the same definitions for an RG-lattice, meaning a finitely generated RG-module that is R-torsion free. For example, for a prime number p, the localization $R = \mathbf{Z}_{(p)}$ or the completion $R = \mathbf{Z}_p$ come up naturally.

For a profinite group L (such as the étale fundamental group of a scheme), we define an R-module M with continuous L-action to be invertible, coflasque, or flasque if there is a finite quotient group G of L such that M is an RG-lattice with the corresponding property.

For a finite group G, every finitely generated RG-module M has a *coflasque resolution*

$$0 \to Q \to P \to M \to 0,$$

meaning that P is a permutation module over R and Q is coflasque [8, Lemma 0.6]. Moreover, Q is determined by M up to direct sums with permutation modules.

Theorem 3.1. *Let X be a k-scheme of finite type, and let $0 \to A \to B \to C \to 0$ be an exact sequence of locally constant étale sheaves on X. Let i be an integer.*

(1) If A is coflasque, then $CH_i(X, B) \to CH_i(X, C) \to 0$ is exact.

(2) If A is invertible, then $CH_i(X, A) \to CH_i(X, B) \to CH_i(X, C) \to 0$ is exact.

Example 3.2. In some cases, Theorem 3.1 describes Chow groups with twisted coefficients in terms of the usual Chow groups of varieties. For example, let G be a finite group, and let $M = \mathbf{Z}G/\mathbf{Z}$. (For example, if $G = \mathbf{Z}/2$, then M is $\mathbf{Z}$ with G acting by ± 1.) Let X be the quotient of a k-scheme Y by a free G-action. (One could assume that Y is quasi-projective to ensure that X is a scheme, or use Remark 5.1.) Then applying Theorem 3.1 to the coflasque resolution $0 \to \mathbf{Z} \to \mathbf{Z}G \to M \to 0$ gives an exact sequence

$$CH_i X \to CH_i Y \to CH_i(X, M) \to 0,$$

where the first homomorphism is pullback. This describes $CH_i(X, M)$ in terms of Chow groups of varieties.

More generally, for any finite group G, every finitely generated $\mathbf{Z}G$-module M has a resolution $0 \to Q \to P \to M \to 0$ with P a permutation module and Q coflasque. By Theorem 3.1, given a homomorphism $\pi_1 X \to G$, $CH_i(X, M)$ is always a quotient of the usual Chow group CH_i of some covering space of X (possibly with several connected components). (This also follows from Theorem 8.1, below.) When G is cyclic, Q is invertible by Endo–Miyata's result above; in that case, Theorem 3.1 expresses $CH_i(X, M)$ more explicitly as a cokernel of a homomorphism between usual Chow groups.

Remark 3.3. In Theorem 3.1, if A is coflasque (but not invertible), $CH_i(X, A) \to CH_i(X, B) \to CH_i(X, C)$ need not be exact (Theorem 14.1).

Proof. (Theorem 3.1) Consider the diagram of étale cohomology groups, with exact columns:

$$\begin{array}{ccc}
\oplus_{x\in X_{(i+1)}} H^1(k(x), A(1)) & \longrightarrow & \oplus_{x\in X_{(i)}} H^0(k(x), A) \\
\downarrow & & \downarrow \\
\oplus_{x\in X_{(i+1)}} H^1(k(x), B(1)) & \longrightarrow & \oplus_{x\in X_{(i)}} H^0(k(x), B) \\
\downarrow & & \downarrow \\
\oplus_{x\in X_{(i+1)}} H^1(k(x), C(1)) & \longrightarrow & \oplus_{x\in X_{(i)}} H^0(k(x), C) \\
\downarrow & & \downarrow \\
\oplus_{x\in X_{(i+1)}} H^2(k(x), A(1)) & \longrightarrow & \oplus_{x\in X_{(i)}} H^1(k(x), A).
\end{array}$$

The cokernels of the first three horizontal maps are $CH_i(X, A)$, $CH_i(X, B)$, and $CH_i(X, C)$.

Proof of (1): Suppose that A is coflasque. Then $H^1(k(x), A) = 0$ for every point x in X. It follows that $H^0(k(x), B) \to H^0(k(x), C)$ is surjective for each point x in X. Therefore, $CH_i(X, B) \to CH_i(X, C)$ is surjective.

Proof of (2): Suppose that A is invertible. Then A is coflasque, and so (1) gives that $CH_i(X, B) \to CH_i(X, C)$ is surjective. Furthermore, since A is $\mathbf{Z}$-torsion free, we have $H^2(k(x), A(1)) \cong H^1(k(x), A \otimes_{\mathbf{Z}} G_m)$ for every point x in X. For $R = \mathbf{Z}$, that group is H^1 with coefficients in an algebraic torus over $k(x)$. Since A is invertible, this H^1 group is zero, using Hilbert's Theorem 90 that $H^1(F, G_m) = 0$ for every field F.

Then a diagram chase implies that $CH_i(X, A) \to CH_i(X, B) \to CH_i(X, C)$ is exact. In more detail, let u be an element of $CH_i(X, B)$ that maps to zero in $CH_i(X, C)$. Choose a representative for u in $Z_i(X, B) := \oplus_{x\in X_{(i)}} H^0(k(x), B)$. Then the image of u in $Z_i(X, C)$ is the boundary of some element y in $\oplus_{x\in X_{(i+1)}} H^1(k(x), C(1))$. Since $H^2(k(x), A(1)) = 0$ by the previous paragraph, y comes from some element z in $\oplus_{x\in X_{(i+1)}} H^1(k(x), B(1))$. Then $u - \partial z$ in $Z_i(X, B)$ maps to zero in $Z_i(X, C)$. Since the columns in the diagram above are exact, $u - \partial z$ comes from an element of $Z_i(X, A)$, as we want. □

4 Twisted motivic cohomology

In this section we define twisted motivic cohomology associated to a locally constant étale sheaf, following Heller–Voineagu–Østvær [22, section 5.2]. We show that twisted motivic cohomology $H^{2i}_{\mathrm{M}}(X, E(i))$ can be described as a twisted higher Chow group $CH^i(X, E, 0)$ (Corollary 4.4). A striking point is that this is not always isomorphic to the twisted Chow group $CH^i(X, E)$ from section 1 (Theorem 14.1). Both theories deserve to be investigated; we compare their advantages in Remark 14.2.

Heller, Voineagu, and Østvær consider an action of a finite group G on a smooth scheme Y over a field k, with the order of G invertible in k. They define *Bredon motivic cohomology* with coefficients in a cohomological Mackey functor M for G, $H^i_G(Y, M(j))$. In this paper, we only consider the case where G acts freely on Y; equivalently, we are considering invariants of $X := Y/G$ with its given G-torsor. Then most of the information in the Mackey functor is irrelevant. Every $\mathbf{Z}G$-module E determines a cohomological Mackey functor M by setting $M(G/H) = E^H$. In this case, Heller–Voineagu–Østvær's theory coincides with twisted motivic cohomology $H^i_{\mathrm{M}}(X, E(j))$, as defined below.

We can imitate Hoyois's simple definition of motivic cohomology and the construction by Kahn–Levine and Elmanto–Nardin–Yakerson of motivic cohomology twisted by an Azumaya algebra [24, 26, 14]. Namely, let X be an arbitrary scheme, and let E be a locally constant étale sheaf on X. Then E determines a presheaf (also called E) on the category Sm_X of smooth schemes over X, taking a smooth morphism $\pi\colon Y \to X$ to $H^0(Y, \pi^*E)$. Since this is an étale sheaf, it is a Nisnevich sheaf on Sm_X. (The Nisnevich topology is defined for arbitrary schemes in [23, Appendix C].) Also, E has transfers for finite locally free morphisms in Sm_X (cf. [30, Lemma 6.11]). A fortiori, E has framed transfers (that is, transfers for finite syntomic morphisms with a trivialization of the cotangent complex). As such, E defines a space E_X in the framed motivic homotopy category $\mathrm{H}^{\mathrm{fr}}(X)$. So E defines a motivic spectrum $HE_X := \Sigma^\infty_{\mathrm{T,fr}} E_X$ in the stable homotopy category $\mathrm{SH}(X)$. (This is not the suspension spectrum of a motivic space. Rather, a *framed* motivic space is analogous to an E_∞ space in topology, and $\Sigma^\infty_{\mathrm{T,fr}}$ denotes the left adjoint to the functor $\Omega^{\infty,\mathrm{fr}}_{\mathrm{T}}$ from $\mathrm{SH}(X)$ to $\mathrm{H}^{\mathrm{fr}}(X)$ (not to $\mathrm{H}(X)$).) For the constant sheaf $E = \mathbf{Z}_X$, Hoyois showed that this object $H\mathbf{Z}_X$ coincides with Spitzweck's motivic cohomology spectrum in $\mathrm{SH}(X)$, for every scheme X [24, Theorem 21].

In particular, this gives a definition of E-twisted motivic cohomology:

$$H^i_{\mathrm{M}}(X, E(j)) = \pi_{2j-i} \operatorname{map}_{\mathrm{SH}(X)}(\Sigma^\infty_{\mathrm{T}} X_+, \Sigma^j_{\mathrm{T}} HE_X).$$

This is analogous to the definition of motivic cohomology twisted by an Azumaya algebra [14, Definition 5.17]. In particular, this agrees with Spitzweck's definition of motivic cohomology when E is a constant sheaf, and with Voevodsky's definition when in addition X is defined over a field.

Lemma 4.1. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of locally constant étale sheaves on a scheme X. Suppose that A is coflasque. Then, for each integer j, we have a long exact sequence of twisted motivic cohomology groups:*

$$\cdots \to H^i_{\mathrm{M}}(X, A(j)) \to H^i_{\mathrm{M}}(X, B(j)) \to H^i_{\mathrm{M}}(X, C(j)) \to H^{i+1}_{\mathrm{M}}(X, A(j)) \to \cdots$$

Remark 4.2. Using this lemma, we can always describe $H^{2i}_{\mathrm{M}}(X, C(i))$ as the cokernel of an explicit map between the usual Chow groups of covering spaces of X (not necessarily connected). Namely, assume that C is a finitely generated $\mathbf{Z}$-module with an action of $\pi_1^{\mathrm{et}} X$, and let $0 \to A \to B \to C \to 0$ be a coflasque resolution. We have $H^{2i+1}_{\mathrm{M}}(X, A(i)) = 0$ by the relation to twisted higher Chow groups (Corollary 4.4 below), and so Lemma 4.1 expresses $H^{2i}_{\mathrm{M}}(X, C(i))$ as the cokernel of $H^{2i}_{\mathrm{M}}(X, A(i)) \to H^{2i}_{\mathrm{M}}(X, B(i))$. By taking a coflasque resolution of A, we can describe $H^{2i}_{\mathrm{M}}(X, A(i))$ as the image of a homomorphism from $H^{2i}_{\mathrm{M}}(X, P(i))$, for some permutation module P. Since P and B are permutation modules, we have expressed $H^{2i}_{\mathrm{M}}(X, C(i))$ as the cokernel of a map between the usual Chow groups of covering spaces of X.

The twisted Chow groups can likewise be expressed as the image of a homomorphism from the usual Chow groups of some covering space by Theorem 3.1 (or by the surjective homomorphism $H^{2i}_{\mathrm{M}}(X, C(i)) \to CH^i(X, C)$ from Theorem 4.5, below). I don't know a simple description of the kernel, though.

Proof. For any exact sequence $0 \to A \to B \to C \to 0$ of locally constant étale sheaves on X, we have an exact sequence $0 \to A_X \to B_X \to C_X$ of Nisnevich sheaves with transfer. Suppose in addition that A is coflasque. Let Y be a smooth scheme over X, and let L be the henselization of the local ring of Y at a point x in Y. (We are not taking the *strict* henselization. So the residue field of L is $k(x)$, not the separable closure of $k(x)$.) I claim that $H^1_{\text{et}}(L, A) = 0$. Indeed, we have $H^1_{\text{et}}(L, A) \cong H^1(k(x), A)$. Let G be the image of $\pi_1(\operatorname{Spec} L) = \operatorname{Gal}(k(x)_s/k(x))$ acting on A, which is a finite group. Let H be the kernel of $\operatorname{Gal}(k(x)_s/k(x)) \to G$. Then the Lyndon–Hochschild–Serre spectral sequence gives an exact sequence

$$0 \to H^1(G, A) \to H^1(k(x), A) \to H^1(H, A)^G.$$

Here $H^1(G, A)$ is zero since A is coflasque, and $H^1(H, A) = \operatorname{Hom}(H, A)$ is zero since H is profinite and A is a discrete torsion-free abelian group. So $H^1_{\text{et}}(L, A) = H^1(k(x), A) = 0$, as claimed.

For each henselian local ring L of a scheme Y in Sm_X, we have an exact sequence of étale cohomology:

$$0 \to A(L) \to B(L) \to C(L) \to H^1_{\text{et}}(L, A).$$

Therefore, the previous paragraph gives that $0 \to A(L) \to B(L) \to C(L) \to 0$ is exact. That is, $0 \to A_X \to B_X \to C_X \to 0$ is an exact sequence of Nisnevich sheaves on X [30, p. 90]. As a result, we get an exact triangle $HA_X \to HB_X \to HC_X$ in $\mathrm{SH}(X)$. That implies the desired long exact sequence. □

We can also define twisted higher Chow groups. These groups should agree with twisted motivic cohomology for a regular scheme. We prove this in bidegrees $(2j, j)$, and in any bidegree when X is smooth over a perfect field k and the étale sheaf E is pulled back from k (Lemma 4.3 and Corollary 4.4). Namely, for an equidimensional scheme X with a locally constant étale sheaf E, define the jth *twisted Bloch cycle complex* $z^j(X, E)$ as the simplicial abelian group (or the associated chain complex) which in degree d is given by

$$\oplus_{z \in X \times \Delta^d} H^0(k(z), E),$$

where the sum is over all codimension-j points of $X \times \Delta^d$ whose closure intersects all faces of $X \times \Delta^d$ in the expected dimension. (Here Δ^d denotes the algebraic simplex $\{x_0 + \cdots + x_d = 0\}$ in A^{d+1}, as in Bloch's definition of higher Chow groups.) The boundary maps in the chain complex are given by intersections with faces of $X \times \Delta^d$. Write $CH^j(X, E, d)$ for the homotopy group π_d of the simplicial abelian group $z^j(X, E)$ (or, equivalently, H_d of the associated chain complex).

Lemma 4.3. *Let X be an equidimensional smooth scheme of finite type over a perfect field k. Let E be an étale sheaf over k, pulled back to X. Then twisted motivic cohomology agrees with twisted higher Chow groups:*

$$H^i_{\mathrm{M}}(X, E(j)) \cong CH^j(X, E, 2j - i).$$

Proof. We follow the proof of the analogous isomorphism for motivic cohomology twisted by an Azumaya algebra [14, Proposition 5.15]. The key point is that the A^1-invariant Nisnevich sheaf E_k on Sm_k is birational, in the sense that $E_k(Y) \cong E_k(U)$

for a dense open subset U of Y in Sm_k. (This follows from the fact that Y is normal [36, Tag 0BQI].) In terms of Voevodsky's slice filtration, it follows that HE_k is a 0-slice in $\mathrm{SH}(k)$, meaning that $HE_k \simeq s_0 HE_k$ [14, Proposition 5.3]. We can rewrite the spectrum used to define the twisted motivic cohomology of X in terms of the morphism $f\colon X \to \operatorname{Spec} k$:

$$\begin{aligned}\mathrm{map}_{SH(X)}(\Sigma^\infty_{\mathrm{T}} X_+, \Sigma^j_{\mathrm{T}} HE_X) &\simeq \mathrm{map}_{SH(X)}(\Sigma^\infty_{\mathrm{T}} X_+, f^*(\Sigma^j_{\mathrm{T}} HE_k))\\ &\simeq \mathrm{map}_{SH(k)}(Lf_\#(\Sigma^\infty_{\mathrm{T}} X_+), \Sigma^j_{\mathrm{T}} HE_k)\\ &= \mathrm{map}_{SH(k)}(\Sigma^\infty_{\mathrm{T}} X_+, \Sigma^j_{\mathrm{T}} HE_k),\end{aligned}$$

where in the last line we write $\Sigma^\infty_{\mathrm{T}} X_+$ for the object of $\mathrm{SH}(k)$ associated to X. By Levine's results on the homotopy coniveau tower, using the smoothness of X over k, it follows that this mapping spectrum in $\mathrm{SH}(k)$ is equivalent to the simplicial spectrum

$$\oplus_{z \in X \times \Delta^\bullet} (s_0 HE_k)(k(z)),$$

where the sum is indexed by every codimension-j point of $X \times \Delta^\bullet$ whose closure meets all faces in the expected dimension [28, Corollary 5.3.2 and Theorem 9.0.3]. Since $s_0 HE_k \simeq HE_k$, the latter spectrum is equivalent to the twisted Bloch cycle complex. $\square$

Corollary 4.4. *Let X be an equidimensional smooth scheme of finite type over a perfect field k. Let E be a locally constant étale sheaf on X such that the exponential characteristic of k acts invertibly on E. Then twisted motivic cohomology in degree $(2j, j)$ agrees with twisted higher Chow groups:*

$$H^{2j}_{\mathrm{M}}(X, E(j)) \cong CH^j(X, E, 0).$$

Proof. Let X^a be the "union of all subvarieties of codimension at least i in X". Precisely, the twisted motivic cohomology of $X - X^a$ means the direct limit of the motivic twisted cohomology of $X - S$ over all closed subsets of codimension at least a in X. Then, by taking a direct limit of localization sequences, we have an exact sequence for any i, j, a:

$$\begin{aligned}\cdots \to \oplus_{x \in X^{(a)}} H^{i-2a}_{\mathrm{M}}(k(x), E(j-a)) &\to H^i_{\mathrm{M}}(X - X^{a+1}, E(j))\\ &\to H^i_{\mathrm{M}}(X - X^a, E(j)) \to \oplus_{x \in X^{(a)}} H^{i-2a+1}_{\mathrm{M}}(k(x), E(j-a)) \to \cdots .\end{aligned}$$

Here $X^{(a)}$ denotes the set of points whose closure has codimension a in X.

The invariants for fields (on the left and right) can be identified with twisted higher Chow groups, by Lemma 4.3. Many of these groups are trivially zero. In particular, the groups contributing to $H^{2i}_{\mathrm{M}}(X, E(i))$ are $CH^{i-b}(k(x), E, 0)$ for points x of codimension b in X, and this group is zero for $b \neq i$. Explicitly, the exact sequence above for $a = i$ gives:

$$H^{2i-1}_{\mathrm{M}}(X - X^i, E(i)) \to \oplus_{x \in X^{(i)}} CH^0(k(x), E, 0) \to H^{2i}_{\mathrm{M}}(X, E(i)) \to 0.$$

To clarify this, we also want generators for $H^{2i-1}_{\mathrm{M}}(X - X^i, E(i))$. The groups contributing to this are $CH^{i-b}(k(x), E, 1)$ for points x of codimension b in X with $0 \leq b \leq i-1$. This is zero for $b \neq i-1$. For $a = i-1$, the exact sequence above shows that $\oplus_{x\in X^{(i-1)}} CH^1(k(x), E, 1)$ maps onto $H^{2i-1}_{\mathrm{M}}(X - X^i, E(i))$. Thus we have an exact sequence:

$$\oplus_{x\in X^{(i-1)}} CH^1(k(x), E, 1) \to \oplus_{x\in X^{(i)}} CH^0(k(x), E, 0) \to H^{2i}_{\mathrm{M}}(X, E(i)) \to 0.$$

For a field F with an étale sheaf E, we have $CH^0(F, E, 0) \cong H^0(F, E)$. Likewise, $CH^1(F, E, 1)$ is generated by $H^0(L, E)$ for closed points $\operatorname{Spec} L$ in $(\operatorname{Spec} F) \times \Delta^1$. These are the same as the generators and relations for $CH^i(X, E, 0)$, and so we have shown that

$$H^{2i}_{\mathrm{M}}(X, E(i)) \cong CH^i(X, E, 0).$$

□

Corollary 4.5. *Let X be a smooth variety over a field k and E a locally constant étale sheaf on X. Assume that the exponential characteristic of k acts invertibly on E. Then there is a natural surjection*

$$H^{2i}_{\mathrm{M}}(X, E(i)) \to CH^i(X, E).$$

Proof. By Corollary 4.4, $H^{2i}_{\mathrm{M}}(X, E(i))$ is isomorphic to the twisted higher Chow group $CH^i(X, E, 0)$. The generators for this group are the same as for $CH^i(X, E)$, and so the natural homomorphism $CH^i(X, E, 0) \to CH^i(X, E)$ is surjective. □

We will see that this surjection is not always an isomorphism (Theorem 14.1). To describe the difference in the definitions: the relations in $CH^i(X, E)$ are $H^1(k(x), E(1))$ for points x of codimension $i-1$ in X, while the relations in $CH^i(X, E, 0)$ are the transfers to $H^1(k(x), E(1))$ of products of $H^0(F, E)$ with $H^1(F, \mathbf{Z}(1)) = F^*$, for finite extensions F of $k(x)$.

5 Equivariant Chow groups with coefficients

Following the definition of equivariant Chow groups [38, 12, 39], Guillot, Di Lorenzo, and Pirisi defined equivariant Chow groups with coefficients in a cycle module [21, 11]. We focus here on the special case of equivariant Chow groups with coefficients in a $\mathbf{Z}G$-module M, $CH^G_*(X, M)$. In particular, that gives a definition of the Chow groups of the classifying space with coefficients in M, $CH^*(BG_k, M) = CH^*_G(\operatorname{Spec} k, M)$.

Namely, let X be a scheme of finite type over a field k with an action of a finite group G. Let M be a $\mathbf{Z}G$-module. Let i be an integer. Let V be any representation of G over k such that G acts freely on a Zariski-closed subset $S \subset V$ with $\operatorname{codim}(S \subset V) > \dim(X) - i$. Then we define the ith *equivariant Chow group with coefficients in* M by:

$$CH^G_i(X, M) = CH_{i+\dim(V)}((X \times (V - S))/G, M).$$

By the same proof as for $CH^G_i(X)$ (using homotopy invariance and the localization sequence), this group is independent of the choice of (V, S).

Remark 5.1. If X is quasi-projective over k, then $(X \times (V - S))/G$ is also a quasi-projective scheme, and so its twisted Chow groups have been defined in section 1. If X is not quasi-projective, then $(X \times (V - S))/G$ is (in general) only an algebraic space. In that case, one has to use that the definition of twisted Chow groups works without change for algebraic spaces, as in [12, section 6.1].

When X is smooth and equidimensional over k, define

$$CH^i_G(X,M) = CH^G_{\dim(X)-i}(X,M).$$

(If X is smooth but has components of different dimensions, define $CH^i_G(X,M)$ to be the direct sum of these groups for each G-orbit of components.)

As in the references above, the formal properties of equivariant Chow groups with coefficients follow immediately from the properties of Chow groups with coefficients. In particular, we have proper pushforward, flat pullback, homotopy invariance, the localization sequence, and (in the smooth case) products. Namely, for smooth k-schemes, $CH^*_G(X,M)$ pulls back under arbitrary G-equivariant morphisms, and $CH^*_G(X,M)$ is a module over the ring $CH^*_G X$.

6 The étale cycle map

We now construct the cycle map from twisted Chow groups to étale motivic cohomology. For smooth varieties, we define the cycle map in full generality, without having to invert the exponential characteristic of k. In that generality, the construction is subtle: étale motivic cohomology need not satisfy the localization sequence in the usual form, and we use instead a new purity result, Corollary B.3. For singular varieties, the cycle map takes values in étale Borel-Moore motivic homology, which we only consider with the exponential characteristic inverted.

Theorem 6.1. *Let X be a scheme of finite type over a field k, and let M be a locally constant étale sheaf on X.*

(1) Suppose that X is regular. Then we define a natural homomorphism

$$CH^r(X,M) \to H^{2r}_{\text{et}}(X, M(r)).$$

(2) Without assuming that X is regular, suppose that the exponential characteristic of k acts invertibly on M. Then we define a natural homomorphism

$$CH_j(X,M) \to H^{BM}_{2j,\text{et}}(X, M(j))$$

to étale motivic Borel-Moore homology.

Remark 6.2. In Theorem 6.1(2), we use a version of étale motivic Borel-Moore homology defined by Cisinski and Déglise. They also suggested that the "right" definition of étale Borel-Moore motivic homology, especially if we do not want to invert the exponential characteristic of k, would be étale cohomology with coefficients in Bloch's higher Chow complexes [6, Remark 7.1.4]. In more detail, we should define

$$H^{BM}_{i,\text{et}}(X, \mathbf{Z}(j)) = H^{-i}_{\text{et}}(X, D_X(-j)),$$

where $D_X(-j)$ is a shift of Bloch's cycle complex, numbered by dimension of cycles. Namely, for U étale over X, $D_U(-j) = z_{j,U}[2j]$, where $z_{j,U}$ is the cochain complex in which $(z_{j,U})^i$ is the free abelian group on the set of $(j-i)$-dimensional subvarieties in $X \times \Delta^{-i}$ that meet all faces in the expected dimension.

For a locally constant étale sheaf M, we can therefore define

$$H^{BM}_{i,\text{et}}(X, M(j)) = H^{-i}_{\text{et}}(X, M \otimes^L_{\mathbf{Z}} D_X(-j)).$$

I conjecture that there is a cycle map

$$CH_i(X, M) \to H^{BM}_{2i,\mathrm{et}}(X, M(i)),$$

without having to invert the exponential characteristic of k.

Proof. (1) We first define the cycle map on generators. Let Y be a subvariety of codimension r in X, and let u be an element of $H^0(k(Y), E)$. Write $i\colon Y \to X$ for the inclusion. First suppose that Y is regular. Then $H^0_{\mathrm{et}}(Y, E) \cong H^0(k(Y), E)$, and so we can view u as an element of $H^0_{\mathrm{et}}(Y, E)$ [36, Tag 0BQI]. Then the morphism $E[-2r] \to i^! E(r)$ in the derived category $D_{\mathrm{et}}(Y)$ (Corollary B.3) gives a homomorphism $H^0_{\mathrm{et}}(Y, E) \to H^0_{\mathrm{et}}(Y, i^! E(r)[2r]) \cong H^{2r}_{\mathrm{et}}(Y, i^! E(r)) \cong H^{2r}_{Y,\mathrm{et}}(X, E(r))$. We can map further to $H^{2r}_{\mathrm{et}}(X, E(r))$, as we want.

Next, let Y be a subvariety of codimension r in X, possibly singular, and let u be an element of $H^0(k(Y), E)$. Let Y' be the singular locus of Y, which has codimension at least $r + 1$ in X. The previous paragraph defines an element of $H^{2r}_{Y-Y'}(X - Y', E(r))$, and so it suffices to show that $H^{2r}_{Y}(X, E(r)) \to H^{2r}_{Y-Y'}(X - Y', E(r))$ is an isomorphism. Consider the exact sequence:

$$H^{2r}_{Y'}(X, E(r)) \to H^{2r}_{Y}(X, E(r)) \to H^{2r}_{Y-Y'}(X - Y', E(r)) \to H^{2r+1}_{Y'}(X, E(r)).$$

So it suffices to show that for a closed subset Y' of codimension everywhere at least s in X and any $a < s$, $H^{2a}_{Y'}(X, E(a)) = H^{2a+1}_{Y'}(X, E(a))$. This follows from Corollary B.3, which says in terms of $f\colon Y' \hookrightarrow X$ that $\tau_{\leq 2a+1} f^! E(a) = 0$.

Next, let us show that this map on generators passes to a well-defined homomorphism $CH^r(X, E) \to H^{2r}_{\mathrm{et}}(X, E(r))$. A relation is given by a closed subvariety Y of codimension $r-1$ in X and an element $u \in H^1(k(Y), E(1))$. We want to show that $\partial u \in Z^r(X, E)$ maps to zero in $H^{2r}_{\mathrm{et}}(X, E(r))$. More precisely, we will show that u maps to zero in $H^{2r}_{Y,\mathrm{et}}(X, E(r))$. Write $i\colon Y \hookrightarrow X$ for the inclusion.

Here u comes from an element of $H^1_{\mathrm{et}}(Y - D, E(1))$ for some reduced divisor D in Y, say $D = D_1 + \cdots + D_s$. Let D_{sing} be the singular locus of D, so $D - D_{\mathrm{sing}} = \coprod_{j=1}^{s} D^0_j$ for some regular codimension-1 subvarieties D^0_j of $Y - D_{\mathrm{sing}}$. Since $Y - D$ is a regular subvariety of the regular scheme $X - D$, the morphism $E(1)[-2r+2] \to i^! E(2r)$ in $D_{\mathrm{et}}(Y - D)$ (Corollary B.3) gives a Gysin homomorphism $H^1_{\mathrm{et}}(Y - D, E(1)) \to H^{2r-1}_{Y-D}(X - D, E(r))$. So we can view u as an element of the latter group. Consider the exact sequence of étale cohomology groups:

$$H^{2r-1}_{Y-D}(X - D, E(r)) \to \oplus_{j=1}^{s} H^{2r}_{D^0_j}(X - D_{\mathrm{sing}}, E(r)) \to H^{2r}_{Y-D_{\mathrm{sing}}}(X - D_{\mathrm{sing}}, E(r)).$$

Since D^0_j is a regular subscheme of the regular scheme $X - D_{\mathrm{sing}}$, we have the purity isomorphism $H^0(D^0_j, E) \cong H^{2j}_{D^0_j}(X - D_{\mathrm{sing}}, E(r))$ (Corollary B.3). So the exact sequence above shows that ∂u in $Z^r(X, E)$ maps to zero in $H^{2r}_{Y-D_{\mathrm{sing}}}(X - D_{\mathrm{sing}}, E(r))$. Finally, the restriction map $H^{2r}_{Y}(X, E(r)) \to H^{2r}_{Y-D_{\mathrm{sing}}}(X - D_{\mathrm{sing}}, E(r))$ is an isomorphism, since D_{sing} has codimension at least $r+1$ in X (by Corollary B.3 again). So the image of ∂u in $H^{2r}_{Y}(X, E(r))$ is zero, as we want. Thus the cycle map $CH^i(X, E) \to H^{2i}_{\mathrm{et}}(X, E(r))$ is well-defined.

(2) Now allow X to be singular, but assume that the exponential characteristic e of k is invertible in M. In this case, Cisinski and Déglise defined one version of

the étale motivic Borel-Moore homology of X over k [6, Remark 7.1.12(4)]. Write $f\colon X \to \operatorname{Spec} k$. Namely, they set

$$H_i^{BM}(X, \mathbf{Z}[1/e](j)) = H_{\mathrm{et}}^{-i}(X, B_X(-j))$$

for an object $B_X(-j)$ in $D_{\mathrm{et}}(X)$ (the object $f^!1_k(-j)$ in the category of h-motives $DM_h(X, \mathbf{Z}[1/e])$, which has a functor $R\alpha_*$ to $D_{\mathrm{et}}(X)$). Therefore, we can define twisted Borel-Moore homology by

$$H_i^{BM}(X, M(j)) = H_{\mathrm{et}}^{-i}(X, M \otimes^L_{\mathbf{Z}} B_X(-j)),$$

where we assume that M is a locally constant étale sheaf on which e acts invertibly.

We have $H^i_{\mathrm{et}}(X, M(j)) \cong H^{BM}_{2n-i,\mathrm{et}}(X, M(n-j))$ for X smooth over k, everywhere of dimension n. Also, for a closed subscheme S of a scheme X over k, we have a localization exact sequence:

$$\cdots \to H^{BM}_{i,\mathrm{et}}(S, M(j)) \to H^{BM}_{i,\mathrm{et}}(X, M(j)) \\ \to H^{BM}_{i,\mathrm{et}}(X - S, M(j)) \to H^{BM}_{i-1,\mathrm{et}}(S, M(j)) \to \cdots .$$

Let X_a be the "union of all subvarieties of dimension at most a in X", as in the proof of Corollary 4.4. By taking a direct limit of localization sequences, we have an exact sequence for any i, j, a:

$$\cdots \to \oplus_{x\in X_{(a)}} H^{2a-i}_{\mathrm{et}}(k(x), M(a-j)) \to H^{BM}_{i,\mathrm{et}}(X - X_{a-1}, M(j)) \\ \to H^{BM}_{i,\mathrm{et}}(X - X_a, M(j)) \to \oplus_{x\in X_{(a)}} H^{2a-i+1}_{\mathrm{et}}(k(x), M(a-j)) \to \cdots .$$

We first define the cycle map $CH_b(X, M) \to H^{BM}_{2b,\mathrm{et}}(X, M(b))$ on generators. So let x be a point in X whose closure has dimension b, and let u be an element of $H^0(k(x), M)$. By the localization sequence above, this maps to an element u in $H^{BM}_{2b,\mathrm{et}}(X - X_{b-1}, M(b))$. For each $0 \leq c \leq b-1$, the restriction map

$$H^{BM}_{2b,\mathrm{et}}(X - X_{c-1}, M(b)) \to H^{BM}_{2b,\mathrm{et}}(X - X_c, M(b))$$

is an isomorphism by the localization sequence, using that $H^i_{\mathrm{et}}(F, M(j)) = 0$ for fields F when i and j are negative. (Indeed, the complex of étale sheaves $\mathbf{Z}[1/e](j)$ with $j < 0$ is concentrated in cohomological degree 1 by section 1, and so $M(j) = M \otimes^L_{\mathbf{Z}} \mathbf{Z}[1/e](j)$ is concentrated in degrees ≥ 0.) Therefore, u comes from a unique element of $H^{BM}_{2b,\mathrm{et}}(X, M(b))$.

It remains to show that the cycle map vanishes on the relations defining $CH_b(X, M)$. So let x be a point in X whose closure has dimension $b+1$, and let u be an element of $H^1(k(x), M(1))$. We want to show that the boundary ∂u in $Z_b(X, M)$ maps to zero in $H^{BM}_{2b,\mathrm{et}}(X, M(b))$. It suffices to prove this with X replaced by the closure of the point x; that is, we can assume that X is a variety of dimension $b+1$ over k. As above, we have the localization sequence

$$H^{BM}_{2b+1,\mathrm{et}}(X - X_b, M(b)) \to \oplus_{x\in X_{(b)}} H^0_{\mathrm{et}}(k(x), M) \to H^{BM}_{2b,\mathrm{et}}(X - X_{b-1}, M(b)).$$

Since X has dimension $b+1$, the first group here is $H^1_{\mathrm{et}}(k(X), M(1))$. So the sequence shows that ∂u in $\oplus_{x\in X_{(b)}} H^0_{\mathrm{et}}(k(x), M)$ maps to zero in $H^{BM}_{2b,\mathrm{et}}(X - X_{b-1}, M(b))$. The previous paragraph defines the image of ∂u in the finer group $H^{BM}_{2b,\mathrm{et}}(X, M(b))$. But we showed in the previous paragraph that the restriction from $H^{BM}_{2b,\mathrm{et}}(X, M(b))$ to $H^{BM}_{2b,\mathrm{et}}(X - X_{b-1}, M(b))$ is an isomorphism. So ∂u is zero in $H^{BM}_{2b,\mathrm{et}}(X, M(b))$. Thus we have a well-defined homomorphism $CH_b(X, M) \to H^{BM}_{2b,\mathrm{et}}(X, M(b))$. $\square$

7 The cycle map for complex varieties

Theorem 7.1. *Let X be a scheme of finite type over $\mathbf{C}$, M a locally constant étale sheaf on X, and i an integer. Then we define a natural homomorphism*

$$CH_i(X, M) \to H_{2i}^{BM}(X, M)$$

to Borel-Moore homology (using the classical topology). For X smooth over $\mathbf{C}$, we can rephrase this as a homomorphism

$$CH^j(X, M) \to H^{2j}(X, M).$$

Proof. Let X be a locally compact space of finite dimension. To define Borel-Moore homology with twisted coefficients, recall that $H_i^{BM}(X, \mathbf{Z})$ can be described as $H^{-i}(X, D_X)$, where D_X is the dualizing complex and we consider cohomology in the classical topology [25, Equation IX.4.1]. For a locally constant étale sheaf M on X, we can define $H_i^{BM}(X, M) = H^{-i}(X, M \otimes_{\mathbf{Z}}^L D_X)$.

Let X be a scheme of finite type over $\mathbf{C}$. We first define the cycle map on generators. Let y be a point of the scheme X whose closure Y has dimension i, and let u be an element of $H^0(k(y), M)$. Let S be the singular set of Y. Since S has dimension at most $i-1$ in X, the localization sequence

$$\to H_{2i}^{BM}(S, M) \to H_{2i}^{BM}(X, M) \to H_{2i}^{BM}(X - S, M) \to H_{2i-1}^{BM}(S, M) \to$$

shows that $H_{2i}^{BM}(X, M) \to H_{2i}^{BM}(X - S, M)$ is an isomorphism. So it suffices to define an element of $H_{2i}^{BM}(X-S, M)$ associated to u. Replacing X by $X-S$, we can assume that Y is smooth. Then $H^0(Y, M) \cong H^0(k(y), M)$ since Y is normal [36, Tag 0BQI]. So we can view u as an element of $H^0(Y, M) \cong H_{2i}^{BM}(Y, M)$. Proper pushforward gives a homomorphism $H_{2i}^{BM}(Y, M) \to H_{2i}^{BM}(X, M)$. So u gives an element of $H_{2i}^{BM}(X, M)$, as we want.

It remains to show that for a point w of the scheme X whose closure has dimension $i+1$ in X and an element t of $H^0(k(w), M(1))$, the boundary of t maps to zero in $H_{2i}^{BM}(X, M)$. Because Galois cohomology commutes with direct limits, t comes from a cohomology class in some Galois submodule of M that is finitely generated as a $\mathbf{Z}$-module. So we can assume that M is finitely generated as a $\mathbf{Z}$-module, and we want to show that ∂t maps to zero in $H_{2i}^{BM}(X, M)$. We can assume that X is connected. In this case, in terms of the classical topology, M is a module for the fundamental group $\pi_1 X$ that factors through some finite quotient group G of $\pi_1 X$.

Since the dualizing complex D_X is constructible, so is $M \otimes_{\mathbf{Z}}^L D_X$, and hence the group $H_{2i}^{BM}(X, M) \cong H^{-2i}(X, M \otimes_{\mathbf{Z}}^L D_X)$ is finitely generated. So if we can show that ∂t in $H_{2i}^{BM}(X, M)$ is divisible, then it is zero, as we want. The image of ∂t is zero in $H_{2i}^{BM}(X, M/n)$ for every positive integer n, because we know that ∂t maps to zero in $H_{2i,\text{et}}^{BM}(X, M(i))$ (Theorem 6.1), hence in $H_{2i,\text{et}}^{BM}(X, M/n(i))$, which (as we are over $\mathbf{C}$) can be identified with $H_{2i}^{BM}(X, M/n)$. By the exact sequence

$$H_{2i}^{BM}(X, nM) \to H_{2i}^{BM}(X, M) \to H_{2i}^{BM}(X, M/n),$$

it follows that ∂t lies in the image of $H_{2i}^{BM}(X, nM)$ for every positive integer n. If M is $\mathbf{Z}$-torsion-free (so $nM \cong M$), then it follows that ∂t is divisible, hence zero as we want.

In general, let a be the order of the torsion subgroup of M. Then aM is a torsion-free $\mathbf{Z}G$-submodule of M. We know that ∂t in $H_{2i}^{BM}(X,M)$ is in the image of $H_{2i}^{BM}(X,naM)$ for every positive integer n. So for every positive integer n, ∂t is the image of an element of $H_{2i}^{BM}(X,aM)$ that is divisible by n, and hence ∂t itself is divisible by n. Thus $\partial t = 0$ in $H_{2i}^{BM}(X,M)$, as we want. Finally, for X smooth over k, the cycle map we have constructed can be rewritten as $CH^j(X,M) \to H^{2j}(X,M)$, via Poincaré duality. □

8 Twisted Chow groups and transfers

We now observe that twisted Chow groups have explicit generators in terms of transfers. This generalizes Merkurjev–Scavia's description of the negligible subgroup of group cohomology in degree 2 [31, Theorem 1.3, Corollary 4.2]. It is also related to Theorem 3.1, since it shows again that twisted Chow groups are a quotient of the usual Chow groups of a suitable covering space of X (possibly with several connected components).

Theorem 8.1. *Let X be a k-scheme of finite type, G a finite group, $Y \to X$ a G-torsor, and M a $\mathbf{Z}G$-module. Then $CH_i(X,M)$ is generated by the images of the homomorphisms*

$$M^H \otimes_{\mathbf{Z}} CH_i(Y/H) \to CH_i(Y/H,M) \to CH_i(X,M)$$

over all subgroups H of G, where the last map is the transfer or pushforward.

More strongly, one does not need to use all subgroups of G. For each element $x \in M$, let G_x be the centralizer of x in G. Then $CH_i(X,M)$ is generated by the elements $\operatorname{tr}_{G_x}^G(xy)$ for all $x \in M$ and all $y \in CH_i(Y/G_x)$.

Proof. The group $CH_i(X,M)$ is generated by the groups $H^0(k(z),M)$ for all points z in X with closure Z of dimension i. Given such a point, let H be the image of the composition $\pi_1(\operatorname{Spec} k(z)) \to \pi_1 X \to G$. Then the restriction of the covering map $f\colon Y/H \to X$ over z has a section, $z_1 \in Y/H$. Let Z_1 be its closure in Y/H (which maps birationally to Z). Let u be any element of $H^0(k(z),M) = M^H$. Then the element of $CH_i(X,M)$ associated to the pair (z,u) is the pushforward of the product $u[z_1]$ in $M^H \otimes_{\mathbf{Z}} CH_i(Y/H) \to CH_i(Y/H,M)$.

That proves the first statement, that $CH_i(X,M)$ is generated by $\operatorname{tr}_H^G(xy)$ for all subgroups H in G, $x \in M^H$ and $y \in CH_i(Y/H)$. Here H is contained in the centralizer G_x of x in G. By the projection formula,

$$\begin{aligned}\operatorname{tr}_H^G(xy) &= \operatorname{tr}_{G_x}^G \operatorname{tr}_H^{G_x}(\operatorname{res}_H^{G_x}(x)y)\\ &= \operatorname{tr}_{G_x}^G(x \operatorname{tr}_H^{G_x} y).\end{aligned}$$

Thus $CH_i(X,M)$ is generated by the elements $\operatorname{tr}_{G_x}^G(xz)$ for all $x \in M$ and all $z \in CH_i(Y/G_x)$. □

To deduce Merkurjev–Scavia's statement, we need the following simple observation.

Lemma 8.2. *Let X be a smooth k-variety, G a finite group, $Y \to X$ a G-torsor, and M a $\mathbf{Z}G$-module. Assume that the exponential characteristic of k acts invertibly on M. Then, for $i > 0$, the cycle class homomorphism $CH^i(X, M) \to H^{2i}_{\text{et}}(X, M(i))$ maps into the kernel of restriction to $H^{2i}(k(X), M(i))$. For $i = 1$, the image of $CH^1(X, M) \to H^2_{\text{et}}(X, M(1))$ is equal to the kernel of restriction to $H^2(k(X), M(1))$.*

Proof. For $i > 0$, it is clear that the homomorphism $CH^i(X, M) \to H^{2i}_{\text{et}}(X, M(i))$ maps into the kernel of restriction to $H^{2i}(k(X), M(i))$, because it gives classes supported on a codimension-i subset of X.

Let $i = 1$. Let u be an element of $H^2_{\text{et}}(X, M(1))$ that restricts to zero in $H^2(k(X), M(1))$. Then there is a closed subset S of codimension at least 1 in X such that u restricts to zero on $X - S$. By the localization sequence for étale motivic cohomology (section 1), $H^2_{\text{et}}(X, M(1))$ does not change after removing a subset of codimension at least 2 from X. After doing so, we can assume that S is a disjoint union of regular codimension-1 subvarieties of X.

For a regular codimension-1 subvariety Z of X, the localization sequence for étale motivic cohomology has the form

$$H^0_{\text{et}}(Z, M) \to H^2_{\text{et}}(X, M(1)) \to H^2_{\text{et}}(X - Z, M(1)).$$

By definition, elements of $H^0_{\text{et}}(Z, M) = H^0_{\text{et}}(k(Z), M)$ are in the image of $CH^1(X, M)$. Thus we have shown that the kernel of restriction to the generic point is contained in the image of $CH^1(X, M)$; so it is equal to that image. $\square$

Corollary 8.3. *Let G be a finite group and M a finite G-module. Let k be a field such that $|G|$ and $|M|$ are invertible in k and k contains the $|G||M|$ roots of unity. Then the subgroup of elements of $H^2(G, M)$ that are negligible over k is generated by the images of the maps*

$$M^H \otimes H^2(H, \mathbf{Z}) \to H^2(H, M) \xrightarrow[\operatorname{tr}_H^G]{} H^2(G, M),$$

where H runs over all subgroups of G.

More strongly, one does not need to use all subgroups of G. For each element $x \in M$, let G_x be the centralizer of x in G. Then the negligible subgroup of $H^2(G, M)$ is generated by the elements $\operatorname{tr}_{G_x}^G(xy)$ for all $x \in M$ and all $y \in H^2(G_x, \mathbf{Z})$.

Proof. By definition, an element u of $H^i(G, M)$ is called *negligible over k* if for every field K containing k and every continuous homomorphism $\operatorname{Gal}(K_s/K) \to G$, the induced homomorphism $H^i(G, M) \to H^i(K, M)$ takes u to zero. Let V be a faithful representation of G; then the resulting G-torsor over $k(V/G)$ is versal in Serre's sense. In particular, an element u of $H^i(G, M)$ is negligible over k if and only if it pulls back to zero in $H^i(k(V/G), M)$ [18, Proposition 2.1, Corollary 2.2].

Merkurjev and Scavia showed, using that k contains the $|G||M|$ roots of unity, that the transferred classes in $H^2(G, M)$ described in the Corollary are negligible over k [31, proof of Theorem 1.3]. This is the easier direction, using Kummer theory.

Conversely, let u be an element of $H^2(G, M)$ which is negligible over k; we want to show that u is in the subgroup of transferred classes as above. A fortiori, u is negligible over a separable closure of k; so we can assume that k is separably closed.

Let U be an open subset of V such that G acts freely on U. For a suitable choice of representation V, we can assume that $V - U$ has codimension at least 2 in V. Then $CH^1(U/G, M) \cong CH^1(BG_k, M)$ by definition of the latter group. Likewise, $H^2_{\text{et}}(U/G, M) \cong H^2(G, M)$ by the Hochschild–Serre spectral sequence, using that k is separably closed [32, Theorem III.2.20]. By Theorem 8.1, an element u in $H^2(G, M)$ is negligible if and only if, as an element of $H^2(G, M) \cong H^2_{\text{et}}(U/G, M) \cong H^2_{\text{et}}(U/G, M(1))$, it is in the image of $CH^1(U/G, M) \cong CH^1(BG_k, M)$. Note that $CH^1(BH_k) \cong H^2(H, \mathbf{Z}) \cong \text{Hom}(H, k^*)$ for each subgroup H of G [39, Lemma 2.26]. Therefore, Theorem 8.1 and Lemma 8.2 yield the two statements we want. $\square$

Corollary 8.3 is Merkurjev–Scavia's result [31, Theorem 1.3, Corollary 4.2]. As they explain, the assumption that k contains the $|G||M|$ roots of unity is sharp. That suggests the following question, which has a positive answer when $i = 1$:

Question 8.4. Let G be a finite group, M a $\mathbf{Z}G$-module which is killed by some positive integer. Let $e(G)$ be the exponent of G and $r := e(M)$ the exponent of M. Let k be a field such that $e(G)e(M)$ is invertible in k and k contains the $e(G)e(M)$ roots of unity. Let $i \geq 0$. Does the cycle map

$$CH^i(BG_k, M) \to H^{2i}_{\text{et}}(BG_k, M(i))$$

factor through the homomorphism

$$H^{2i}(G, M \otimes_{\mathbf{Z}/r} \mu_r(k)^{\otimes i}) \to H^{2i}_{\text{et}}(BG_k, M(i))?$$

By Theorem 8.1, Question 8.4 would follow from the special case $M = \mathbf{Z}/r$ (with trivial action of G). By Gherman and Merkurjev, in the case $i = 1$, the assumption that k contains the $e(G)e(M)$ roots of unity is sharp [18, Theorem 4.2]. Earlier, Grothendieck had considered Question 8.4 in the special case of Chern classes of representations (with $M = \mathbf{Z}/r$), although without the precise hypothesis on the $e(G)e(M)$ roots of unity [20, section 5].

9 Twisted Chow groups of a cyclic group

In this section, for a finite cyclic group G, we compute the Chow groups of BG with arbitrary coefficients. In this case, the twisted Chow groups are periodic and have a simple relation to group cohomology.

Theorem 9.1. *Let G be the cyclic group of order m, and let M be a finitely generated $\mathbf{Z}G$-module. Let k be a field such that m is invertible in k and k contains the mth roots of unity. Then*

$$CH^i(BG_k, M) \cong \begin{cases} M^G & \text{if } i = 0 \\ M^G/\operatorname{tr}(M) & \text{if } i > 0, \end{cases}$$

where the trace tr *is $\sum_{g \in G} g$. For $k = \mathbf{C}$, we can also say that the natural homomorphism*

$$CH^i(BG_{\mathbf{C}}, M) \to H^{2i}(BG, M)$$

is an isomorphism for all i. (The right side is usually written as $H^{2i}(G, M)$, group cohomology with coefficients.)

Thus $CH^*(BG_{\mathbf{C}}, M) \to H^{\mathrm{ev}}(BG, M)$ is an isomorphism for G cyclic. Note that the cohomology may be nonzero in odd degrees for G cyclic. Namely, for $i \geq 1$ odd, $H^i(G, M) \cong \ker(\mathrm{tr}\colon M \to M)/\operatorname{im}(1-\sigma)$, where $G = \langle \sigma : \sigma^m = 1\rangle \cong \mathbf{Z}/m$ and $\mathrm{tr} = 1 + \sigma + \cdots + \sigma^{m-1}$. For example, for $G = \mathbf{Z}/2$, $H^1(G, M)$ is nonzero for $M = \mathbf{F}_2$, or for $M = \mathbf{Z}G/\mathbf{Z}$ (which is $\mathbf{Z}$ with G acting by ± 1).

Proof. (Theorem 9.1) Since k contains the mth roots of unity, G has a faithful representation V of dimension 1. Clearly G acts freely on $V - 0$, and so we can apply the following lemma.

Lemma 9.2. *Let G be a finite group. Suppose that G has a representation V of dimension $n > 0$ over a field k such that G acts freely on $V - 0$. Let M be a finitely generated $\mathbf{Z}G$-module. Then multiplication by the Euler class c_nV on $CH^i(BG_k, M)$ is surjective for $i \geq 0$ and an isomorphism for $i \geq 1$.*

Proof. Use the localization sequence for equivariant Chow groups with coefficients, applied to the inclusion $\{0\} \subset V$:

$$CH^i_G(\{0\}, M) \to CH^{i+n}_G(V, M) \to CH^{i+n}_G(V - 0, M) \to 0.$$

By homotopy invariance of twisted Chow groups, the first two groups can be identified with $CH^i(BG, M)$ and $CH^{i+n}(BG, M)$, and the homomorphism is multiplication by the Euler class of V, $c_nV \in CH^nBG$. Also, the third group in the exact sequence is $CH^{i+n}((V-0)/G, M)$, because G acts freely on $V - 0$. So the exact sequence says that $CH^*((V-0)/G, M) \cong CH^*(BG, M)/(c_n(V)CH^*(BG, M))$.

We have $CH^i((V-0)/G, M) = 0$ for $i > n$. More subtly, it is also zero for $i = n$. By Theorem 8.1, it suffices to show that $CH_0((V-0)/H)$ is zero for every subgroup H of G. The point is that H commutes with the action of the multiplicative group G_m on V by scalars, and so we have an action of G_m on $(V-0)/H$ with finite stabilizer groups. It follows that $CH_0((V-0)/H) = 0$ [39, Lemma 5.3]. We conclude that $CH^i((V-0)/G, M) = 0$ for $i \geq n$. Therefore, multiplication by c_nV is surjective on $CH^i(BG_k, M)$ for $i \geq 0$.

To prove the injectivity statement, use the previous term in the localization sequence. In Rost's notation:

$$\begin{aligned} &A_{1-i}((V-0)/G, M)_i \\ &\qquad \to A^G_{-i}(\{0\}, M)_i \to A^G_{-i}(V, M)_i \to A_{-i}((V-0)/G, M)_i \to 0. \end{aligned}$$

The first group is zero for $i > 1$, and so multiplication by c_nV is injective on $CH^i(BG, M)$ for $i > 1$.

In fact, this injectivity extends to the case $i = 1$. The point is that the group $A_0((V-0)/G, M)_1$ may not be zero, but its boundary map to $A^G_{-1}(\{0\}, M)_1$ is zero. By definition of equivariant Chow groups, for an approximation U/G to BG of dimension N, this is identified with the boundary map from $A_N(((V-0) \times U)/G, M)_{1-N}$ to $A_{N-1}((\{0\} \times U)/G, M)_{1-N}$. But the inverse image of $((V-0) \times U)/G$ of each closed point in $(V-0)/G$ has closure in $(V \times U)/G$ disjoint from $(\{0\} \times U)/G$, and so the boundary map is zero. Thus multiplication by c_nV is injective on $CH^i(BG, M)$ for $i \geq 1$ (not just for $i > 1$). Lemma 9.2 is proved. □

We can now prove Theorem 9.1. By definition, $CH^i(BG_k, M)$ means $CH^i(U/G, M)$ for any Zariski open subset U of a finite-dimensional representation W of G over k such that G acts freely on U and $W - U$ has codimension $> i$ in W. In particular, $CH^0(BG, M) \cong CH^0(U/G, M) \cong H^0(k(U/G), M) \cong M^G$, as we want.

Applying Lemma 9.2 to the 1-dimensional faithful representation V of G shows that $CH^*(BG, M)$ is generated by $CH^0(BG, M) = M^G$ as a module over $\mathbf{Z}[c_1 V]$. For each positive integer i, let U/G be an approximation to BG as above, and write $f\colon U \to U/G$ for the covering. Then $CH^i(BG, M) = CH^i(U/G, M)$ is generated by $c_1(V)^i M^G$. Moreover, $f^*(c_1(V)^i) = 0$ in $CH^i(U) = 0$, and so $0 = f_*(f^*(c_1(V)^i)x) = c_1(V)^i \operatorname{tr}(x)$ for every x in $CH^0(U, f^*M) = M$. That is, $M^G/\operatorname{tr}(M)$ maps onto $CH^i(BG, M)$ for $i > 0$.

Let $k = \mathbf{C}$. Then, for $i > 0$, the surjection $M^G/\operatorname{tr}(M) \to CH^i(BG_{\mathbf{C}}, M)$ must be an isomorphism, by mapping further to $H^{2i}(BG, M)$. Indeed, it is a standard calculation in group cohomology that for a cyclic group G, the map $M^G/\operatorname{tr}(M) \to H^{2i}(BG, M)$ (given by multiplication by $c_1(V)^i$) is an isomorphism for all $i > 0$.

For a general field k under our assumptions (m invertible in k and k containing the mth roots of unity), let us show that the surjection $M^G/\operatorname{tr}(M) \to CH^i(BG_k, M)$ for $i > 0$ is also injective. Since both sides commute with direct limits, we can assume that M is a finitely generated $\mathbf{Z}G$-module. Also, it suffices to prove this injectivity after replacing k by its separable closure. For $i > 0$ and each prime number l dividing $m = |G|$, it suffices to show that the homomorphism $(M^G/\operatorname{tr}(M))_{(l)} \to CH^i(BG_k, M)_{(l)} \to H^{2i}_{\text{et}}(BG_k, M^{\wedge l}(i))$ (continuous or pro-étale cohomology) is injective. By the Hochschild–Serre spectral sequence, using that k is separably closed, we have $H^{2i}_{\text{et}}(BG_k, M^{\wedge l}(i)) \cong H^{2i}(G, M^{\wedge l} \otimes \mathbf{Z}_l(i))$ [32, Theorem III.2.20]. Since G is cyclic, this is isomorphic to $(M^G/\operatorname{tr}(M))_{(l)}$, as we want. $\square$

10 Twisted Chow groups in codimension 1

We now show that the codimension-1 Chow group with any twist injects into étale motivic cohomology. This fails in higher codimension, even for $CH^2(X, \mathbf{Z}/r) = CH^2(X)/r$ (for example, see [35, 40]). It also fails for twisted motivic cohomology in codimension 1 (Theorem 14.1).

Theorem 10.1. *Let X be a smooth scheme of finite type over a field k. Let M be a locally constant étale sheaf on X. Assume that the exponential characteristic of k acts invertibly on M. Then the cycle map $CH^i(X, M) \to H^{2i}_{\text{et}}(X, M(i))$ is an isomorphism for $i = 0$ and injective for $i = 1$.*

Proof. We can assume that X is connected, hence irreducible. By definition, we have $CH^0(X, M) \cong H^0(k(X), M) \cong H^0_{\text{et}}(X, M)$, as we want. (This follows from the fact that X is normal [36, Tag 0BQI].) Next, let y be an element of $CH^1(X, M)$ that maps to zero in $H^2_{\text{et}}(X, M(1))$. Then y is represented by finitely many distinct irreducible divisors $D_1, \ldots, D_s$ on X together with an element of $H^0(k(D_j), M)$ for $j = 1, \ldots, s$. Let S be the singular locus of $D := D_1 \cup \cdots \cup D_s$; then S has codimension at least 2 in X. Clearly y maps to zero in $H^2_{\text{et}}(X - S, M(1))$, and we want to show that y is zero in $CH^1(X, M) \cong CH^1(X - S, M)$. Thus, we can replace X by $X - S$; then $D_1, \ldots, D_s$ are regular and disjoint in X. Let D be the union of $D_1, \ldots, D_s$.

Consider the commutative diagram, with top row exact:

$$\begin{tikzcd}
H^1_{\text{et}}(X-D, M(1)) \arrow[r] \arrow[d] & \oplus_j H^0_{\text{et}}(D_j, M) \arrow[r] \arrow[d, "\cong"] & H^2_{\text{et}}(X, M(1)) \\
H^1(k(X), M(1)) \arrow[r] & \oplus_j H^0(k(D_j), M) &
\end{tikzcd}$$

Then y in $\oplus_j H^0(k(D_j), M)$ comes from an element of $\oplus_j H^0(D_j, M)$ that maps to zero in $H^2_{\text{et}}(X, M(1))$. By the top row, that element is in the image of $H^1_{\text{et}}(X - D, M(1))$, and so y is in the image of $H^1_{\text{et}}(k(X), M(1))$. That is, $y = 0$ in $CH^1(X, M)$, as we want. □

Corollary 10.2. *(1) Let G be a finite group and M a $\mathbf{Z}G$-module. Then*

$$CH^1(BG_{\mathbf{C}}, M) \to H^2(BG_{\mathbf{C}}, M) = H^2(G, M)$$

is injective.

(2) Let G be a finite group, k a field, M a finitely generated $\mathbf{Z}G$-module, and l a prime number invertible in k. Then the kernel of $CH^1(BG_k, M) \to H^2_{\text{et}}(BG_k, M^{\wedge l}(i))$ (continuous or pro-étale cohomology) is torsion of order prime to l.

Proof. In proving (1), we write BG for $BG_{\mathbf{C}}$. Let y be an element of $CH^1(BG, M)$ that maps to zero in $H^2(BG, M)$ (cohomology for the classical topology); we want to show that y is zero. Since both sides commute with direct limits, we can assume that M is a finitely generated $\mathbf{Z}G$-module. Since y maps to zero in $H^2(BG, M)$, it maps to zero in $H^2(BG, M/nM) \cong H^2_{\text{et}}(BG, (M/nM)(1))$ for every positive integer n. By the exact sequence

$$H^2_{\text{et}}(BG, nM(1)) \to H^2_{\text{et}}(BG, M(1)) \to H^2_{\text{et}}(BG, (M/nM)(1)),$$

the class of y in $H^2_{\text{et}}(BG, M(1))$ is in the image of $H^2_{\text{et}}(BG, nM(1))$ for every positive integer n. If M is $\mathbf{Z}$-torsion-free, this means that y in $H^2_{\text{et}}(BG, M(1))$ is divisible. Next, observe that $H^2_{\text{et}}(EG, M(1)) \cong H^2_{\text{et}}(\mathbf{C}, M(1)) = 0$, since $\mathbf{C}$ is separably closed and $M(1) = M \otimes^L_{\mathbf{Z}} G_m$ is in cohomological degrees $[-1, 0]$. By pullback and pushforward for $EG \to BG$, it follows that $H^2_{\text{et}}(BG, M(1))$ is killed by $|G|$. In the case where M is $\mathbf{Z}$-torsion-free, it follows that y is zero in $H^2_{\text{et}}(BG, M(1))$. By Theorem 10.1, y is zero in $CH^1(BG, M)$, as we want.

In general, we have arranged that M is a finitely generated $\mathbf{Z}G$-module, possibly with torsion. The previous argument shows that y in $H^2_{\text{et}}(BG, M(1))$ is in the image of $H^2_{\text{et}}(BG, nM(1))$ for every positive integer n. Let a be the order of the torsion subgroup of M. Then aM is a $\mathbf{Z}$-torsion-free $\mathbf{Z}G$-submodule of M. We know that y in $H^2_{\text{et}}(BG, M(1))$ is in the image of $H^2_{\text{et}}(BG, naM(1))$ for every positive integer n. Therefore, y in $H^2_{\text{et}}(BG, M(1))$ is in the image of n times an element of $H^2_{\text{et}}(BG, aM(1))$, and so y is a multiple of n in $H^2_{\text{et}}(BG, M(1))$. The latter group is killed by $|G|$, and so (taking $n = |G|$) it follows that y is zero in $H^2_{\text{et}}(BG, M(1))$. By Theorem 10.1, y is zero in $CH^1(BG, M)$, as we want.

For an arbitrary field k, essentially the same argument proves (2). Here we assume that M is a finitely generated $\mathbf{Z}G$-module. We can replace M by $M[1/e]$ without changing what we are trying to prove, where e is the exponential characteristic of k. Let y be an element of $CH^1(BG_k, M)$ that maps to zero in $H^2_{\text{et}}(BG_k, M^{\wedge l}(1))$.

Then y maps to zero in $H^2_{\text{et}}(BG_k, (M/l^r M)(1))$ for every positive integer r. If M is $\mathbf{Z}$-torsion-free, it follows as above that y is l-divisible in $H^2_{\text{et}}(BG_k, M(1))$.

By homotopy invariance for étale motivic cohomology (using that e acts invertibly on M), the composition

$$H^2_{\text{et}}(k, M(1)) \to H^2_{\text{et}}(BG_k, M(1)) \to H^2_{\text{et}}(EG_k, M(1))$$

is an isomorphism. Moreover, y pulls back to zero in $CH^1(EG_k, M) = 0$, hence in $H^2_{\text{et}}(EG_k, M(1))$. That is, y is in the summand $\ker(H^2_{\text{et}}(BG_k, M(1)) \to H^2_{\text{et}}(EG_k, M(1)))$, and so y is l-divisible in this summand. This summand is killed by $|G|$, by pullback and pushforward. So y in $H^2_{\text{et}}(BG_k, M(1))$ is killed by a positive integer N prime to l. By Theorem 10.1, it follows that y in $CH^1(BG_k, M)$ is killed by the positive integer N prime to l. The extra argument when M has torsion works without change. □

11 The generalized quaternion groups

The generalized quaternion group Q_{2^m} of order 2^m with $m \geq 3$ plays a special role in finite group theory, as these are the only non-cyclic p-groups of p-rank 1 [1, Proposition IV.6.6]. We now compute the Chow groups of the generalized quaternion groups with arbitrary coefficients. The answer is periodic, similar to group cohomology but not identical. Here

$$Q_{2^m} = \langle x, y : x^{2^{m-1}} = 1, y^2 = x^{2^{m-2}}, yxy^{-1} = x^{-1} \rangle.$$

Theorem 11.1. *Let G be the generalized quaternion group Q_{2^m} of order 2^m, $m \geq 3$, and let M be a finitely generated $\mathbf{Z}G$-module. Let k be a field of characteristic not 2 that contains the 2^{m-1}st roots of unity. Then $CH^0(BG_k, M) \cong M^G$,*

$$\begin{aligned} CH^i(BG_k, M) \cong \operatorname{im}\big(&M^{\langle x\rangle} \otimes_{\mathbf{Z}} CH^1 B\langle x\rangle \\ &\oplus M^{\langle y\rangle} \otimes_{\mathbf{Z}} CH^1 B\langle y\rangle \oplus M^{\langle xy\rangle} \otimes_{\mathbf{Z}} CH^1 B\langle xy\rangle \to H^2_{\text{et}}(BG_k, M^{\wedge 2}(1))\big)\end{aligned}$$

if i is odd and $i > 0$, and

$$CH^i(BG_k, M) \cong M^G / \operatorname{tr}(M)$$

if i is even and $i > 0$, where the trace tr *is $\sum_{g \in G} g$. Here $\langle x\rangle \cong \mathbf{Z}/2^{m-1}$, $\langle y\rangle \cong \mathbf{Z}/4$, and $\langle xy\rangle \cong \mathbf{Z}/4$. The homomorphism in the formula for $CH^1(BG_k, M)$ is the sum of the transfers from these three subgroups to G, followed by the cycle map to cohomology. Finally, for $k = \mathbf{C}$, we can equivalently describe $CH^1(BG_{\mathbf{C}}, M)$ as the image of the groups above in cohomology for the classical topology, $H^2(BG, M) = H^2(G, M)$.*

Thus the cycle map $CH^i(BG_{\mathbf{C}}, M) \to H^{2i}(BG, M)$ is injective for a generalized quaternion group G. It is surjective for i even, but not always for i odd. For example, for the quaternion group $G = Q_8$, $CH^1(BG, M)$ is always killed by 4, because it is transferred from the subgroups of order 4 in G, whereas the $\mathbf{Z}G$-module $M := \Omega^2\mathbf{Z}$ has $H^2(G, M) \cong \mathbf{Z}/8$. (By definition, for a $\mathbf{Z}G$-module M, the *syzygy module* ΩM denotes the kernel of any chosen surjection from a projective $\mathbf{Z}G$-module to M. Then, in terms of Tate cohomology, $\widehat{H}^i(G, \Omega M) \cong \widehat{H}^{i-1}(G, M)$ for all integers i [4, section VI.5.4].)

Proof. (Theorem 11.1) The assumption on k ensures that $G = Q_{2^m}$ has a faithful representation V of dimension 2 over k. (Namely, G has a cyclic subgroup H of index 2. Let L be a faithful 1-dimensional representation of H over k; then we can take V to be the induced representation $\mathrm{Ind}_H^G L$.) Here G acts freely on $V - 0$.

Let M be a finitely generated $\mathbf{Z}G$-module. By Lemma 9.2, $CH^*(BG_k, M)$ is generated by elements of degree less than 2 as a module over $\mathbf{Z}[c_2V]$ in CH^*BG. By definition, $CH^0(BG, M)$ is isomorphic to the G-fixed subgroup M^G. For any finite group H and any field k, the first Chern class gives an isomorphism $\mathrm{Hom}(H, k^*) \cong CH^1BH_k$ [39, Lemma 2.26]. So, for $G = Q_{2^m}$, $CH^1BG \cong (\mathbf{Z}/2)^2$.

The main issue is to compute $CH^1(BG, M)$. As a first step, let us show that $CH^1BG \cong (\mathbf{Z}/2)^2$ is generated by transfers from the subgroups $\langle x\rangle \cong \mathbf{Z}/2^{m-1}$ and $\langle y\rangle \cong \mathbf{Z}/4$. Let u be a generator of $CH^1B\langle x\rangle$ and v a generator of $CH^1B\langle y\rangle$. The double coset formula describes the restriction of a transfer in the Chow ring, as in group cohomology [39, Lemma 2.15]. We find that

$$\mathrm{res}^G_{\langle x\rangle} \mathrm{tr}^G_{\langle x\rangle} u = u + y(u) = u - u = 0,$$

while

$$\begin{aligned}\mathrm{res}^G_{\langle y\rangle} \mathrm{tr}^G_{\langle x\rangle} u &= \mathrm{tr}^{\langle y\rangle}_{\langle y^2\rangle} \mathrm{res}^{\langle x\rangle}_{\langle x^{2^{m-2}}\rangle} u \\ &= \mathrm{tr}^{\langle y\rangle}_{\langle y^2\rangle}(\text{generator of } CH^1B\langle y^2\rangle) \\ &\neq 0 \in CH^1B\langle y\rangle \cong \mathbf{Z}/4.\end{aligned}$$

It follows that $\mathrm{tr}^G_{\langle x\rangle} u$ is the nonzero element of $CH^1BG \cong (\mathbf{Z}/2)^2$ whose restriction to $\langle x\rangle$ is zero. Next,

$$\begin{aligned}\mathrm{res}^G_{\langle x\rangle} \mathrm{tr}^G_{\langle y\rangle} v &= \mathrm{tr}^{\langle x\rangle}_{\langle x^{2^{m-2}}\rangle} \mathrm{res}^{\langle y\rangle}_{\langle y^2\rangle} v \\ &= \mathrm{tr}^{\langle x\rangle}_{\langle x^{2^{m-2}}\rangle}(\text{generator of } CH^1B\langle x^{2^{m-2}}\rangle) \\ &\neq 0 \in CH^1B\langle x\rangle \cong \mathbf{Z}/2^{m-1}.\end{aligned}$$

So $\mathrm{tr}^G_{\langle y\rangle} v$ in $CH^1BG \cong (\mathbf{Z}/2)^2$ has nonzero restriction to $\langle x\rangle$. It follows that CH^1BG is generated by $\mathrm{tr}^G_{\langle x\rangle} u$ and $\mathrm{tr}^G_{\langle y\rangle} v$.

By the projection formula, it follows that the image of the product map $M^G \otimes CH^1BG \to CH^1(BG, M)$ is contained in the sum of the images of the transfers from $CH^1(B\langle x\rangle, M)$ and $CH^1(B\langle y\rangle, M)$. By Theorem 8.1, it follows that $CH^1(BG, M)$ is generated by transfers from the three maximal subgroups of G, $H_1 := \langle x\rangle$, $H_2 := \langle x^2, y\rangle$, and $H_3 := \langle x^2, xy\rangle$. Moreover, by Corollary 10.2, $CH^1(BG, M)$ is isomorphic to the sum of the images of these three transfers in $H^2_{\mathrm{et}}(BG_k, M^{\wedge 2}(1))$, or (when $k = \mathbf{C}$) in $H^2(BG, M) \cong H^2(G, M)$. For $m = 3$, the three subgroups H_i are isomorphic to $\mathbf{Z}/4$, and so Theorem 9.1 gives that $CH^1(BH_j, M)$ is generated by $M^{H_j} \otimes_{\mathbf{Z}} CH^1BH_j$ for $j = 1, 2, 3$. That gives the description of $CH^1(BG, M)$ in the theorem, for $m = 3$.

For $m \geq 4$, we prove the description of $CH^1(BG, M)$ in the theorem by induction on m. Here H_1 is isomorphic to $\mathbf{Z}/2^{m-1}$, while H_2 and H_3 are isomorphic to $Q_{2^{m-1}}$. We know that $CH^1(BH_1, M)$ is generated by the image of $M^{H_1} \otimes_{\mathbf{Z}} CH^1BH_1$, by Theorem 9.1. By induction on m, $CH^1(BH_2, M)$ is generated by transfers from

$\langle x^2\rangle \cong \mathbf{Z}/2^{m-2}$, $\langle y\rangle \cong \mathbf{Z}/4$, and $\langle x^2y\rangle \cong \mathbf{Z}/4$. Likewise, $CH^1(BH_3, M)$ is generated by transfers from $\langle x^2\rangle \cong \mathbf{Z}/2^{m-2}$, $\langle xy\rangle \cong \mathbf{Z}/4$, and $\langle x^3y\rangle \cong \mathbf{Z}/4$. So $CH^1(BG, M)$ is generated by transfers from $\langle x\rangle$, $\langle y\rangle$, $\langle xy\rangle$, $\langle x^2y\rangle$, and $\langle x^3y\rangle$. However, $xyx^{-1} = x^2y$, and so the subgroup $\langle y\rangle$ is conjugate to $\langle x^2y\rangle$, and likewise $\langle xy\rangle$ is conjugate to $\langle x^3y\rangle$. Therefore, $CH^1(BG, M)$ is generated by transfers from $\langle x\rangle$, $\langle y\rangle$, and $\langle xy\rangle$. That completes the induction, computing $CH^1(BG, M)$.

Recall that $G = Q_{2^m}$ has a representation V of dimension 2 over k such that G acts freely on $V - 0$. By Theorem 9.2, multiplication by $(c_2V)^i$ is an isomorphism from $CH^1(BG_k, M)$ to $CH^{2i+1}(BG_k, M)$ for $i \geq 0$, as we want. Also, multiplication by $(c_2V)^i$ is surjective from $CH^0(BG_k, M) = M^G$ to $CH^{2i}(BG_k, M)$ for $i > 0$. Since c_2V restricts to zero in CH^2 of the trivial group, $\mathrm{tr}_1^G(M)(c_2V)^i$ is zero in $CH^{2i}(BG, M)$ for $i > 0$; so $CH^{2i}(BG, M)$ is generated by $(M^G/\operatorname{tr}(M))(c_2V)^i$ for $i > 0$. Since G has periodic cohomology with period 4, with periodicity generated by c_2V, we have $H^{4i}_{\text{et}}(BG_{k_s}, M^{\wedge 2}(2i)) \cong M^G/\operatorname{tr}(M)$ for all $i \geq 1$ and $H^{2i+2}_{\text{et}}(BG_{k_s}, M^{\wedge 2}(i+1)) \cong H^2_{\text{et}}(BG, M^{\wedge 2}(1))$ for all $i \geq 0$ [4, Theorem VI.9.1]. Via the cycle map to these cohomology groups over the separable closure k_s, it follows that $CH^{2i}(BG, M)$ is isomorphic to $M^G/\operatorname{tr}(M)$ for all $i \geq 1$, as we want. $\square$

12 Bounds on equivariant Chow groups with coefficients

Using Theorem 8.1, some earlier bounds on the degrees of equivariant Chow groups generalize to arbitrary coefficient modules.

Theorem 12.1. *(1) Let G be a finite group with a faithful representation V of dimension n over a field k with $|G|$ invertible in k. Let M be a $\mathbf{Z}G$-module. Then $CH^*(BG_k, M)$ is generated by elements of at most $n(n-1)/2$ as a module over the Chern classes $\mathbf{Z}[c_1V, \ldots, c_nV]$ in CH^*BG_k.*

(2) Under the same assumptions, let $x_1, \ldots, x_m$ be homogeneous elements of positive degree in CH^*BG_k such that $c_1V, \ldots, c_nV$ are in the subring of CH^*BG_k generated by $x_1, \ldots, x_m$. Then $CH^*(BG_k, M)$ is generated by elements of degree at most $\sum(|x_i| - 1)$ as a module over $\mathbf{Z}[x_1, \ldots, x_m]$.

(3) Suppose in addition that G acts on a smooth scheme X of finite type over k. Then $CH^*_G(X, M)$ is generated by elements of degree at most $\dim(X) + n^2$ as a module over $\mathbf{Z}[c_1V, \ldots, c_nV]$.

Remark 12.2. Theorem 12.1 marks a strong contrast between Chow groups with coefficients and cohomology with coefficients. For a finite group G with a faithful complex representation V of dimension n, Symonds showed that $H^*(G, \mathbf{F}_p)$ is generated by elements of degree at most n^2 as a module over $\mathbf{F}_p[c_1V, \ldots, c_nV]$ [37], [39, Corollary 4.3]. But unlike what happens for twisted Chow groups, there is no uniform bound for the degrees of module generators of $H^*(G, M)$ for all $\mathbf{F}_pG$-modules M. For example, for $G = \mathbf{Z}/2 \times \mathbf{Z}/2$ and an $\mathbf{F}_2G$-module M, define the syzygy module ΩM as the kernel of the surjection from a projective cover of M to M. Then, for $m \geq 0$ and $M = \Omega^m\mathbf{F}_2$, $H^*(G, M)$ needs a generator of degree m as a module over $H^*(G, \mathbf{F}_2)$, by Benson and Carlson's results on products in Tate cohomology [3, Lemma 2.1 and Theorem 3.1]. See the proof of Theorem 13.1 for more details.

Proof. (Theorem 12.1) Statement (1) is known for $M = \mathbf{Z}$ [39, Theorem 5.2]. (Moreover, the bound $n(n-1)/2$ is optimal [39, section 5.2].) The same statement applies to all subgroups H of G, as modules over the same ring $\mathbf{Z}[c_1V, \ldots, c_nV]$. The transfer from $CH^i(BH, M)$ to $CH^i(BG, M)$ is linear over CH^*BG, hence over $\mathbf{Z}[c_1V, \ldots, c_nV]$. Therefore, Theorem 8.1 gives that $CH^*(BG, M)$ is generated by elements of at most $n(n-1)/2$ as a module over $\mathbf{Z}[c_1V, \ldots, c_nV]$.

For statement (2), observe that $CH^i(BG_k, M)$ is killed by the order of G for $i > 0$, by pullback and pushforward along $EG_k \to BG_k$. Therefore, it suffices to prove the desired bounds for generators of $CH^*(BG_k, M)/l$ for each prime number l. Use that for each finite group G, the graded ring $CH^*(BG)/l$ has Castelnuovo–Mumford regularity at most 0 [39, Theorem 6.5]. Under the assumptions of (2), it follows that $CH^*(BG)/l$ is generated by elements of degree at most $\sum(|x_i| - 1)$ as a module over $\mathbf{F}_l[x_1, \ldots, x_m]$ [39, Lemma 3.10, Theorem 3.14]. The same bound applies to every subgroup of G. As in part (1), it follows that $CH^*(BG, M)/l$ is generated by elements of degree at most $\sum(|x_i|-1)$ as a module over $\mathbf{F}_l[x_1, \ldots, x_m]$.

Likewise, statement (3) is known for $M = \mathbf{Z}$ [39, Lemma 6.3]. By Theorem 8.1, the same bound holds for any $\mathbf{Z}G$-module M. $\square$

Thus we have strong bounds for generators of the twisted Chow groups of a finite group, stronger than what is true for cohomology. We may ask the same question about relations.

Question 12.3. Let G be a finite group, with a faithful complex representation V of dimension n. By Theorem 12.1, $CH^*(BG_{\mathbf{C}}, M)$ is generated in degrees at most $n(n-1)/2$ as a module over $\mathbf{Z}[c_1V, \ldots, c_nV]$, for all $\mathbf{Z}G$-modules M. Is there also a bound for the degrees of relations in $CH^*(BG_{\mathbf{C}}, M)$ that depends only on G?

One natural approach fails: the Castelnuovo–Mumford regularity of $CH^*(BG, M)$ can be arbitrarily large, for a fixed group G (Remark 13.3).

13 The group $\mathbf{Z}/2 \times \mathbf{Z}/2$

We now compute the Chow groups of $G = \mathbf{Z}/2 \times \mathbf{Z}/2$ with coefficients in any $\mathbf{F}_2G$-module. Here $\mathbf{Z}/2 \times \mathbf{Z}/2$ is the simplest finite group with p-rank greater than 1, and hence with non-periodic cohomology. The calculations show some new phenomena (Remarks 13.2 and 13.3, and Theorem 14.1). To understand the statement, note that for any field k of characteristic not 2, CH^*BG_k is isomorphic to $\mathbf{Z}[u, v]/(2u, 2v)$, where u and v are first Chern classes of 1-dimensional representations of G [39, Theorem 2.10 and Lemma 2.12].

Theorem 13.1. *Let $G = \mathbf{Z}/2 \times \mathbf{Z}/2$, and let M be an $\mathbf{F}_2G$-module. Let k be a separably closed field of characteristic not 2. Then*

$$CH^*(BG_k, M) \cong \operatorname{im}(M^G \otimes_{\mathbf{Z}} CH^*BG_k \to H^*(BG, M)).$$

Proof. First let $k = \mathbf{C}$. It suffices to show two statements: the product map

$$M^G \otimes CH^*BG = CH^0(BG, M) \otimes CH^*BG \to CH^*(BG, M)$$

is surjective, and $CH^*(BG, M)$ injects into $H^*(BG, M) = H^*(G, M)$ (using the classical topology). Let F be an algebraic closure of $\mathbf{F}_2$. Replacing M by $M \otimes_{\mathbf{F}_2} F$

changes all these groups by tensoring up to F. Therefore, it suffices to prove both statements for $M \otimes_{\mathbf{F}_2} F$. That is, we can assume from now on that M is an FG-module.

Write $G = \langle g, h : g^2 = 1, h^2 = 1, gh = hg\rangle$. Let L_1 and L_2 be the 1-dimensional representations of G over k given by $g \mapsto -1$, $h \mapsto 1$ (for L_1) and $g \mapsto 1$, $h \mapsto -1$ (for L_2). Let $u = c_1 L_1$ and $v = c_1 L_2$ in $CH^1 BG$; then $CH^* BG = \mathbf{Z}[u, v]/(2u, 2v)$. The representation $L_1 \oplus L_2$ of G is faithful, and its Chern classes are polynomials in u and v. By Theorem 12.1(2), it follows that $CH^*(BG, M)$ is generated by elements of degree 0 as a module over $CH^* BG$. That is, the product map

$$M^G \otimes CH^* BG \to CH^*(BG, M)$$

is surjective.

For later use, the cohomology ring of G is the polynomial ring $H^*(BG, F) \cong F[x, y]$ with $|x| = |y| = 1$. We can choose the generators so that the cycle map $CH^*(BG) \otimes F \to H^*(BG, F)$ takes u to x^2 and v to y^2.

It remains to show that $CH^*(BG, M) \to H^*(BG, M)$ is injective. These groups commute with direct limits of FG-modules; so we can assume that M is an FG-module of finite dimension over F. Using direct sums, we can also assume that M is indecomposable. Then we can use the classification of indecomposable FG-modules, as follows [2, v. 1, Theorem 4.3.3]. For a finite-dimensional FG-module M, define the syzygy module ΩM as the kernel of the surjection from a projective cover of M to M; then ΩM is well-defined up to isomorphism. (Likewise, define the shift $\Omega^{-1} M$ as the cokernel of the inclusion from M to its injective hull.) Each element f of $H^n(G, F)$ with $n > 0$ is represented by a map $\Omega^n F \to F$ of FG-modules; for $f \neq 0$, define L_f to be the kernel. Then, for $G = \mathbf{Z}/2 \times \mathbf{Z}/2$, every indecomposable FG-module is isomorphic to either FG, $\Omega^n F$ for some $n \in \mathbf{Z}$, or L_{ζ^n} for some $n > 0$ and some nonzero $\zeta \in H^1(G, F)$. Here ζ only matters up to scalars, and so the last type of module is determined (up to isomorphism) by $n > 0$ and a point in $\mathbf{P}(H^1(G, F)) \cong \mathbf{P}^1(F)$. Here $\Omega^n F$ has dimension $2|n| + 1$ and L_{ζ^n} has dimension $2n$.

In each case, we use that $CH^0(BG, M) = H^0(BG, M) = M^G$ and $CH^1(BG, M) \to H^2(BG, M)$ is injective (Corollary 10.2). First, let $M = FG$. Then $H^2(G, M) = 0$, and so $CH^1(BG, M) = 0$. Since $M^G \otimes_F CH^* BG \to CH^*(BG, M)$ is surjective (and $CH^*(BG) \otimes F = F[u, v]$ is generated in degree 1), it follows that $CH^i(BG, M) = 0$ for $i > 0$. Thus $CH^*(BG, M) \to H^*(BG, M)$ is injective.

It will be useful to recall the description of Tate cohomology for finite groups [4, section VI.4]: $\widehat{H}^j(G, M)$ is isomorphic to $H^j(G, M)$ if $j > 0$, to $H_{-1-j}(G, M)$ if $j < -1$, and $\widehat{H}^{-1}(G, M)$ and $\widehat{H}^0(G, M)$ are the kernel and cokernel of the trace map:

$$0 \to \widehat{H}^{-1}(G, M) \to M_G \underset{\mathrm{tr}}{\longrightarrow} M^G \to \widehat{H}^0(G, M) \to 0.$$

For each indecomposable FG-module other than FG, I claim that the trace map $\mathrm{tr}\colon M \to M$ is zero. (This holds more generally for any p-group.) If not, let x be an element of M with $\mathrm{tr}(x) \neq 0$. Then there is an FG-linear map $f\colon FG \to M$ that takes 1 to x, and hence $\mathrm{tr}(1)$ to $\mathrm{tr}(x) \neq 0$. But $\mathrm{tr}(1)$ in FG spans the socle, $(FG)^G \cong F$. It follows that f is injective. Since the FG-module FG is injective as well as projective [2, Proposition 3.1.10], it follows that M contains FG as a summand. Thus, for M indecomposable and not isomorphic to FG, the trace is zero on M. Equivalently, $M^G = H^0(G,M)$ maps isomorphically to $\widehat{H}^0(G,M)$. This is relevant because we have more direct access to $\widehat{H}^0(G,M)$ in the following calculations, and hence we can read off M^G.

Next, let $M = F$. We have $H^*(BG,F) \cong F[x,y]$, and $CH^*(BG) \to H^*(BG,F)$ sends $u \mapsto x^2$ and $v \mapsto y^2$. It follows that $CH^*(BG,M) = F[u,v]$ injects into $H^*(BG,F)$, as we want.

Next, let $M = \Omega^{-m}F$ with $m > 0$. Then $\widehat{H}^i(G,M) \cong \widehat{H}^{i+m}(G,F) \cong F^{i+m+1}$ for $i \geq 0$. In particular, $CH^1(BG,M)$ is the image of $M^G \otimes CH^1BG$ in $H^2(BG,M)$, thus the image of $F\{x^m, x^{m-1}y, \ldots, y^m\} \otimes_F F\{x^2,y^2\}$, which is all of $H^2(BG,M) \cong F\{x^{m+2}, x^{m+1}y, \ldots, y^{m+2}\}$. Therefore, $CH^*(BG,M)$ is a quotient of the $F[u,v]$-module $F[u,v]\{e_0,\ldots,e_m\}/(ue_{i+2} - ve_i)$ for $0 \leq i \leq m-2$, where e_i maps to $x^{m-i}y^i$ in $M^G/\operatorname{tr}(M)$. But we compute that this $F[u,v]$-module maps isomorphically to $H^{\mathrm{ev}}(BG,M)$ (that is, to the subspace of $F[x,y]$ spanned by homogeneous polynomials of degree at least m and congruent to m modulo 2). Therefore, $CH^*(BG,M) \to H^{\mathrm{ev}}(BG,M)$ is an isomorphism (hence injective, as we want).

Next, let $M = \Omega^m F$ with $m > 0$. Then $\widehat{H}^i(G,M) \cong \widehat{H}^{i-m}(G,F)$. These vector spaces decrease from dimension m (when $i = 0$) to dimension 1 (when i is $m-1$ or m) and then increase again, by the description of Tate cohomology above. For $i \geq 0$ and $j < -i$, the product $\widehat{H}^i(G,F) \times \widehat{H}^j(G,F) \to \widehat{H}^{i+j}(G,F)$ can be identified with the cap product of cohomology with homology (which is dual to the product on cohomology in positive degrees, hence usually nonzero). On the other hand, products from negative degree into nonnegative degree are zero. Namely, Benson and Carlson showed that for $i > 0$ and $-i \leq j < 0$, for $G = \mathbf{Z}/2 \times \mathbf{Z}/2$ (as for many other groups of p-rank at least 2), the product $\widehat{H}^i(G,F) \times \widehat{H}^j(G,F) \to \widehat{H}^{i+j}(G,F)$ is zero [3, Lemma 2.1 and Theorem 3.1].

For $m = 1$ (that is, $M = \Omega F$), it follows that the image of the product map $M^G \otimes CH^1BG \to H^2(BG,M)$ is zero, and hence $CH^1(BG,M) = 0$. Since $M^G \otimes CH^*BG \to CH^*(BG,M)$ is surjective (and CH^*BG is generated in degree 1), it follows that $CH^i(BG,M) = 0$ for all $i > 0$. Thus $CH^*(BG,M) \to H^*(BG,M)$ is injective, as we want.

For $m > 1$, let $R = CH^*(BG) \otimes F = F[u,v]$. Then in degrees at most 1, $CH^*(BG,M)$ agrees with the R-module $N := R\{e_0,\ldots,e_{m-1}\}/(ue_{i+2} = ve_i$ for $0 \leq i \leq m-3, ue_0 = 0, ue_1 = 0, ve_{m-2} = 0, ve_{m-1} = 0)$, using that $CH^1(BG,M)$ is the image of $M^G \otimes CH^1BG \to H^2(BG,M)$. Since $M^G \otimes CH^*BG \to CH^*(BG,M)$ is surjective in all degrees, we have a surjection of R-modules from N to $CH^*(BG,M)$. But we compute that N^i maps isomorphically to $H^{2i}(BG,M)$ for $0 \leq i < m/2$, and N is zero in higher degrees. In particular, N injects into $H^*(BG,M)$. Therefore, N maps isomorphically to $CH^*(BG,M)$, and $CH^*(BG,M)$ injects into $H^*(BG,M)$ as we want.

Finally, let M be the FG-module L_{ζ^n}, for a positive integer n and a nonzero element ζ in $H^1(BG,F) = F\{x,y\}$. Choose one of x or y which is linearly independent of ζ, say x (without loss of generality). Then $\widehat{H}^*(G,M)$ is periodic with period 1; namely, multiplication by x is an isomorphism on this $F[x,y]$-module [2, v. 1, Corollary 5.10.7].

To describe the cohomology of $M = L_{\zeta^n}$ in more detail: by the exact sequence

$$0 \to M \to \Omega^n F \to F \to 0,$$

we have $H^{i+1}(G,M) \cong H^i(G,F)/\zeta^n H^{i-n}(G,F)$ for $i \geq n-1$. (This uses that ζ^n

is a non-zero-divisor in $H^*(G,F)=F[x,y]$.) Since ζ is linearly independent of x, we can also view $H^*(G,F)$ as the polynomial ring $F[x,\zeta]$. Using the periodicity of $\widehat{H}^*(G,M)$, let us identify $M^G=\widehat{H}^0(G,M)$ with $H^n(G,M)\cong H^{n-1}(G,F)$. As such, M^G has a basis $e_i=x^{n-1-i}\zeta^i$ with $0\leq i\leq n-1$. Let $w=\zeta^2$; then $CH^*(BG)\otimes F=F[u,w]$. Then $CH^1(BG,M)\cong \operatorname{im}(M^G\otimes_F(CH^1(BG)\otimes F)\to H^2(BG,M))$ is spanned by ue_i and we_i for $0\leq i\leq n-1$, modulo the relations $ue_{i+2}=we_i$ for $0\leq i\leq n-3$, $we_{n-2}=0$, and $we_{n-1}=0$.

Let $R=CH^*(BG)\otimes F=F[u,w]$. Since $M^G\otimes_F R\to CH^*(BG,M)$ is surjective, $CH^*(BG,M)$ is a quotient of the graded R-module

$$N:=R\{e_0,\ldots,e_{n-1}\}/(ue_{i+2}=we_i \text{ for } 0\leq i\leq n-3, we_{n-2}=0, we_{n-1}=0).$$

But we compute that this module N maps isomorphically to $H^{\mathrm{ev}}(BG,M)$, viewed as the part of $F[x,\zeta]/(\zeta^n)$ in degrees at least $n-1$. Therefore, N maps isomorphically to $CH^*(BG,M)$, and $CH^*(BG,M)$ maps isomorphically to $H^{\mathrm{ev}}(BG,M)$, hence injectively, as we want.

That completes the proof for $k=\mathbf{C}$. More generally, let k be a separably closed field of characteristic not 2. Then the Hochschild–Serre spectral sequence for $EG_k\to BG_k$ shows that $H^*_{\mathrm{et}}(BG_k,M)$ is isomorphic to $H^*(G,M)$, for every $\mathbf{F}_2G$-module M [32, Theorem III.2.20]. Also, Theorem 10.1 shows that $CH^1(BG_k,M)\to H^2_{\mathrm{et}}(BG_k,M(1))$ is injective. (The twist here is irrelevant, since the étale sheaf μ_2 is canonically isomorphic to $\mathbf{Z}/2$ over k.) Given that, the arguments over $\mathbf{C}$ work without change over k. ☐

Remark 13.2. For the group $G=\mathbf{Z}/2\times\mathbf{Z}/2$, Theorem 13.1 shows that $CH^*(BG_{\mathbf{C}},M)$ maps injectively to $H^*(BG,M)$, for all $\mathbf{F}_2G$-modules M. But for some modules M, this map is far from an isomorphism. In particular, for $m>0$, we have shown that $CH^*(BG,\Omega^m\mathbf{F}_2)$ is zero in degrees at least $m/2$, whereas $H^*(BG,\Omega^m\mathbf{F}_2)$ contains $H^*(BG,\mathbf{F}_2)=\mathbf{F}_2[x,y]$ (shifted in degree) as a submodule. Thus the "support variety" of $H^*(BG,M)$ is all of $\operatorname{Spec} H^*(BG,\mathbf{F}_2)=A^2_{\mathbf{F}_2}$, while the support variety of $CH^*(BG,M)$ is only the origin in $\operatorname{Spec} CH^*(BG)/2=A^2_{\mathbf{F}_2}$.

Remark 13.3. For a finite group G and a prime number p, the Castelnuovo–Mumford regularity of $CH^*(BG_{\mathbf{C}})/p$ is at most zero [39, Theorem 6.5]. In terms of a faithful representation V of G over $\mathbf{C}$, with $n:=\dim(V)$, this regularity bound amounts to an upper bound for the degrees of generators, relations, relations between relations, and so on, for $CH^*(BG_{\mathbf{C}})/p$ as a graded module over the Chern classes $\mathbf{F}_p[c_1V,\ldots,c_nV]$.

We have seen that there is a bound for the degrees of generators of $CH^*(BG_{\mathbf{C}},M)$ as a module over $\mathbf{F}_p[c_1V,\ldots,c_nV]$ for all $\mathbf{F}_pG$-modules M, depending only on G (Theorem 12.1). However, the regularity of $CH^*(BG_{\mathbf{C}},M)$ does not have such a bound. Take $G=\mathbf{Z}/2\times\mathbf{Z}/2$ and $M=\Omega^m\mathbf{F}_2$ for $m\geq 2$. Let $R=CH^*(BG)/2=\mathbf{F}_2[u,v]$. By the properties of regularity, $CH^*(BG,M)$ has the same regularity over R as over the Chern classes of a faithful representation [39, Lemma 3.10].

By the calculation of $CH^*(BG,M)$ in the proof of Theorem 13.1 plus the Hilbert syzygy theorem [13, Corollary 19.7], $CH^*(BG,M)$ has a graded free resolution over R of the form

$$0\to R^{\oplus 2}\to R^{\oplus m+2}\to R^{\oplus m}\to CH^*(BG,M)\to 0.$$

Here the generators of $R^{\oplus m}$ are in degree 0, and the generators of $R^{\oplus m+2}$ (corresponding to the relations in $CH^*(BG, M)$) are in degree 1. From the Hilbert series of $CH^*(BG, M)$, we compute that the module of "relations between relations" $R^{\oplus 2}$ has generators in degrees $\lfloor (m+2)/2 \rfloor$ and $\lfloor (m+3)/2 \rfloor$. Therefore, $CH^*(BG, M)$ has Castelnuovo–Mumford regularity $\lfloor (m+3)/2 \rfloor - 2 = \lfloor (m-1)/2 \rfloor$ [39, Theorem 3.14]. In particular, the regularity of $CH^*(BG, M)$ cannot be bounded in terms of G.

14 Twisted Chow groups vs. twisted motivic cohomology

We now show that the surjection from twisted motivic cohomology $H^{2i}_{\mathrm{M}}(X, M(i))$ to twisted Chow groups $CH^i(X, M)$ is not always an isomorphism. As a result, one might think that the definition of twisted Chow groups from section 1 is "wrong", and that the definition should be changed to agree with twisted motivic cohomology. Given the good properties of twisted Chow groups from section 1, however, I believe that twisted Chow groups are worth studying. They form a nontrivial intermediary between twisted motivic cohomology and twisted étale cohomology. An advantage of twisted Chow groups is that $CH^1(X, M)$ injects into étale motivic cohomology $H^2_{\mathrm{et}}(X, M(1))$, whereas (as we will see) $H^2_{\mathrm{M}}(X, M(1))$ does not always inject into $H^2_{\mathrm{et}}(X, M(1))$.

Theorem 14.1. *(1) There is a smooth complex variety X with a locally constant étale sheaf M such that the maps*

$$H^2_{\mathrm{M}}(X, M(1)) \to H^2_{\mathrm{et}}(X, M(1))$$

and

$$H^2_{\mathrm{M}}(X, M(1)) \to CH^1(X, M)$$

are not injective.

(2) There is a short exact sequence $0 \to A \to B \to C \to 0$ of locally constant étale sheaves on X such that A is coflasque but the sequence

$$CH^1(X, A) \to CH^1(X, B) \to CH^1(X, C)$$

is not exact.

Proof. Let $k = \mathbf{C}$. The idea is to compare twisted Chow groups with twisted motivic cohomology for BG_k with $G = \mathbf{Z}/2 \times \mathbf{Z}/2 = \langle g, h : g^2 = 1, h^2 = 1, gh = hg \rangle$. (This is the smallest group G that has a coflasque $\mathbf{Z}G$-lattice that is not invertible. That is relevant because of Theorem 3.1.) We can take the smooth variety X to be U/G for any open subset U of a representation V of G over k such that G acts freely on U and $V - U$ has codimension at least 2 in V.

Let M be the G-module $\Omega^{-m}\mathbf{F}_2$ with $m \geq 2$. Then M is the vector space $(\mathbf{F}_2)^{2m+1}$ with basis $e_1, \ldots, e_{2m+1}$, and G acts by: $g(e_i) = e_i$ for $1 \leq i \leq m+1$, $g(e_{m+1+i}) = e_i + e_{m+1+i}$ for $1 \leq i \leq m$, $h(e_i) = e_i$ for $1 \leq i \leq m+1$, and $h(e_{m+1+i}) = e_{i+1} + e_{m+1+i}$ for $1 \leq i \leq m$ [2, v. 1, Theorem 4.3.3].

Then M^G is spanned by $e_1, \ldots, e_{m+1}$. Let $H_1 = \langle g \rangle$, $H_2 = \langle h \rangle$, and $H_3 = \langle gh \rangle$ be the subgroups of order 2 in G. For $a = 1, 2, 3$, the subspace M^{H_a} is also spanned by $e_1, \ldots, e_{m+1}$. Define a $\mathbf{Z}G$-linear map from the permutation module $B := (\mathbf{Z}G)^m \oplus \mathbf{Z}^{m+1}$ to M, sending the $\mathbf{Z}G$-module generators of B by: $f_i \mapsto e_{m+1+i}$ for $1 \leq i \leq m$ and $f_{m+i} \mapsto e_i$ for $1 \leq i \leq m+1$. Then $B^H \to M^H$ is surjective for every subgroup H of G, and so the kernel A is a coflasque $\mathbf{Z}G$-lattice.

A $\mathbf{Z}$-basis for $A \cong \mathbf{Z}^{5m+1}$ is given by the elements $s_i = 2f_i$ for $1 \leq i \leq 2m+1$, $s_{2m+1+i} = gf_i - f_i - f_{m+i}$ for $1 \leq i \leq m$, $s_{3m+1+i} = hf_i - f_i - f_{m+1+i}$ for $1 \leq i \leq m$, and $s_{4m+1+i} = ghf_i - f_i - f_{m+i} - f_{m+1+i}$ for $1 \leq i \leq m$. Here A^G has rank $2m+1$ and A^{H_a} has rank $3m+1$ for $a = 1, 2, 3$. In more detail, a $\mathbf{Z}$-basis for A^G is given by $s_{m+1}, \ldots, s_{2m+1}$, and $2s_i + s_{2m+1+i} + s_{3m+1+i} + s_{4m+1+i}$ for $1 \leq i \leq m$. A $\mathbf{Z}$-basis for A^{H_1} is given by the basis for A^G together with $s_i + s_{2m+1+i}$ for $1 \leq i \leq m$. For A^{H_2}, we have the basis for A^G together with $s_i + s_{3m+1+i}$ for $1 \leq i \leq m$, and for A^{H_3} we have the basis for A^G together with $s_i + s_{4m+1+i}$ for $1 \leq i \leq m$,

Define a $\mathbf{Z}G$-linear map from $P := \oplus_{a=1}^3 \mathbf{Z}[G/H_a]^{\oplus m}$ to A, sending the $\mathbf{Z}G$-module generators to $s_i + s_{2m+1+i}$, $s_i + s_{3m+1+i}$, and $s_i + s_{4m+1+i}$. We read off that $P^H \to A^H$ is surjective for every subgroup H of G. Therefore, the kernel of $P \to A$ is coflasque. By Lemma 4.1, it follows that $H^{2i}_{\mathrm{M}}(BG, P(i)) \to H^{2i}_{\mathrm{M}}(BG, A(i))$ is surjective. Using that lemma again for the coflasque resolution $0 \to A \to B \to \Omega^{-m}\mathbf{F}_2 \to 0$, we find that

$$H^{2i}_{\mathrm{M}}(BG, P(i)) \to H^{2i}_{\mathrm{M}}(BG, B(i)) \to H^{2i}_{\mathrm{M}}(BG, \Omega^{-m}\mathbf{F}_2(i)) \to 0$$

is exact.

Since P and B are permutation modules, we can rewrite this exact sequence as

$$\oplus_{a=1}^3 CH^i(BH_a)^{\oplus m} \to CH^i(\operatorname{Spec} k)^{\oplus m} \oplus CH^i(BG)^{\oplus m+1} \to H^{2i}_{\mathrm{M}}(BG, \Omega^{-m}\mathbf{F}_2(i)) \to 0.$$

For $i > 0$, we have $CH^i(\operatorname{Spec} k) = 0$, and the maps from $CH^i BH_a$ to $CH^i BG$ are multiples of the transfer map. But transfer from $CH^i BH_a$ to $CH^i BG$ is zero for $i > 0$, using that these groups are killed by 2 and restriction from $CH^i BG$ to $CH^i BH_a$ is surjective. Therefore, we have an isomorphism

$$CH^i(BG)^{\oplus m+1} \cong H^{2i}_{\mathrm{M}}(BG, \Omega^{-m}\mathbf{F}_2(i))$$

for $i > 0$. (By inspection, this also holds for $i = 0$.)

In particular, $H^2_{\mathrm{M}}(BG, \Omega^{-m}\mathbf{F}_2(1)) \cong (\mathbf{F}_2)^{2m+2}$. By Theorem 13.1, we have $CH^1(BG, \Omega^{-m}\mathbf{F}_2) \cong (\mathbf{F}_2)^{m+3}$. Thus, for $m \geq 2$, the surjection

$$H^2_{\mathrm{M}}(BG, \Omega^{-m}\mathbf{F}_2(1)) \to CH^1(BG, \Omega^{-m}\mathbf{F}_2)$$

from Corollary 4.5 is not injective. The map

$$H^2_{\mathrm{M}}(BG, \Omega^{-m}\mathbf{F}_2(1)) \to H^2_{\mathrm{et}}(BG, \Omega^{-m}\mathbf{F}_2(1))$$

factors through $CH^1(BG, \Omega^{-m}\mathbf{F}_2)$, and so it is also not injective. We have now proved two parts of the theorem.

It remains to show that the sequence $CH^1(X, A) \to CH^1(X, B) \to CH^1(X, \Omega^{-m}\mathbf{F}_2)$ is not exact, even though A is coflasque. Because the surjection $P \to A$ has coflasque kernel, we know that $CH^1(X, P) \to CH^1(X, A)$ is surjective (Theorem 3.1). So it is equivalent to show that $CH^1(X, P) \to CH^1(X, B) \to CH^1(X, \Omega^{-m}\mathbf{F}_2)$ is not exact. Since P and B are permutation modules, we have to show that the sequence

$$\oplus_{a=1}^3 CH^1(BH_a)^{\oplus m} \to CH^1(\operatorname{Spec} k)^{\oplus m} \oplus CH^1(BG)^{\oplus m+1} \to CH^1(BG, \Omega^{-m}\mathbf{F}_2) \to 0.$$

The first map is a linear combination of transfers from the subgroups H_a to G, and so it is zero, as shown above. Therefore, we want to show that $CH^1(BG)^{\oplus m+1} \to CH^1(BG, \Omega^{-m}\mathbf{F}_2)$ is not an isomorphism. The first group is isomorphic to $(\mathbf{F}_2)^{2m+2}$ and the second is $(\mathbf{F}_2)^{m+3}$. Since $m \geq 2$, this is not an isomorphism. The third part of the theorem is proved. $\square$

Remark 14.2. Let us compare the advantages of twisted Chow groups $CH^i(X, E)$ and twisted motivic cohomology $H^{2i}_{\mathrm{M}}(X, E(i))$. Assume here that E is a locally constant étale sheaf on a smooth variety X over a field k.

(1) When E is the constant sheaf corresponding to an abelian group, both theories agree with the usual Chow groups, $CH^i(X) \otimes_{\mathbf{Z}} E$.

(2) Twisted motivic cohomology has a long exact sequence associated to a short exact sequence of locally constant étale sheaves $0 \to A \to B \to C \to 0$ if A is coflasque (Lemma 4.1). For twisted Chow groups, we can only say that $CH^i(X, A) \to CH^i(X, B) \to CH^i(X, C)$ is exact if A is invertible (Theorems 3.1 and 14.1).

(3) The cycle map $CH^1(X, E) \to H^2_{\mathrm{et}}(X, E(1))$ is injective, whereas $H^2_{\mathrm{M}}(X, E(1)) \to H^2_{\mathrm{et}}(X, E(1))$ need not be injective (Theorems 10.1 and 14.1). More broadly, twisted Chow groups should be closer to étale cohomology than twisted motivic cohomology is.

A The residue on étale motivic cohomology

Here we construct the residue homomorphism on étale motivic cohomology twisted by a locally constant sheaf E,

$$\partial_v \colon H^a(F, E(a)) \to H^{a-1}(k(v), E(a-1))$$

(Corollary A.3). This requires extra effort when $k(v)$ has characteristic $p > 0$ and p does not act invertibly on E. We use the residue homomorphism in section 1 to show that Chow groups with twisted coefficients have the desired formal properties in full generality. This appendix uses no results from the rest of the paper.

Lemma A.1. *Let O_v be a discrete valuation ring, and let $i \colon \operatorname{Spec} k(v) \to \operatorname{Spec} O_v$ be the inclusion of the closed point. For each $a \geq 1$,*

$$\mathbf{Z}(a-1)_{k(v)}[-2] \cong \tau_{\leq a+2} i^! \mathbf{Z}(a)$$

in $D_{\mathrm{et}}(k(v))$.

Here $i^!$ is the exceptional inverse image functor on derived categories, sometimes called $Ri^!$.

Proof. First, $\mathbf{Z}(a-1)_{k(v)}$ is concentrated in degrees at most $a-1$ in $D_{\mathrm{et}}(k(v))$, and so $\mathbf{Z}(a-1)[-2]$ is concentrated in degrees at most $a+1$. Next, using the Bloch-Kato conjecture (Voevodsky's theorem), Geisser showed that the canonical map

$$\mathbf{Z}(a-1)_{k(v)}[-2] \to \tau_{\leq a+1} i^! \mathbf{Z}(a)$$

in $D_{\mathrm{et}}(k(v))$ is an isomorphism. Also, the truncation is unnecessary after inverting the exponential characteristic e [16, Theorem 1.4]. (These results imply the lemma when $k(v)$ has characteristic zero.)

It remains to show that $M := \mathcal{H}^{a+2}(i^!\mathbf{Z}(a)_{O_v})$ is zero. Here M is an étale sheaf on $\operatorname{Spec} k(v)$, and so it suffices to show that the stalk of M at the separable closure $k(v)_s$ is zero. This stalk is isomorphic to $H^{a+2}(k(v)_s, i^!\mathbf{Z}(a)_{O_{\mathrm{nr}}})$, where O_{nr} is the maximal unramified extension of O_v (that is, the strict henselization of O_v), and we use the same name i for the inclusion $i\colon \operatorname{Spec} k(v)_s \to \operatorname{Spec} O_{\mathrm{nr}}$ [36, Theorem 03Q9]. Let $j\colon \operatorname{Spec} F_{\mathrm{nr}} \to \operatorname{Spec} O_{\mathrm{nr}}$ be the inclusion of the generic point. There is an exact triangle

$$i_*i^!E \to E \to j_*j^*E$$

for every object E in $D_{\mathrm{et}}(O_{\mathrm{nr}})$. Applying this to $E = \mathbf{Z}(a)_{O_{\mathrm{nr}}}$, we have a long exact sequence

$$\cdots \to H^{j-1}_{\mathrm{et}}(F_{\mathrm{nr}}, \mathbf{Z}(a)) \to H^j_{\mathrm{et}}(k(v)_s, i^!\mathbf{Z}(a)) \to H^j_{\mathrm{et}}(O_{\mathrm{nr}}, \mathbf{Z}(a)) \to H^j_{\mathrm{et}}(F_{\mathrm{nr}}, \mathbf{Z}(a)) \to \cdots.$$

Consider the map from Zariski to étale cohomology:

$$\begin{array}{ccccccccc}
H^{j-1}_{\mathrm{Zar}}(O_{\mathrm{nr}}, \mathbf{Z}(a)) & \to & H^{j-1}_{\mathrm{Zar}}(F_{\mathrm{nr}}, \mathbf{Z}(a)) & \to & H^j_{\mathrm{Zar}}(k(v)_s, i^!\mathbf{Z}(a)) & \to & H^j_{\mathrm{Zar}}(O_{\mathrm{nr}}, \mathbf{Z}(a)) & \to & H^j_{\mathrm{Zar}}(F_{\mathrm{nr}}, \mathbf{Z}(a)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^{j-1}_{\mathrm{et}}(O_{\mathrm{nr}}, \mathbf{Z}(a)) & \to & H^{j-1}_{\mathrm{et}}(F_{\mathrm{nr}}, \mathbf{Z}(a)) & \to & H^j_{\mathrm{et}}(k(v)_s, i^!\mathbf{Z}(a)) & \to & H^j_{\mathrm{et}}(O_{\mathrm{nr}}, \mathbf{Z}(a)) & \to & H^j_{\mathrm{et}}(F_{\mathrm{nr}}, \mathbf{Z}(a))
\end{array}$$

Voevodsky proved the Beilinson-Lichtenbaum conjecture for smooth schemes over a field, and Geisser deduced it for smooth schemes over a discrete valuation ring (in particular, for a DVR itself) [41, Theorem 6.18], [16, Theorem 1.2(2)]. Thus, for both O_{nr} and F_{nr}, the map from Zariski to étale cohomology is an isomorphism for $j \leq a+1$ and injective for $j = a+2$. By the commutative diagram above, it follows that $H^j_{\mathrm{Zar}}(k(v)_s, i^!\mathbf{Z}(a)) \to H^j_{\mathrm{et}}(Y, i^!\mathbf{Z}(a))$ is an isomorphism for $j \leq a+1$. But, by localization in Zariski motivic cohomology, the first group is $\cong H^j_{\mathrm{Zar}}(k(v)_s, \mathbf{Z}(a-1)[-2]) \cong H^j_{\mathrm{et}}(k(v)_s, \mathbf{Z}(a-1)[-2])$ for $j \leq a+1$, using the Beilinson-Lichtenbaum conjecture again. Thus, in $D_{\mathrm{et}}(k(v)_s)$, the map $\mathcal{H}^j(\mathbf{Z}(a-1)[-2]) \to \mathcal{H}^j(i^!\mathbf{Z}(a))$ is an isomorphism for $j \leq a+1$.

Next, consider the diagram above for $j = a+1$. In this case, we have the extra information that $H^{a+2}_{\mathrm{Zar}}(O_{\mathrm{nr}}, \mathbf{Z}(a)) = H^{a+2}_{\mathrm{et}}(O_{\mathrm{nr}}, \mathbf{Z}(a)) = 0$, because O_{nr} is strictly henselian and $\mathbf{Z}(a)$ is concentrated in degrees at most a. Then the diagram above implies that $H^{a+2}_{\mathrm{Zar}}(k(v)_s, i^!\mathbf{Z}(a)) \to H^{a+2}_{\mathrm{et}}(k(v)_s, i^!\mathbf{Z}(a))$ is an isomorphism. The first group is $\cong H^{a+2}_{\mathrm{Zar}}(k(v)_s, \mathbf{Z}(a-1)[-2]) = 0$, using that $\mathbf{Z}(a-1)[-2]$ is concentrated in degrees at most $a+1$. This completes the proof that the map

$$\mathbf{Z}(a-1)[-2] \to \tau_{\leq a+2} i^!\mathbf{Z}(a)$$

in $D_{\mathrm{et}}(k(v))$ is an isomorphism. ∎

Lemma A.2. *Let O_v be a discrete valuation ring, and let E be a locally constant étale sheaf on* $\operatorname{Spec} O_v$. *For each $a \geq 1$,*

$$E(a-1)_{k(v)}[-2] \cong \tau_{\leq a+1} i^!E(a)$$

in $D_{\mathrm{et}}(k(v))$.

Proof. Let $M = i^!\mathbf{Z}(a)$ in $D_{\mathrm{et}}(Y)$. Note that $E \otimes^L_{\mathbf{Z}} M$ is isomorphic to $i^!E(a)$, and that $E(a-1)_{k(v)}[-2]$ is concentrated in degrees at most $a+1$. The result follows from Lemma A.1, by the universal coefficient theorem applied to stalks at any geometric point:

$$0 \to E \otimes_{\mathbf{Z}} \mathcal{H}^j(M) \to \mathcal{H}^j(E \otimes^L_{\mathbf{Z}} M)) \to \mathrm{Tor}_1^{\mathbf{Z}}(E, \mathcal{H}^{j+1}(M)) \to 0.$$

□

Corollary A.3. *Let O_v be a discrete valuation ring, $k(v)$ the residue field, and F the fraction field. Let E be a locally constant étale sheaf on* $\operatorname{Spec} O_v$. *Then, for each $a \geq 1$, we define a residue homomorphism*

$$\partial_v \colon H^a(F, E(a)) \to H^{a-1}(k(v), E(a-1)).$$

Proof. By the basic exact triangle for $i^!$, there is a natural map $H^a(F, E(a)) \to H^{a+1}(k(v), i^! E(a))$. The latter group is (trivially) isomorphic to $H^{a+1}(k(v), \tau_{\leq a+1} i^! E(a))$. By Lemma A.2, that is isomorphic to $H^{a+1}(k(v), E(a-1)[2]) \cong H^{a-1}(k(v), E(a-1))$. □

B Purity for étale motivic cohomology

We prove here some purity properties of étale motivic cohomology. The subtleties occur only for varieties in characteristic $p > 0$. The point is that we only have the localization sequence in its usual form for étale motivic cohomology after inverting p (by Cisinski–Déglise). Nonetheless, we prove some purity results without inverting p, building on work of Geisser, Gros, and Levine [16, 17, 19]. We use these results to define the étale cycle map for twisted Chow groups on regular schemes, without inverting p (Theorem 6.1).

Lemma B.1. *Let X be a regular noetherian scheme of finite type over a field k. Let $i \colon Y \to X$ be the inclusion of a regular subscheme of codimension r. For each $a \geq r$, the canonical morphism*

$$\mathbf{Z}(a-r)[-2r] \to \tau_{\leq a+r+1} i^! \mathbf{Z}(a)$$

is an isomorphism in $D_{\mathrm{et}}(Y)$.

Proof. We can reduce to the case where the field k is perfect. Indeed, if k has characteristic zero, then k is already perfect. If k has characteristic $p > 0$, then X and Y can be defined over some finitely generated field k over $\mathbf{F}_p$. We can view k as the function field of a variety B over $\mathbf{F}_p$. After shrinking B, X is the generic fiber of a regular scheme U of finite type over B, and likewise Y is the generic fiber of a regular subscheme V of codimension r in U. Then U and V are smooth over the perfect field $\mathbf{F}_p$, and it suffices to prove the lemma for V inside U.

So we can assume that X and Y are smooth over a perfect field k. In the Zariski topology, we have $\mathbf{Z}(a-r)[-2r] \cong i^! \mathbf{Z}(a)$ in $D_{\mathrm{Zar}}(Y)$; that is a reformulation of the localization sequence for motivic cohomology. This determines a morphism $\varphi \colon \mathbf{Z}(a-r)[-2r] \to i^! \mathbf{Z}(a)$ in $D_{\mathrm{et}}(Y)$. The object $\mathbf{Z}(a-r)$ in $D_{\mathrm{et}}(Y)$ is concentrated in degrees at most $a-r$, and so $\mathbf{Z}(a-r)[-2r]$ is concentrated in degrees at most $a+r$.

By Cisinski and Déglise, φ becomes an isomorphism after inverting the exponential characteristic of k. That completes the proof for k of characteristic zero. So we now assume that k has characteristic $p > 0$. Let C be the cofiber of φ. Then we know that $C[1/p] = 0$; that is, $\mathcal{H}^j(C)$ is p-power torsion for each integer j.

Tensoring φ over $\mathbf{Z}$ with $\mathbf{F}_p$ gives a morphism $\mathbf{F}_p(a-r)[-2r] \to i^!\mathbf{F}_p(a)$ in $D_{\mathrm{et}}(Y)$. The object $\mathbf{F}_p(a)$ in $D_{\mathrm{et}}(X)$ is concentrated in degree p; namely, by Geisser–Levine, it is isomorphic in $D_{\mathrm{et}}(X)$ to $\Omega^a_{X,\log}[-a]$, where $\Omega^a_{X,\log}$ is the subsheaf of Ω^a_X generated locally by logarithmic differentials $df_1/f_1 \wedge \cdots df_a/f_a$ for units $f_1, \ldots, f_a$ [17]. (This is a sheaf of $\mathbf{F}_p$-vector spaces, not an O_X-module.)

In thsee terms, using that k is perfect, Gros showed that the morphism $\mathbf{F}_p(a-r)[-2r] \to \tau_{\leq a+r} i^!\mathbf{F}_p(a)$ in $D_{\mathrm{et}}(Y)$ is an isomorphism [19, eq. II.3.5.3, Th. II.3.5.8]. (In his notation, this is the statement that $\underline{H}^j_Y(X, \Omega^a_{X,\log})$ is zero for $j < r$ and isomorphic to $\Omega^{a-r}_{Y,\log}$ for $j = r$.) By the octahedral axiom for triangulated categories, we have an exact triangle $\mathbf{F}_p(a-r)[-2r] \to i^!\mathbf{F}_p(a) \to C/p$ in $D_{\mathrm{et}}(Y)$. It follows that $\mathcal{H}^j(C/p) = 0$ for $j \leq a+r$, where in the case $j = a+r$ we use that $\mathcal{H}^{a+r+1}(\mathbf{F}_p(a-r)[-2r]) = 0$. By the exact triangle $C \underset{p}{\to} C \to C/p$, it follows that multiplication by p is an isomorphism on $\mathcal{H}^j(C)$ for $j \leq a+r$ and is injective on $\mathcal{H}^{a+r+1}(C)$. But $\mathcal{H}^j(C)$ is p-power torsion for all j. So $\mathcal{H}^j(C) = 0$ for $j \leq a+r+1$. By definition of C, it follows that $\mathcal{H}^j(\mathbf{Z}(a-r)[-2r]) \to \mathcal{H}^j(i^!\mathbf{Z}(a))$ is an isomorphism for $j \leq a+r+1$. We conclude that

$$\mathbf{Z}(a-r)[-2r] \to \tau_{\leq a+r+1} i^!\mathbf{Z}(a)$$

is an isomorphism in $D_{\mathrm{et}}(Y)$. $\square$

Lemma B.2. *Let X be a regular noetherian scheme of finite type over a field k. Let $i\colon Y \to X$ be the inclusion of a closed subset of codimension at least r everywhere. For each $a < r$,*

$$\tau_{\leq 2a+2} i^!\mathbf{Z}(a) = 0$$

in $D_{\mathrm{et}}(Y)$.

Proof. As in the proof of Lemma B.1, we can reduce to the case where X is smooth over a perfect field k. In this case, we can stratify Y into pieces that are smooth over k. By induction, it suffices to consider the case where Y is a smooth subvariety of codimension at least r in X.

By the localization sequence for étale motivic cohomology, with the exponential characteristic e inverted, for $a < r$, we have $Ri^!\mathbf{Z}(a)[1/e] \cong \oplus_{l \leq e} \mathbf{Q}_l/\mathbf{Z}_l(a-r)[-1-2r]$. It follows that $\tau_{\leq 2a+2} i^!\mathbf{Z}(a)$ becomes zero after inverting e. That completes the proof for k of characteristic zero. So we can assume that k has characteristic $p > 0$. In this case, we have shown that $\mathcal{H}^j(i^!\mathbf{Z}(a))$ is p-power torsion for each $j \leq 2a+2$.

We use again Geisser–Levine's result that $\mathbf{F}_p(a)$ is isomorphic to $\Omega^a_{\log}[-a]$ in $D_{\mathrm{et}}(X)$. Using that $a < r$, Gros showed (using that $a < r$) that $\tau_{\leq a+r} i^!\mathbf{F}_p(a) = 0$ [19, eq. II.3.5.3, Th. II.3.5.8]. (In his terms, when $a < r$, he showed that $\underline{H}^j_Y(X, \Omega^a_{\log}) = 0$ for $j \leq r$.) By the exact triangle $\mathbf{Z}(a) \underset{p}{\to} \mathbf{Z}(a) \to \mathbf{F}_p(a)$ in $D_{\mathrm{et}}(X)$, it follows that the p-torsion subgroup of $\mathcal{H}^j(i^!\mathbf{Z}(a))$ is zero for $j \leq a+r+1$. We have shown that this group is p-power torsion, and so in fact $\mathcal{H}^j(i^!\mathbf{Z}(a))$ is zero for $j \leq a+r+1$. That is, $\tau_{\leq a+r+1} i^!\mathbf{Z}(a) = 0$ in $D_{\mathrm{et}}(Y)$. Since $a < r$, it follows that $\tau_{\leq 2a+2} i^!\mathbf{Z}(a) = 0$. $\square$

Corollary B.3. *Let X be a regular scheme of finite type over a field k. Let E be a locally constant étale sheaf on X.*

(1) Let $i\colon Y \to X$ be the inclusion of a regular subscheme of codimension r. For each $a \geq r$, the canonical morphism

$$E(a-r)[-2r] \to \tau_{\leq a+r} i^! E(a)$$

is an isomorphism in $D_{\text{et}}(Y)$.

(2) Let $i\colon Y \to X$ be the inclusion of a closed subset of codimension at least r everywhere. For each $a < r$,

$$\tau_{\leq 2a+1} i^! E(a) = 0$$

in $D_{\text{et}}(Y)$.

Proof. By definition, $E(a) = E \otimes^L_{\mathbf{Z}} \mathbf{Z}(a)$ in $D_{\text{et}}(X)$. Since $\mathbf{Z}(a-r)[-2r]$ in $D_{\text{et}}(Y)$ is concentrated in degrees at most $(a-r)+2r = a+r$, so is $E(a-r)[-2r]$. Given that, (1) and (2) follow from Lemmas B.1 and B.2, together with the universal coefficient theorem (applied to the stalks at any geometric point):

$$0 \to E \otimes_{\mathbf{Z}} \mathcal{H}^j(i^! \mathbf{Z}(a)) \to \mathcal{H}^j(i^! E(a)) \to \operatorname{Tor}_1^{\mathbf{Z}}(E, \mathcal{H}^{j+1}(i^! \mathbf{Z}(a))) \to 0.$$

Here we have used that $E \otimes^L_{\mathbf{Z}} i^! \mathbf{Z}(a) \cong i^! E(a)$. □

C Twisted motivic cohomology and twisted Chow groups: conjectures

In this section, we propose a general notion of "twist" which should make it possible to define twisted motivic cohomology and twisted Chow groups; the two theories do not always agree. Namely, it should be possible to twist by any birational sheaf with transfers E in the sense of Kahn–Sujatha [27, Definition 2.3.1]. Some cases have been worked out, including the notion of twisting by an Azumaya algebra [26, 14], as well as by a locally constant étale sheaf. This paper has focused on the latter case.

This section is not logically necessary for the rest of the paper. In this section, we assume that the exponential characteristic e of the base field k acts invertibly on E, in order to use the good properties of categories of motives with e inverted. (By definition, $e = 1$ if k has characteristic zero, and $e = p$ if k has characteristic $p > 0$.) I hope that inverting e can be avoided. When E is a locally constant étale sheaf (the main focus of this paper), section 1 defines twisted Chow groups without inverting e.

Let X be a noetherian scheme of finite dimension. Building on Voevodsky's ideas, Cisinski and Déglise defined the derived category of motives $DM(X)$ [5, Definition 11.1.1]. The definition is based on an abelian category $\operatorname{Sh}^{\text{tr}}(X, \mathbf{Z})$, the category of Nisnevich sheaves with transfers [5, Definition 10.4.2]. These are Nisnevich sheaves of abelian groups on the category of smooth separated schemes of finite type over X, with transfers for finite correspondences in a precise sense.

For a presheaf with transfers E over X, we define the *contraction* E_{-1} (following Voevodsky) by

$$E_{-1}(Y) := \operatorname{coker}(E(Y \times A^1) \to E(Y \times G_m))$$

for Y smooth over X. This is also a presheaf with transfers [30, Lecture 23], [27, Definition 2.4.1]. (These references assume that $X = \operatorname{Spec} k$ for a field k, but the same argument applies.) If E is a homotopy invariant sheaf with transfers, then so is E_{-1}.

Define a homotopy invariant sheaf with transfers E over X to be *birational* if $E_{-1} = 0$. The name is motivated by Kahn–Sujatha's result that for a perfect field k, a homotopy invariant sheaf with transfers E over k has $E_{-1} = 0$ if and only if $E(Y) \xrightarrow{\cong} E(U)$ for every dense open subset U of a smooth k-scheme Y [27, Proposition 2.5.2]. It is clear over any base scheme X that a homotopy invariant sheaf with transfers that has the latter property has $E_{-1} = 0$, hence is "birational" in our sense.

Write $HI(X)$ for the full subcategory of homotopy invariant Nisnevich sheaves with transfers. By construction, there is a fully faithful functor $HI(X) \to DM(X)$. As a result, we get a definition of motivic cohomology twisted by an object E in $HI(X)$:

$$H^i_{\mathrm{M}}(X, E(j)) := \operatorname{Hom}_{DM(X)}(1_X, E(j)[i]),$$

for integers i and j. In particular, we have this definition for E a birational sheaf with transfers.

On the other hand, we can also define twisted Chow groups. One could define these to be equal to $H^{2i}_{\mathrm{M}}(X, E(i))$; but we consider a different notion, inspired by Rost's ideas, which mixes the étale and Zariski topologies. Then it becomes an interesting question to compare twisted motivic cohomology and twisted Chow groups; see the examples below.

The idea is that étale sheafification gives a functor from $HI(X)$ to $HI_{\mathrm{et}}(X)$, the category of homotopy invariant étale sheaves with transfers over X. (For $X = \operatorname{Spec} k$, this is [30, Theorem 6.17].) Moreover, $(E_{\mathrm{et}})_{-1} \cong (E_{-1})_{\mathrm{et}}$, and so this functor takes birational Nisnevich sheaves with transfer to birational étale sheaves with transfer. There is a tensor product on the abelian category of étale sheaves with transfer, and hence a derived tensor product on the derived category of étale sheaves with transfer, written $\otimes^{\mathrm{tr}}_{L,\mathrm{et}}$, or $\otimes^{\mathrm{tr}}$ for short. (These tensor products are part of the structure of "premotivic category" constructed in [6, Corollary 2.1.12 and section 2.2.4].)

Suppose that X is a separated scheme of finite type over a field k. In this case, Rost defined the abelian category of cycle modules over X [34].

Conjecture C.1. Let E be a birational Nisnevich sheaf with transfers over X. For every field F over X, define

$$H^*[E](F) = \oplus_{j \geq 0} H^j_{\mathrm{et}}(F, E(j)).$$

Then this is a cycle module over X.

Here $\mathbf{Z}(j)$ denotes Voevodsky's motivic cohomology complex, and $E(j) := E \otimes^{\mathrm{tr}} \mathbf{Z}(j)$. (The derived tensor product is meant there, since $\mathbf{Z}(j)$ with $j \geq 0$ is a complex of sheaves with transfer, not just a sheaf. Indeed, $\mathbf{Z}(j)$ itself can be defined as the derived tensor product of j copies of $\mathbf{Z}(1) \cong G_m[-1]$, $\mathbf{Z}(j) = \mathbf{Z}(1) \otimes^{\mathrm{tr}} \cdots \otimes^{\mathrm{tr}} \mathbf{Z}(1)$.) Conjecture C.1 involves the étale sheafification of these objects, considered over fields. Part of the difficulty for the conjecture is that the relation between Rost's cycle modules and the derived category of motives has only been worked out over a field, not over a more general scheme X [9, 10].

Given the conjecture, Rost's theory gives a definition of twisted Chow groups $CH_i(X,E)$, meaning $A_i(X,H^*[E])_{-i}$ in Rost's notation. That is, $CH_i(X,E)$ is defined as the cokernel of the residue homomorphism

$$\oplus_{x\in X_{(i+1)}} H^1_{\text{et}}(k(x),E(1)) \to \oplus_{x\in X_{(i)}} H^0_{\text{et}}(k(x),E).$$

When X is smooth over k, we define $CH^i(X,E)$ likewise in terms of codimension; so $CH^i(X,E)\cong CH_{n-i}(X,E)$ if X is smooth of dimension n everywhere.

One could define a different notion of twisted Chow groups using the Nisnevich rather than the étale topology on fields; but that would be less interesting, as it would always coincide with twisted motivic cohomology (in bidegree $(2i,i)$). Our definition of twisted Chow groups sits between twisted motivic cohomology and twisted étale cohomology, as follows. Given Conjecture C.1, the proof is the same as that of Corollary 4.5 and Theorem 6.1.

Lemma C.2. *Let X be a smooth scheme of finite type over a field k. Assume Conjecture C.1. Then for every birational Nisnevich sheaf with transfers E over X, we have natural homomorphisms*

$$H^{2i}_{\text{M}}(X,E(i)) \to CH^i(X,E) \to H^{2i}_{\text{et}}(X,E(i)).$$

Neither map is an isomorphism, in general. In the following examples, let X be a smooth scheme over k.

- Let E be the constant sheaf $\mathbf{Z}_X$. Then both $H^{2i}_{\text{M}}(X,\mathbf{Z}(i))$ and $CH^i(X,\mathbf{Z}_X)$ can be identified with the usual Chow group, $CH^i(X)$. (For $CH^i(X,\mathbf{Z}_X)$, this follows from the definition by generators and relations: $H^0_{\text{et}}(k(x),\mathbf{Z}_X)\cong\mathbf{Z}$ and $H^1_{\text{et}}(k(x),\mathbf{Z}_X(1))\cong H^0_{\text{et}}(k(x),G_m)\cong k(x)^*$.) The homomorphism from $CH^i(X)$ to $H^{2i}_{\text{et}}(X,\mathbf{Z}(i))$ is rationally an isomorphism, but not integrally, in general. For example, there are smooth complex varieties X with $CH^2(X)/l$ infinite for a prime number l [35, 38], whereas $H^4_{\text{et}}(X,\mathbf{Z}(2))/l$ is contained in $H^4_{\text{et}}(X,\mathbf{Z}/l(2))$, which is finite.
- Let A be an Azumaya algebra over X, and let $E=\mathbf{Z}^A$ be the Nisnevich sheaf over X associated to K_0^A. By Kahn–Levine and Elmanto–Nardin–Yakerson, this can also be described as the subsheaf of $\mathbf{Z}_X$ that is the image of the rank homomorphism $K_0^A\to\mathbf{Z}_X$ [14, Lemma 2.17]. Then the étale sheafification of $\mathbf{Z}^A$ is simply $\mathbf{Z}_X$, because the Azumaya algebra A is étale-locally trivial. So the homomorphism $H^{2i}_{\text{M}}(X,\mathbf{Z}^A(i))\to CH^i(X,\mathbf{Z}^A)$ is a homomorphism to the usual Chow groups, CH^iX. This homomorphism was considered by Kahn and Levine [26, section 5.9]. It is not always an isomorphism, even for $i=0$. Namely, for a smooth variety X over k, the image of $H^0_{\text{M}}(X,\mathbf{Z}^A(0))\to CH^0(X)=\mathbf{Z}$ is a subgroup of index equal to the index of A over the function field $k(X)$, which can be greater than 1.
- Let E be a birational *étale* sheaf with transfers over X. Then E is in particular a birational Nisnevich sheaf with transfers over X, and so the definitions above apply. In this case, $H^{2i}_{\text{M}}(X,E(i))\to CH^i(X,E)$ is surjective, by inspection

of the generators of $CH^i(X,E)$. One main result of this paper is that this surjection need not be an isomorphism, even for $i=1$.

In particular, let E be a locally constant étale sheaf over X. (We only consider sheaves of abelian groups.) Then E has transfers in a natural way; for $X = \operatorname{Spec} k$, this is [30, Lemma 6.11]. So every locally constant étale sheaf E can be viewed as a birational étale sheaf with transfers. We have seen that $H^2_{\mathrm{M}}(X,E(1)) \to CH^1(X,E)$ need not be an isomorphism, even in this special case (Theorem 14.1).

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