Math. Z. 212, 527–532 (1993)

Mathematische Zeitschrift

The total Chern class is not a map of multiplicative cohomology theories

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Received January 14, 1991; in final form May 15, 1992

1 Introduction

Grothendieck [4] pointed out that, for any topological space X, one can make the set $\prod_{i\geq 0} H^{2i}(X;\mathbf{z})$ into a commutative ring, denoted $M(X)$, in such a way that the augmented total Chern class

$$C\colon\ K^0X \to M(X)$$

becomes a homomorphism of rings. Here the *augmented* total Chern class c: $K^0X \to \prod_{i\geq 0} H^{2i}(X;\mathbf{Z})$ sends a vector bundle E over X to its rank in $H^0(X;\mathbf{Z})$ and to its ith Chern class $c_i \in H^{2i}(X;\mathbf{Z})$ for $i \geqq 1$. The definition of the ring structure on $M(X)$ uses the standard formulas for the total Chern class of a direct sum and tensor product of vector bundles.

It is well known that the ring K^0X is the 0th group of a multiplicative cohomology theory K^*X, and it is natural to ask whether the ring $M(X)$ has a similar extension to a cohomology theory. Segal [5] defined a cohomology theory whose 0th group was isomorphic to $M(X)$, but this theory did not have the desired property that the Chern class map $K^0X \to M(X)$ should extend to a map of cohomology theories. Recently Boyer et al. [3], using Chow varieties, defined a different cohomology theory M^*X such that the abelian group M^0X is naturally isomorphic to $M(X)$, and such that the Chern class map $K^0X \to M^0X$ extends to a map of cohomology theories $k^*X \to M^*X$. (Here k^*X denotes connective K-theory.) This solved Segal's problem (b) in [5].

In this paper we show that the ring-homomorphism c: $K^0X \to M^0X$ does not extend to any map of multiplicative cohomology theories $k^*X \to M^*X$. In fact, the abelian group M^0X is not the 0-term of any multiplicative cohomology theory. This solves Segal's problem (a) in [5]. The proof is an elementary transfer calculation.

2 Statement, and outline of proof

To begin, we define the abelian group $M(X)$. For any space X, let

$$M(X)=H^0(X;\mathbf{Z})\times(1\times\prod_{i\geq 1} H^{2i}(X;\mathbf{Z})),$$

with the group structure on $H^0(X;\mathbf{Z})$ being addition, and with the group structure $\oplus$ on $1\times\prod_{i\geq 1} H^{2i}(X;\mathbf{Z})$ given by the cup product in ordinary cohomology:

$$\begin{aligned}&(x_0;1+x_1+x_2+\ldots)\oplus(y_0;1+y_1+y_2+\ldots)\\&\quad=(x_0+y_0;1+(x_1+y_1)+(x_2+x_1y_1+y_2)+\ldots).\end{aligned}$$

Theorem 1 *Let M^* be a cohomology theory such that there is a natural isomorphism $M^0X\cong M(X)$ of abelian groups. Then M^* cannot be given the structure of a multiplicative cohomology theory.*

As explained in the introduction, the hypothesis of the theorem applies to at least two examples, namely the cohomology theory defined by Segal and the one defined by Boyer et al.

Our references for the notion of a multiplicative cohomology theory, or equivalently of a ring-spectrum, are Adams's books [1] and [2]. In particular, a multiplicative cohomology theory's multiplication is not assumed to be either commutative or associative.

The strategy of the proof is to show that the projection formula $f_*(xf^*y)=(f_*x)y$, which is the basic property of the transfer homomorphism in a multiplicative cohomology theory, cannot hold. (Here $f\colon X\to Y$ is a covering map of topological spaces.)

Specifically, let $G=\mathbf{Z}/2\oplus\mathbf{Z}/2$, $H=\mathbf{Z}/2\subset G$, and consider the double cover $f\colon BH\to BG$. We will exhibit an element $y\in M^0BG$ such that $f^*y=0\in M^0BH$, but (assuming that there is a multiplication $\otimes$ on the cohomology theory M^*) $f_*1\otimes y\neq 0\in M^0BG$. This contradicts the projection formula. So, in fact, there is no multiplicative cohomology theory M^* with the given 0th group.

The inconvenience in this proof is that we have not made any explicit assumptions about the transfer f_* and the product $\otimes$ in M^0X. So we need to derive some information about them from the general situation in order to check that $f_*1\otimes y\neq 0$ in our specific covering.

3 Analysis of the product $\otimes$ on M^0X

The reader is advised to skip this section. Lemma 2, at the end of the section, gives some technical information about an arbitrary product on M^0X which is needed for the proof of Theorem 1. The whole section could be omitted if we were willing to assume that the product $\otimes$ on M^0X was given by the usual formula for the Chern classes of a tensor product, but it seems preferable to prove Theorem 1 as stated.

In this section we analyze the product $\otimes$ on M^0X. We will make the weak assumption that $(M^0,\otimes)$ is a contravariant functor from spaces to (possibly) non-associative rings. That is, the abelian group $M^0 X$ has a bilinear product $M^0 X \otimes_{\mathbf{Z}} M^0 X \to M^0 X$ with a multiplicative identity $1 \in M^0 X$.

For example, consider the case $X = \text{point}$. Then the additive group $M^0 X = H^0(X; \mathbf{Z}) = \mathbf{Z}$. Since the multiplication on this group is bilinear over addition, it must have the form $a \otimes b = kab$ for some fixed $k \in \mathbf{Z}$. In order to have a multiplicative identity, we must have $k = \pm 1$. If $k = -1$, then by reversing the sign of our identification of $M^0 X$ with $M(X)$ we can arrange that $k = 1$; so we can assume that $k = 1$. That is, $M^0(pt.) \cong \mathbf{Z}$ as a ring.

By functoriality, for any space X, the element 1 in the ring $M^0 X$ is the pullback via the map $X \to pt.$ of $1 \in M^0(pt.)$. That is, it is the element

$$1 = (1; 1+0+0+\ldots) \in M^0 X \cong H^0(X; \mathbf{Z}) \times \prod_{n \geqq 1} H^{2n}(X; \mathbf{Z}).$$

Lemma 1 *Suppose that M^0 is a contravariant functor from spaces to rings such that $M^0 X \cong M(X)$ as abelian groups. (The rings $M^0 X$ are not assumed to be commutative or even associative.) Then there are integers p, q such that, for every space X, the multiplication on $M^0 X$ satisfies:*

$$\begin{aligned}(0; 1+x_1+x_2+\ldots) &\otimes (0; 1+0+y_2+0+\ldots) \\ &= (0; 1+0+0+p x_1 y_2 + q(x_1^2 - 2x_2) y_2 + \ldots).\end{aligned}$$

Here the various ellipses "..." denote arbitrary cohomology classes of dimension greater than the dimension of the classes shown.

Proof. The tensor product we are considering, $\otimes: M^0 X \times M^0 X \to M^0 X$, corresponds to a map of classifying spaces

$$\Big(\prod_{n \geqq 0} K(\mathbf{Z}, 2n)\Big) \times \Big(\prod_{n \geqq 0} K(\mathbf{Z}, 2n)\Big) \to \prod_{n \geqq 0} K(\mathbf{Z}, 2n).$$

For the lemma, we consider only the restriction of this map to the product $A \times B$ defined as

$$(*) \qquad A \times B := \Big(\prod_{n \geqq 1} K(\mathbf{Z}, 2n)\Big) \times \Big(K(\mathbf{Z}, 4) \times \prod_{n \geqq 4} K(\mathbf{Z}, 2n)\Big) \to \prod_{n \geqq 1} K(\mathbf{Z}, 2n).$$

In what follows, we will denote by $x = (0; 1+x_1+x_2+\ldots)$ elements of $[X, A] := \prod_{n \geqq 1} H^{2n}(X; \mathbf{Z})$, and by $y = (0; 1+0+y_2+0\ldots)$ elements of $[X, B]$. The tensor product map $(*)$ can be described by cohomology classes $z_n \in H^{2n}(A \times B; \mathbf{Z})$, $n \geqq 1$. The form of such classes is clear, since the cohomology of Eilenberg-MacLane spaces is known:

$$\begin{aligned}
z_1 &= a x_1 \\
z_2 &= b x_1^2 + c x_2 + d y_2 \\
z_3 &= e x_1^3 + f x_1 x_2 + g x_3 + h x_1 y_2 \\
z_4 &= i x_1^4 + j x_1^2 x_2 + k x_1 x_3 + l x_4 + m y_2^2 + n x_1^2 y_2 + o x_2 y_2 \\
&\quad + p y_4 + q t(y_2) + r t(x_2).
\end{aligned}$$

Here a, b, etc. are integers. Also $t \in H^8(K(\mathbf{Z}, 4); \mathbf{Z})_{\text{tors}} \cong \mathbf{Z}/2$ is the only relevant torsion class.

We now use bilinearity of the product $\otimes$ over $\oplus$. Note that, for any space $X, A(X)$ and $B(X)$ are subgroups of $M^0 X$ under $\oplus$. In particular, z_n must be 0 if all of the x_i are 0, or if all of the y_i are 0. This simple observation suffices to show that the z_i must have the following special form:

$$\begin{aligned}
z_1 &= 0\\
z_2 &= 0\\
z_3 &= p x_1 y_2\\
z_4 &= q x_1^2 y_2 + r x_2 y_2,
\end{aligned}$$

$p, q, r \in \mathbf{Z}$. The addition $\oplus$ has a simple form on such z's in dimensions $\leqq 8$:

$$\begin{aligned}
&(0; 1+0+0+z_3+z_4+\ldots)\oplus(0; 1+0+0+z'_3+z'_4+\ldots)\\
&\quad=(0; 1+0+0+(z_3+z'_3)+(z_4+z'_4)+\ldots).
\end{aligned}$$

Therefore each of z_3 and z_4 must be bilinear functions of x and y (with respect to $\oplus$).

We can use the linearity of z_4 as a function of x (for fixed y) to produce a relation between q and r in the formula for z_4. The addition in x has the following form.

$$\begin{aligned}
&(0; 1+x_1+x_2+\ldots)\oplus(0; 1+x'_1+x'_2+\ldots)\\
&\quad=(0; 1+(x_1+x'_1)+(x_2+x_1 x'_1+x'_2)+\ldots).
\end{aligned}$$

The linearity of $z_4 = q x_1^2 y_2 + r x_2 y_2 = (q x_1^2 + r x_2) y_2$ in x implies the linearity of $q x_1^2 + r x_2$ in x, that is:

$$q(x_1+x'_1)^2 + r(x_2+x_1 x'_1+x'_2) = q x_1^2 + r x_2 + q(x'_2)^2 + r x_2.$$

So $(2q+r) x_1 x'_1 = 0$. Since this is to be true for all cohomology classes x_1, x'_1 in H^2 of any space, we must have $2q+r=0\in\mathbf{Z}$. So $z_4 = q(x_1^2 - 2x_2) y_2$, which completes the proof of the lemma.

Lemma 2 *Suppose that M^0 is a contravariant functor from spaces to rings such that $M^0 X \cong M(X)$ as abelian groups. (The ring $M^0 X$ is not assumed to be associative.) Then there are integers p, q such that for all spaces X, the product in $M^0 X$ satisfies:*

$$\begin{aligned}
&(2; 1+x_1+x_2+\ldots)\otimes(0; 1+0+y_2)\\
&\quad=(0; 1+0+2y_2+p x_1 y_2+(y_2^2+q(x_1^2-2x_2) y_2)+\ldots).
\end{aligned}$$

Proof. In addition to Lemma 1, we use the fact, mentioned earlier in this section, that $(1;1)\in M^0 X$ is the multiplicative identity, so that $(2;1)\otimes x = x\oplus x$ for any $x\in M^0 X$.

$$\begin{aligned}
&(2; 1+x_1+x_2+\ldots)\otimes(0; 1+0+y_2)\\
&\quad=(2;1)\otimes(0; 1+0+y_2)\oplus(0; 1+x_1+x_2+\ldots)\otimes(0; 1+0+y_2)\\
&\quad=(0;(1+0+y_2)^2(1+0+0+p x_1 y_2+q(x_1^2-2x_2) y_2+\ldots)\\
&\quad=(0; 1+0+2y_2+p x_1 y_2+(y_2^2+q(x_1^2-2x_2) y_2)+\ldots).
\end{aligned}$$

4 Proof of Theorem 1

Let M^* be a cohomology theory such that $M^0X \cong M(X)$ as abelian groups. Suppose that M^0X is given a natural ring structure $\otimes$, which need not be either commutative or associative. Thus M^0X is just an abelian group with a bilinear product $M^0X \otimes_{\mathbf{Z}} M^0X \to M^0X$ which has a multiplicative identity $1 \in M^0X$. We will produce a finite covering map $f: X \to Y$ and an element $y \in M^0Y$ such that $f^*y=0$ but $(f_* 1) \otimes y \neq 0$. This violates the projection formula, and therefore M^* cannot be made into a multiplicative cohomology theory.

It is not obvious how to compute $f_* 1 \otimes Y$, since we have not made any explicit assumption on the transfer map f_* or on the product $\otimes$. But at least we can say that if $f: X \to Y$ is a covering map of degree d, then $f_* 1$ has "rank d" in M^0Y (Lemma 3), and we can try to find a covering $f: X \to Y$ of some degree d and an element $y \in M^0Y$ such that $f^*y=0$ and $y\otimes$ (anything of rank d in M^0Y) is not 0 (Lemma 4).

Lemma 3 *Let M^* be a cohomology theory such that $M^0X \cong M(X)$ as abelian groups. Let $f: X \to Y$ be a covering map of degree d. Suppose Y is connected. Then the transfer $f_*: M^0X \to M^0Y$ satisfies*

$$f_* 1 = (d; 1 + \ldots) \in M^0 Y.$$

We will say that $f_* 1$ has "rank d" in M^0Y.

Proof. Let y be a point in Y. Then the covering becomes trivial over y. But for any trivial covering map $\coprod_{d \text{ copies}} Z \to Z$, the transfer map of a cohomology theory is just the addition in that theory. Since the addition $\oplus$ in M^0X restricts to addition in $M^0(\text{point}) \cong \mathbf{Z}$, the result follows. Q.E.D.

Lemma 4 *Let $(M^0, \otimes)$ be a contravariant functor from spaces to (possibly nonassociative) rings such that $M^0X \cong M(X)$ as abelian groups. There is a space Y with a double cover $f: X \to Y$ and an element $y \in M^0Y$ such that $f^*y=0$ and*

$$y \otimes (\textit{anything of rank } 2 \textit{ in } M^0\, Y) \neq 0.$$

Proof. By Lemma 2, it suffices to find a space Y with a double cover $f: X \to Y$, and a class $y_2 \in H^4(Y; \mathbf{Z})$, such that $f^*y_2 = 0$ and

$$y_2^2 + q(x_1^2 - 2x_2)\, y_2 \neq 0 \in H^8(Y; \mathbf{Z})$$

for any $q \in \mathbf{Z}$, $x_1 \in H^2(Y; \mathbf{Z})$, $x_2 \in H^4(X; \mathbf{Z})$.

Let

$$G = \mathbf{Z}/2 \oplus \mathbf{Z}/2,$$
$$H = \mathbf{Z}/2 \subset G.$$

We will show that the double cover $f: BH \to BG$ has the required properties. (Thus $X = BH$ and $Y = BG$, in the statement of the lemma.) We know that

$$H^2(BG; \mathbf{Z}) \cong \mathbf{Z}/2 \oplus \mathbf{Z}/2.$$

Let a and b the generators for this group such that $f^*a = 0$ in $H^2(BH; \mathbf{Z}) = \mathbf{Z}/2$.

Let $y_2 = ab \in H^4(BG; \mathbf{Z})$; clearly $f^* y_2 = 0$ and $2y_2 = 0$. According to the first paragraph of this proof, we need to show that

$$y_2^2 + q(x_1^2 - 2x_2)\, y_2 = y_2(y_2 + q x_1^2) \neq 0,$$

for any $q \in \mathbf{Z}$, $x_1 \in H^2(BG; \mathbf{Z})$, $x_2 \in H^4(BH; \mathbf{Z})$. Since $2y_2 = 0$, this expression simplifies to $y_2(y_2 + q x_1^2)$. Now the even-dimensional cohomology $H^{\mathrm{ev}}(BG; \mathbf{Z})$ is a $\mathbf{Z}/2$-polynomial algebra generated by a and b. Since $y_2 = ab$ is not a $\mathbf{Z}/2$-multiple of a square in the polynomial ring $\mathbf{Z}/2[a, b]$, we have $y_2 + q x_1^2 \neq 0$ for any $q \in \mathbf{Z}$. Since $H^{\mathrm{ev}}(BG; \mathbf{Z})$ is an integral domain, we have

$$y_2^2 + b(x_1^2 - 2x_2)\, y_2 = y_2(y_2 + b x_1^2) \neq 0 \in H^8(BG; \mathbf{Z}). \quad \text{QED}$$

Suppose that there is a multiplicative cohomology theory $(M^*, \otimes)$ such that $M^0 X \cong M(X)$ as abelian groups. Let $f: BH \to BG$ be the covering map constructed in the proof of Lemma 4. Lemmas 3 and 4 combine to prove that $(0; 1 + 0 + ab) \in M^0 BG$ has the property that $f^*(0; 1 + 0 + ab) = 0 \in M^0 BH$, but that

$$f_* 1 \otimes (0; 1 + 0 + ab) \neq 0 \in M^0 BG.$$

In fact, Lemma 3 shows that $f_* 1$ has rank 2 in $M^0 BG$, and Lemma 3 shows that the product of $(0; 1 + 0 + ab) \in M^0 BG$ with any element of rank 2 in $M^0 BG$ is not zero.

We recall that the projection formula is valid for any multiplicative cohomology theory A^*, by [2, pp. 127–128]. It asserts (about A^0) that, for a finite covering $f: X \to Y$, $x \in A^0 X$, $y \in A^0 Y$,

$$f_*(x f^* y) = (f_* x)\, y.$$

In particular, $(f_* 1)\, y = f_*(f^* y)$ must be 0 if $f^* y = 0$. Therefore the previous paragraph's result contradicts the projection formula. So there is no multiplicative cohomology theory M^* such that $M^0 X \cong M(X)$ as abelian groups. The theorem is proved.

5 Variants

A similar proof shows that the ring $M_{\mathbf{R}}(X) = \prod_{n \geq 0} H^n(X; \mathbf{Z}/2)$, which is related to Stiefel-Whitney classes, cannot be made into a multiplicative cohomology theory. We can use the same covering map $BH \to BG$, and we can find classes $a, b \in H^1(BG; \mathbf{Z}/2)$ such that, for any ring structure on $M_{\mathbf{R}}(X)$, the product in $M_{\mathbf{R}}(BG)$ of $(0; 1 + 0 + ab)$ with any element of rank 2 is not zero. Then the proof goes as before.

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