The torsion index of E_8 and other groups

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The torsion index is a positive integer associated by Grothendieck to any connected compact Lie group G [10]. Knowing the torsion index of a group has direct consequences for the integral cohomology of the classifying space BG, the complex cobordism of BG, the Chow ring of BG, and the classification of G-torsors over fields. These applications of the torsion index are summarized in my paper [25], section 1. The best upper bounds for the torsion index so far have been those of Tits [21]. In this paper, we show in particular that the exceptional group E_8 has torsion index equal to $2^6 3^2 5 = 2880$. Together with my paper on the spin groups [25], this completes the calculation of the torsion index for all simply connected compact Lie groups. We also compute the torsion index of the groups $PSO(2n)$, $E_6/(\mathbf{Z}/3)$, and $E_7/(\mathbf{Z}/2)$ in this paper, which completes the calculation of the torsion index for all compact Lie groups of adjoint type.

The calculation of the torsion index of E_8 implies the optimality of my result that every E_8-torsor over a field becomes trivial over some field extension of degree dividing $2^6 3^2 5$ [24]. (Equivalently, every algebraic group of type E_8 over a field becomes split over some field extension of degree dividing $2^6 3^2 5$.) Indeed, by Grothendieck's theorem on the torsion index [10] together with this paper's calculation, there is a field k and an E_8-torsor X over k such that any finite extension field over which X becomes trivial has degree a multiple of $2^6 3^2 5$ over k. Until now the best estimates of this number were that it must be between $2^2 3 \cdot 5 = 60$ and $2^9 3^3 5 = 69120$, by Tits [21].

The prime numbers p dividing the torsion index of G are precisely those such that the integral cohomology of the classifying space BG has p-torsion, or equivalently those such that the integral cohomology of G itself has p-torsion. These "torsion primes" are known for all compact Lie groups, the final answer being given by Borel in 1961 [5]: they are the primes dividing the order of the torsion subgroup of the fundamental group of G, together with 2 if the universal cover of G has a simple factor of type $Spin(n)$ for $n \geq 7$ or G_2, 2 and 3 in the cases F_4, E_6, and E_7, and 2, 3, and 5 in the case E_8.

To define the torsion index, let T be a maximal torus in a compact Lie group G, and let N be the complex dimension of the flag manifold G/T. Each character of the torus T determines a complex line bundle on G/T. Consider the subring of the integral cohomology of G/T generated by the Chern classes in $H^2(G/T, \mathbf{Z})$ of these line bundles. Then the *torsion index* of G is defined as the smallest positive integer $t(G)$ such that $t(G)$ times the class of a point in $H^{2N}(G/T, \mathbf{Z})$ belongs to this subring.

Let me sum up the calculations that have been made of the torsion index. Probably the most important early result, although it was stated somewhat differently, is an upper bound for the torsion index in terms of the torsion index of a subgroup of maximal rank (Lemma 2.2), due to Borel in 1955 [4]. Demazure gave a different proof of this upper bound (Lemma 7 in [8]), in order to reprove Borel's calculation of the primes dividing the torsion index. Marlin in 1974 gave an upper bound for the torsion index of $SO(n)$ which turns out to be an equality (see section 2), a fairly good upper bound for the torsion index of the spin groups, and calculations of the torsion index for G_2 and F_4, although the result for F_4 is mistaken [14]. Finally, Tits in 1992 found the best results so far on the torsion index [21]. In particular, his Proposition 2 implies the calculations (for the simply connected groups): $t(G_2) = 2$, $t(F_4) = 2 \cdot 3$, $t(E_6) = 2 \cdot 3$, and $t(E_7)|2^2 3$. In fact, we will show without much difficulty that $t(E_7) = 2^2 3$.

After Tits's paper, the remaining simply connected simple groups were the group E_8 and the spin groups. For E_8, the known result that the Dynkin index is 60 implies that the torsion index of E_8 is a multiple of $60 = 2^2 3 \cdot 5$, as Serre showed by a different argument in ([22], Proposition 9). For an upper bound, Tits only shows that the torsion index of E_8 divides $2^9 3^3 5$. He suggests the "hypothèse optimiste" that the torsion index of E_8 is 60. The main result of this paper is that E_8 is much more complex than that, both 2- and 3-locally: the torsion index of E_8 is $2^6 3^2 5$. The hardest part is to prove the lower bound 2^6 for the 2-part of the torsion index of E_8.

This completes the calculation of the torsion index for all simply connected compact Lie groups. First, the torsion index of the product of two groups is the product of the torsion indices, and so it suffices to consider simply connected simple groups. Of these, the groups $SU(n)$ and the symplectic groups have torsion index 1, and we have just stated the torsion indices of the simply connected exceptional groups. The remaining simply connected simple groups are the spin groups. Here is the result, from my paper [25]:

Theorem 0.1 *Let l be a nonnegative integer. The groups $Spin(2l+1)$ and $Spin(2l+2)$ have the same torsion index, of the form $2^{c(l)}$. For all l, $c(l)$ is either*

$$l - \left\lfloor \log_2 \left(\binom{l+1}{2} + 1 \right) \right\rfloor$$

or that expression plus 1. The second case arises only for certain numbers l (initially: $l = 8, 16, 32, 33, \ldots$) which are equal to or slightly larger than a power of 2. Precisely, the second case arises if and only if $l = 2^e + b$ for some nonnegative integers e, b such that $2b - c(b) \leq e - 3$.

Finally, we compute the torsion index for all compact Lie groups of adjoint type. The known cases are $PU(n) = SU(n)/(\mathbf{Z}/n)$, which has torsion index n, and $Sp(2n)/(\mathbf{Z}/2)$ (where $Sp(2n)$ denotes the symplectic group of rank n), which has torsion index $2^{1+\text{ord}_2(n)}$. These two calculations follow, using that the universal cover has torsion index 1, from Merkurjev [16], 4.1 and 4.3. In this paper, we compute that the torsion index of $PSO(2n)$ is 2^{n-1} for all n not a power of 2, and 2^n if n is a power of 2 and $n \geq 2$. Also, we show that $E_6/(\mathbf{Z}/3)$ has torsion index $2 \cdot 3^3$ and $E_7/(\mathbf{Z}/2)$ has torsion index $2^3 3$. These results complete the calculation of the torsion index for all compact Lie groups of adjoint type.

I checked some elementary but long calculations in this paper using the computer algebra program Macaulay 2. My thanks go to the authors, Dan Grayson and Mike Stillman. Also, thanks to Jean-Pierre Serre for some useful comments.

1 Notation

We define a *Levi subgroup* of a compact Lie group G to be the centralizer of a torus in G. This name has become standard, by analogy with complex algebraic groups. The complexification of a Levi subgroup of a compact Lie group G, in this sense, is a Levi subgroup of a parabolic subgroup of the complexification of G.

2 Easy bounds for the torsion index

A convenient fact is that the calculation of the torsion index for any compact connected Lie group reduces easily to the case of semisimple groups, as follows.

Lemma 2.1 *For any compact connected Lie group G, the torsion index of G is equal to the torsion index of the derived group $G_{der} = [G, G] \subset G$.*

Proof. By definition, the torsion index of G is the smallest positive integer $t(G)$ such that $t(G)$ times the class of a point in $H^*(G/T, \mathbf{Z})$ is in the image of $H^*(BT, \mathbf{Z})$, which is the polynomial algebra over the integers generated by the group of characters $X^*(T) = H^2(BT, \mathbf{Z})$. We have a commutative diagram of fibrations,

$$\begin{array}{ccccc} G_{\text{der}}/T \cap G_{\text{der}} & \longrightarrow & B(T \cap G_{\text{der}}) & \longrightarrow & BG_{\text{der}} \\ \downarrow & & \downarrow & & \downarrow \\ G/T & \longrightarrow & BT & \longrightarrow & BG, \end{array}$$

where T is a maximal torus in G. Here $G_{\text{der}}/T \cap G_{\text{der}}$ maps isomorphically to G/T. So the lemma will follow if we can show that the restriction map

$$H^*(BT, \mathbf{Z}) \to H^*(B(T \cap G_{\text{der}}), \mathbf{Z})$$

is surjective. For that, it suffices to show that the homomorphism of groups of characters,

$$X^*(T) \to X^*(T \cap G_{\text{der}}),$$

is surjective. That follows from the inclusion $T \cap G_{\text{der}} \subset T$, since these groups are diagonalizable. QED

The following lemma, essentially due to Borel [4], gives good bounds for the torsion index very easily in some cases.

Lemma 2.2 *Let H be a closed connected subgroup of maximal rank in a compact connected Lie group G. Then the torsion index $t(G)$ divides $t(H)\chi(G/H)$, where χ denotes the topological Euler characteristic.*

Proof. As mentioned in the introduction, there are several ways to prove this. We follow the oldest way, due to Borel [4], which is formulated in somewhat different terms. We consider the fibration $G/H \to BH \to BG$, where G/H is a closed real manifold. The tangent bundle along the fibers is a real vector bundle on BH. In fact, it is the vector bundle associated to the representation of H on the quotient of Lie algebras g/h. Since H is connected, this representation of H maps into $SO(g/h)$, and so we can choose an orientation of the associated real vector bundle on BH.

Let $y \in H^n(BH, \mathbf{Z})$ be the Euler class of this oriented real vector bundle, where n is the dimension of G/H. The restriction of y to G/H is the Euler class of the tangent bundle of G/H, which is $\chi(G/H)$ times the class of a point in $H^n(G/H, \mathbf{Z}) \cong \mathbf{Z}$.

Now consider the fibration of closed manifolds $H/T \to G/T \to G/H$. The restriction of y to BT gives an element $y \in H^n(BT, \mathbf{Z})$ whose image in $H^n(G/T, \mathbf{Z})$ is $\chi(G/H)$ times the class of a fiber $H/T \subset G/T$. Also, let m be the dimension of H/T. By definition of the torsion index, there is an element $z \in H^m(BT, \mathbf{Z})$ whose image in $H^m(H/T, \mathbf{Z})$ is $t(H)$ times the class of a point. So the element $yz \in H^{m+n}(BT, \mathbf{Z})$ restricts to $\chi(G/H)t(H)$ times the class of a point in G/T. Therefore $t(G)|\chi(G/H)t(H)$. QED

Hopf and Samelson in 1940 found a simple algebraic interpretation of the Euler characteristic $\chi(G/H)$: it is nonzero if and only if H has maximal rank in G, and in that case it equals the index of the Weyl group W_H as a subgroup of W_G [11]. See the textbook [17], p. 393, for example.

When it works, Lemma 2.2 is an ideally simple way to bound the torsion index. For example, to estimate the torsion index of $G = SO(2n)$, use the subgroup $H = U(n)$ of maximal rank. We have $|W_{SO(2n)}| = 2^{n-1}n!$, $|W_{U(n)}| = n!$, and so $\chi(SO(2n)/U(n)) = 2^{n-1}$. Since the integral cohomology of $BU(n)$ is torsion-free, we have $t(U(n)) = 1$, and so Lemma 2.2 gives that $t(SO(2n))$ divides 2^{n-1}. Likewise, considering $U(n) \subset SO(2n+1)$, we find that $t(SO(2n+1))$ divides 2^n. These are both equalities, as follows from Merkurjev [16], 4.2 and 4.4; other proofs were given by Reichstein-Youssin ([19], 5.2) and me ([25], 3.2). This calculation clears up a possible misunderstanding: the torsion subgroup of $H^*(BSO(n), \mathbf{Z})$ is killed by 2 for all n (see [17], p. 145), so the torsion index of a group G is only a (multiplicative) upper bound for the order of torsion elements in the integral cohomology of BG. Notice, however, that the primes p dividing the torsion index are exactly those for which there is p-torsion in $H^*(BG, \mathbf{Z})$, by Borel [5].

Let us go through some further cases in which Lemma 2.2 is all we need to find the optimal upper bound for the torsion index. We can use the subgroups of maximal rank in simple compact Lie groups that are listed by Borel and de Siebenthal [6], or more explicitly (giving the group, not just the Lie algebra) in [17], pp. 304–306. For example, the group G_2 has a torsion-free subgroup $SU(3)$, with

$$\chi(G_2/SU(3)) = [W_{G_2} : W_{SU(3)}] = 12/6 = 2,$$

and so $t(G_2)|2$; since G_2 does have 2-torsion, we have $t(G_2) = 2$.

The group $PU(n) = SU(n)/(\mathbf{Z}/n)$ contains the torsion-free subgroup $SU(n-1)$, with $\chi(PU(n)/SU(n-1)) = \chi(\mathbf{CP}^{n-1}) = n$, and so $t(PU(n))$ divides n. This is an equality for the very simple reason that $t(PU(n))$ kills the torsion in $H^*(BPU(n), \mathbf{Z})$ and we have

$$H^3(BPU(n), \mathbf{Z}) \cong H^2(PU(n), \mathbf{Z}) \cong \mathrm{Hom}(\pi_1 PU(n), S^1) \cong \mathbf{Z}/n.$$

Notice, however, that this last argument would only give the lower bound 2 for the torsion index of $SU(4)/(\mathbf{Z}/2) \cong SO(6)$, while in fact that torsion index is 4. More generally, one can check that for any divisor a of n, the torsion index of $SU(n)/(\mathbf{Z}/a)$ is the product of all primes dividing a, raised to the powers to which they occur in n.

For the exceptional groups F_4, E_6, E_7, we get some information about the torsion index from Lemma 2.2. However, Tits has already given the optimal upper bounds in these cases: by his Proposition 2 [21], the torsion index of F_4, E_6, E_7 divides $2 \cdot 3, 2 \cdot 3, 2^2 \cdot 3$ respectively. (This implication uses Grothendieck's interpretation of the torsion index; see [10] or Theorem 1.1 in my paper [25].) Since these groups do have 2- and 3-torsion [5], it follows that F_4 and E_6 have torsion index $2 \cdot 3$. To see that E_7 has torsion index $2^2 \cdot 3$ rather than $2 \cdot 3$, we can use the calculation of its Dynkin index, as follows.

In general, the Dynkin index of a homomorphism $G \to H$ between simply connected simple groups is defined to be the integer corresponding to the homomorphism $H^4(BH, \mathbf{Z}) \to H^4(BG, \mathbf{Z})$, both groups being canonically isomorphic to $\mathbf{Z}$. The Dynkin index of a simply connected simple group G is defined to be the greatest common divisor of the Dynkin indices of all representations $G \to SU(N)$. The Dynkin indices of all simply connected compact Lie groups were computed by Laszlo and Sorger ([13], Proposition 2.6), using Dynkin's calculations. It is 1 for $SU(n)$ and $Sp(2n)$, 2 for $Spin(n)$ for $n \geq 7$, and $2, 2 \cdot 3, 2 \cdot 3, 2^2 3, 2^2 3 \cdot 5$ for the exceptional groups G_2, F_4, E_6, E_7, E_8.

Lemma 2.3 *For any simply connected compact Lie group G, the Dynkin index divides the torsion index.*

Proof. We use the theorem of Atiyah-Hirzebruch-Segal that the topological K-theory of BG is the completed representation ring [3]. Let u denote a generator of $H^4(BG, \mathbf{Z}) \cong \mathbf{Z}$. Then, by Atiyah-Hirzebruch-Segal's theorem, the Dynkin index is the smallest positive integer $d(G)$ such that $d(G)u$ is a permanent cycle in the Atiyah-Hirzebruch spectral sequence $H^*(BG, \mathbf{Z}) \Rightarrow K^*(BG)$.

On the other hand, we know that the product of any element of $H^*(BG, \mathbf{Z})$ with the torsion index $t(G)$ lies in the image of MU^*BG, by Corollary 1.4 in my paper [25]. Moreover, classes in the image of MU^*BG are automatically permanent cycles in the Atiyah-Hirzebruch spectral sequence. Therefore $d(G)$ divides $t(G)$. QED

For E_8, the Levi subgroup $H = (E_7 \times S^1)/(\mathbf{Z}/2)$ has $\chi(E_8/H) = 2^4 3 \cdot 5$. Since E_7 has torsion index $2^2 3$, Lemma 2.1 shows that H also has torsion index $2^2 3$. So the torsion index of E_8 divides $(2^4 3 \cdot 5)(2^2 3) = 2^6 3^2 5$. The upper bound at 5 is optimal since E_8 does have 5-torsion. We will show later that the upper bounds at 2 and 3 are also optimal, the calculation of the torsion index of E_8 at the prime 2 being the central part of this paper. Lemma 2.3 on the Dynkin index only shows that the torsion index of E_8 is a multiple of $2^2 3 \cdot 5 = 60$.

Here is one approach to proving the desired lower bound for the torsion index of E_8 at the prime 2 which seems not to work. Namely, we can consider the semispin group $Ss(16)$, which is a maximal-rank subgroup of E_8. Here $\chi(E_8/Ss(16)) = 3^3 5$, and so the 2-part of the torsion index of E_8 divides that of the semispin group $Ss(16)$. The group $Ss(16)$ is much smaller than E_8, and it is much easier to compute the torsion index of $Ss(16)$: it turns out to be 2^6. It is tempting to think that this implies that the torsion index of E_8 is a multiple of 2^6, but in fact there seems to be no argument in this direction. (When H is a *Levi* subgroup of G, Demazure showed that the torsion index of H divides that of G ([8], Lemma 7).)

In fact, there is a counterexample to a similar statement. The simply connected group E_7 has a maximal-rank subgroup isomorphic to $SL(8)/(\mathbf{Z}/2)$. Here E_7 has torsion index $2^2 \cdot 3$, while $SL(8)/(\mathbf{Z}/2)$ has torsion index 2^3. Thus the torsion index of a maximal-rank subgroup does not in general divide the torsion index of the whole group.

3 The torsion index of E_8 at the prime 3

We showed in section 2 that the 3-part of the torsion index of E_8 divides 3^2; here we will show that it is actually equal to 3^2. We use the following method. By Atiyah-Hirzebruch-Segal, we know that for any compact Lie group G, the topological K-theory of BG is the completed representation ring of G [3]. As a result, if a cohomology class $x \in H^{2i}(BG, \mathbf{Z})$ is a permanent cycle in the Atiyah-Hirzebruch spectral sequence from integral cohomology to K-theory, then there is a virtual complex representation W of G such that x is the bottom term of the Chern character of W (assuming that x is nonzero in rational cohomology). That is, the Chern character $\mathrm{ch}_j W \in H^{2j}(BG, \mathbf{Q})$ is 0 for $j < i$ and equal to x for $j = i$.

On the other hand, we know that multiplying any element of $H^*(BG, \mathbf{Z})$ by the torsion index $t(G)$ lies in the image of MU^*BG, by Corollary 1.4 in my paper [25]. Moreover, classes in the image of MU^*BG are automatically permanent cycles in the Atiyah-Hirzebruch spectral sequence. So, to show that the torsion index of E_8 at the prime 3 is a multiple of 3^2, it suffices to find an $i \geq 0$ and an element of $3 \cdot H^{2i}(BG, \mathbf{Z})$ whose image in $H^{2i}(BG, \mathbf{Q})$ is not the bottom term of the Chern character of any virtual representation of G.

The known calculation that E_8 has Dynkin index $60 = 2^2 3 \cdot 5$ means that $\mathrm{ch}_2(W) \in 60 \cdot H^4(BE_8, \mathbf{Z})$ for every virtual representation W of E_8. In particular, $\mathrm{ch}_2(W) \equiv 0 \pmod 3$ for all virtual representations W of E_8. (We will write $x \equiv y \pmod{3^r}$, for $x, y \in H^*(BE_8, \mathbf{Q})$, to mean that $x - y$ is in the image of $3^r H^*(BE_8, \mathbf{Z})$.) We need to make an analogous calculation in H^8, as expressed in the following lemma. To formulate this, we will use the known calculation of the cohomology of BE_8 in low dimensions. Namely, there is a map $BE_8 \to K(\mathbf{Z}, 4)$ which is 16-connected ([17], Theorem VI.7.15, p. 362). In particular, let u be the generator of $H^4(BE_8, \mathbf{Z}) \cong \mathbf{Z}$ such that the adjoint representation of E_8 has $\mathrm{ch}_2 = 60u$. Then $H^8(BE_8, \mathbf{Z}_{(3)})$ is isomorphic to $\mathbf{Z}_{(3)}$, generated by u^2.

Lemma 3.1 *For any virtual representation W of E_8, we have $ch_2 W \equiv 0 \pmod 3$ and*

$$u\, ch_2 W \equiv ch_4 W \pmod{3^2}.$$

This lemma implies the desired lower bound for the torsion index of E_8 at the prime 3. Indeed, it implies that any virtual representation W of E_8 with $\mathrm{ch}_j W = 0$ for $j < 4$ has $\mathrm{ch}_4 W \equiv 0 \pmod{3^2}$. Thus the element $3u^2 \in H^8(BE_8, \mathbf{Z})$, as well as the product of this element with any integer prime to 3, is not a permanent cycle in the Atiyah-Hirzebruch spectral sequence. Therefore, the 3-part of the torsion index of E_8 is a multiple of 3^2 by the argument above.

Proof of lemma. The fact that all virtual representations W of E_8 have $\mathrm{ch}_2 W \equiv 0 \pmod 3$ follows from the known calculation that E_8 has Dynkin index $60 = 2^2 3 \cdot 5$. We can also prove it at the same time as we prove that $u\,\mathrm{ch}_2 W \equiv \mathrm{ch}_4 W \pmod{3^2}$. As a preliminary step, used in the following calculations, we know that all virtual representations of E_8 have Chern character ch_i equal to 0 in $H^*(BE_8, \mathbf{Q})$ for i odd, just because $H^*(BE_8, \mathbf{Q})$ is zero in these degrees.

We use that the Chern character takes sums of virtual representations to sums in $H^*(BE_8, \mathbf{Q})$ and tensor products to products. It is straightforward to check that the property of the Lemma is preserved under sums and products (using that ch_0 of a virtual representation is its rank, which is an integer). So it suffices to prove the lemma for any set of virtual representations which generate the representation ring $R(E_8)$ as a $\mathbf{Z}$-algebra.

Next, we show that if the property of the Lemma holds for one virtual representation W, then it holds for all exterior powers $\Lambda^i W$. In view of the identity $\Lambda^i(W+1) = \Lambda^i W + \Lambda^{i-1} W$, it suffices to prove this statement for virtual representations W of rank 0. In that case, the formulas for the Chern character are fairly simple, using that the Chern character of W is 0 in odd degrees:

$$\begin{aligned}
\mathrm{ch}_2 \Lambda^i W &= (-1)^{i+1} i \, \mathrm{ch}_2 W \\
\mathrm{ch}_4 \Lambda^i W &= (-1)^{i+1} i^3 \mathrm{ch}_4 W + (-1)^i \frac{i(i^2-1)}{12} (\mathrm{ch}_2 W)^2.
\end{aligned}$$

Thus, suppose that W has rank 0 and satisfies the property of the Lemma. Then, for i a multiple of 3, these formulas show that $\mathrm{ch}_2 \Lambda^i W$ and $\mathrm{ch}_4 \Lambda^i W$ are both 0 modulo 3^2, and so $\Lambda^i W$ satisfies the property of the Lemma. Likewise, if i is not a multiple of 3, then these formulas show that $\mathrm{ch}_2 \Lambda^i W = (-1)^{i+1} i \, \mathrm{ch}_2 W$ and $\mathrm{ch}_4 \Lambda^i W = (-1)^{i+1} i \, \mathrm{ch}_4 W \pmod{3^2}$, using that $i^2 = 1 \pmod 3$. So again $\Lambda^i W$ satisfies the property of the Lemma.

Thus, it suffices to check the Lemma for any virtual representations of E_8 which generate the representation ring as λ-ring. By Adams [1], Corollary 2, the representation ring of E_8 is generated as a λ-ring by 3 irreducible representations, those whose highest weight is a fundamental weight corresponding to an "extremity" of the Dynkin diagram (in Bourbaki's notation [7], ω_8, ω_1, and ω_2). Therefore, it suffices to compute the Chern character of these 3 representations (of dimensions 248, 3875, and 147250) in degrees at most 4.

To make the calculation a little easier, we can identify the representation ring of E_8 with the invariants of the Weyl group W on a maximal torus T. Then it is clear that there is a virtual representation of E_8 whose character is exactly the sum of the weights in the Weyl group orbit of ω_8, or of ω_1, or of ω_2. Also, it is clear that these virtual representations ρ_8, ρ_1, ρ_2 generate the representation ring of E_8 as a λ-ring, since they differ from the above irreducible representations by "lower" terms. So it suffices to compute to compute the Chern characters of these three virtual representations in degrees at most 4. Since the ring $H^*(BE_8, \mathbf{Q})$ is generated by u in this range, we just need to compute the coefficients of 1, u, and u^2, and we can do so by restricting the given virtual representations to a suitable 1-dimensional torus inside T. The result is:

$$\begin{aligned}
\mathrm{ch}(\rho_8) &= 240 + 60u + 6u^2 + \cdots \\
\mathrm{ch}(\rho_1) &= 2160 + 1080u + 216u^2 + \cdots \\
\mathrm{ch}(\rho_2) &= 17280 + 17280u + 6912u^2 + \cdots
\end{aligned}$$

Thus ρ_1 and ρ_2 have ch_2 and ch_4 equal to $0 \pmod{3^2}$, and ρ_8 has ch_2 equal to $0 \pmod 3$ and ch_4 equal to $u\,\mathrm{ch}_2 \pmod{3^2}$. Thus the property of the Lemma holds for these three virtual representations of E_8, and hence for all virtual representations of E_8. QED

4 The torsion index of E_8 at the prime 2

Theorem 4.1 *The torsion index of E_8 is $2^6 3^2 5 = 2880$.*

We know that the torsion index of E_8 divides $2^6 3^2 5$, and also that the 3- and 5-parts of the torsion index are exactly 3^2 and 5. So it remains to show that the torsion index of E_8 is a multiple of 2^6. That will be the main part of this paper.

The basis of the proof is Toda's theorem that $H^*(E_8/T, \mathbf{Z})$ is a complete intersection ring, with generators and relations in known degrees [23]. Building on Toda's result, Kono and Ishitoya gave an explicit calculation of the ring $H^*(E_8/T, \mathbf{Z}/2)$, with an error that we now correct [12].

Kono-Ishitoya's error is in their formula for δ in 3.11 and 5.11: they say that the element γ_{15} of $H^*(E_8/T, \mathbf{Z}/2)$ which they define has square equal to $\gamma_3^2 c_4^4 c_1^8$; in fact, its square is $\gamma_3^2 c_4^4 c_1^8 + \gamma_3^2 c_4^6$. This is the result of some mistaken calculations of Steenrod operations, which one easily corrects. (In their Theorem 5.9, $\mathrm{Sq}^8 \gamma_{15}$ should be $\gamma_3(c_4^4 + c_4^3 c_1^4 + c_4^2 c_1^8)$; $\mathrm{Sq}^{16}\gamma_{15}$ should be $\gamma_3(c_4^5 + c_4^4 + c_1^4 + c_4^3 c_1^8) + \gamma_5 c_6 c_4^2 c_1^4$; and then redoing the proof of Corollary 5.11 yields the above corrected formula for δ.) It is reassuring that this error became visible naturally, upon comparing Kono and Ishitoya's calculation to my calculation of $H^*(E_8/A_8, \mathbf{Z}/2)$, below.

Even after finding an explicit description of the ring $H^*(E_8/T, \mathbf{Z}_{(2)})$ by generators and relations, it would still be a massive task to compute the torsion index; a priori, we would have to consider all monomials of degree 120 (since E_8/T has complex dimension 120) in 8 variables, so that a direct approach would be impossible. We therefore proceed to simplify the problem. First, we note that it suffices to analyze the ring $H^*(E_8/A_8, \mathbf{Z}_{(2)})$, where A_8 denotes the subgroup of E_8 isomorphic to $SU(9)/(\mathbf{Z}/3)$. Next, we compute the relations in this ring explicitly, in fact in a useful form, making up a "Gröbner basis" for the ring. Finally, by inspecting these relations modulo 4, we define a new valuation on the ring $H^*(E_8/A_8, \mathbf{Z}_{(2)})$. This valuation is just barely strong enough to prove that all monomials of top degree in the appropriate elements are multiples of 2^6. This will mean that the torsion index of E_8 is a multiple of 2^6, as we want.

First, we explain how the torsion index of E_8 can be computed using the homogeneous space E_8/A_8. This uses Lemma 2.1 in my paper [25]:

Lemma 4.2 *Let G be a compact connected Lie group, p a prime number, and H a closed connected subgroup of maximal rank in G such that p does not divide the torsion index of H. Then the $\mathbf{Z}_{(p)}$-cohomology of G/H is torsion-free and concentrated in even dimensions, and the p-part of the torsion index of G is equal to the index in the top degree of the image of $H^*(BH, \mathbf{Z}_{(p)})$ in the ring $H^*(G/H, \mathbf{Z}_{(p)})$.*

Clearly the torsion index of the subgroup $A_8 = SU(9)/(\mathbf{Z}/3)$ in E_8 is prime to 2. So the lemma gives that the 2-part of the torsion index of E_8 is equal to the index in the top degree of the image of $H^*(BA_8, \mathbf{Z}_{(2)})$ in the ring $H^*(E_8/A_8, \mathbf{Z}_{(2)})$. To compute this, we will compute the ring $H^*(E_8/A_8, \mathbf{Z}_{(2)})$ explicitly.

First, we note that the natural map $BSU(9) \to BA_8 = B(SU(9)/(\mathbf{Z}/3))$ is an isomorphism on $\mathbf{Z}_{(2)}$-cohomology, and so we have

$$H^*(BA_8, \mathbf{Z}_{(2)}) = \mathbf{Z}_{(2)}[d_2, \ldots, d_9],$$

where the d_i's are the Chern classes of the natural representation of $SU(9)$.

The calculation of the cohomology of E_8/A_8 begins with Toda's theorems on the cohomology of homogeneous spaces ([23], 3.2, 2.1), which give the following statement for the space E_8/A_8. (To be precise, Toda's Proposition 3.2 considers homogeneous spaces G/H with H torsion-free. In the case at hand, the subgroup A_8 is only 2-locally torsion free, and so Toda's arguments apply to $\mathbf{Z}_{(2)}$-cohomology rather than to $\mathbf{Z}$-cohomology.)

Theorem 4.3 *The $\mathbf{Z}_{(2)}$-cohomology ring of E_8/A_8 is generated by the elements $d_2, \ldots, d_9$ together with elements g_3, g_5, g_9, and g_{15}, where g_i is in degree i (meaning H^{2i}). There are relations ρ_i in degrees $2, 8, 12, 14, 18, 20, 24, 30$, these being the fundamental degrees of E_8, together with relations in degrees $3, 5, 9, 15$ of the form $2g_i + \delta_i = 0$, where the elements δ_i can be taken to be polynomials in lower-dimensional generators which reduce modulo 2 to the following: $\delta_3 \equiv Sq^2 \rho_2$, $\delta_5 \equiv Sq^4 \delta_3$, $\delta_9 \equiv Sq^8 \delta_5$, and $\delta_{15} \equiv Sq^{14} \rho_8 \pmod 2$.*

We will combine Toda's theorem with a calculation of the rational cohomology of E_8/A_8 to get an explicit calculation of the $\mathbf{Z}_{(2)}$-cohomology of E_8/A_8. We know that the rational cohomology of E_8/A_8 is the quotient of the polynomial ring $H^*(BA_8, \mathbf{Q}) = \mathbf{Q}[d_2, \ldots, d_9]$ by the image of $H^*(BE_8, \mathbf{Q})$ in positive degrees, thus by a regular sequence in degrees $2, 8, 12, 14, 18, 20, 24, 30$, these being the fundamental degrees of E_8. (In this statement, we continue the convention that H^{2i} has degree i.)

One way to exhibit elements of the rational cohomology of BE_8 is by taking the Chern character of the adjoint representation of E_8, of dimension 248. To see how these cohomology classes restrict to BA_8, we restrict the adjoint representation to $A_8 = SU(9)/(\mathbf{Z}/3)$. It is straightforward to compute that this restriction is the representation

$$E := (V \otimes V^* - 1) + \Lambda^3 V + \Lambda^3 V^*$$

of $SU(9)$, where V denotes the standard 9-dimensional representation of $SU(9)$. (Here $V \otimes V^* - 1$ is the adjoint representation of $SU(9)$.) In particular, we can compute the Chern character ch_i of E for i equal to $2, 8, 12, 14, 18, 20, 24, 30$, although this is only practical by computer. We find that these 8 elements of $H^*(BSU(9), \mathbf{Q}) \cong H^*(BA_8, \mathbf{Q})$ form a regular sequence. This calculation implies that $H^*(BE_8, \mathbf{Q})$, which we know to be a polynomial ring on generators in degrees $2, 8, 12, 14, 18, 20, 24, 30$, is in fact the polynomial ring generated by the Chern character ch_i of the adjoint representation in those degrees i. (A priori, one might have to consider other representations.)

The fact that $H^*(BE_8, \mathbf{Q})$ is generated by the Chern character of the adjoint representation was proved in a computer-free way by Mehta [15]. More precisely, he proved the equivalent statement that one can take as generating set for the ring of invariants of the Weyl group of E_8 the sums of the kth powers of the roots, for $k = 2, 8, \ldots, 30$.

This calculation implies that the rational cohomology of E_8/A_8 is the quotient of $H^*(BA_8, \mathbf{Q}) = \mathbf{Q}[d_2, \ldots, d_9]$ by the Chern character ch_i of the representation E in the above degrees i. That is:

$$H^*(E_8/A_8, \mathbf{Q}) \cong \mathbf{Q}[d_2, \ldots, d_9]/(-60d_2,$$
$$13/840\, d_2^4 + 3/140\, d_2 d_3^2 - 1/35\, d_2^2 d_4 + 1/21\, d_4^2 - 1/14\, d_3 d_5 + 1/35\, d_2 d_6 - 3/7\, d_8, \ldots).$$

The relations take a few pages to write out in this form; I don't write them out here, since I will write them in a shorter and more useful form below.

In particular, since the $\mathbf{Z}_{(2)}$-cohomology of E_8/A_8 is torsion-free, the first relation here implies that $d_2 = 0$ in $H^*(E_8/A_8, \mathbf{Z}_{(2)})$. Applying Steenrod operations to this relation, as in Toda's theorem, produces several more relations in $\mathbf{Z}/2$-cohomology: $d_3 = 0$, $d_5 = 0$, and $d_9 = 0$ in $H^*(E_8/A_8, \mathbf{Z}/2)$. Therefore, the following classes are in $H^*(E_8/A_8, \mathbf{Z}_{(2)})$:

$$\begin{aligned} g_3 &:= d_3/2 \\ g_5 &:= d_5/2 \\ g_9 &:= (d_9 + d_5 d_4 + d_6 d_3)/2 \end{aligned}$$

We have chosen these particular generators so that they reduce modulo 2 to elements in $H^*(E_8/A_8, \mathbf{Z}/2)$ which were considered by Kono and Ishitoya in their partial computation of that ring ([12], Lemma 5.5). They called these classes $\gamma_3, \gamma_5, \gamma_9$.

Similarly, the relation in degree 8 in $H^*(E_8/A_8, \mathbf{Q})$ implies that

$$d_8 = 1/9\, d_4^2 - 2/3\, g_3 g_5$$

in $H^*(E_8/A_8, \mathbf{Q})$. Since $H^*(E_8/A_8, \mathbf{Z}_{(2)})$ is torsion-free, the same relation holds in $\mathbf{Z}_{(2)}$-cohomology. Applying Sq^{14} to this relation, as suggested by Toda's theorem, shows that $d_8 d_7 = 0$ in $H^*(E_8/A_8, \mathbf{Z}/2)$. Therefore the class

$$g_{15} := d_8 d_7/2$$

is in $H^*(E_8/A_8, \mathbf{Z}_{(2)})$. Again, this element reduces modulo 2 to one considered by Kono and Ishitoya, which they called $\overline{\gamma_{15}}$ ([12], Lemma 5.5).

By Toda's theorem, $H^*(E_8/A_8, \mathbf{Z}_{(2)})$ is the $\mathbf{Z}_{(2)}$-subalgebra of $H^*(E_8/A_8, \mathbf{Q})$ generated by the classes $d_2, \ldots, d_9, g_3, g_5, g_9$, and g_{15}. In fact, by the relations we have mentioned, this is also the subring generated by $d_4, d_6, d_7, g_3, g_5, g_9, g_{15}$. We can rewrite the relations defining $H^*(E_8/A_8, \mathbf{Q})$ in terms of these generators. Simplifying these relations in a carefully chosen way (as we will see), we get the following description of $H^*(E_8/A_8, \mathbf{Q})$.

Lemma 4.4 *$H^*(E_8/A_8, \mathbf{Q})$ is isomorphic to the quotient of* $\mathbf{Q}[g_{15}, g_9, g_5, g_3, d_7, d_6, d_4]$

by the relations:

$$
\begin{aligned}
&(d_6^2 - 25/81\, d_4^3 + 2(15 g_9 g_3 + 1/3\, g_3^4 - 5/3\, g_5 d_7 - 125/9\, g_5 g_3 d_4) + 2^2(-23/3\, g_3^2 d_6),\\
&\ d_7^2 - 1/3\, d_6 d_4^2 + 2(-9 g_9 g_5 + 9 g_5 g_3 d_6 + 1/3\, g_3 d_7 d_4) + 2^2(-1/9\, g_3^2 d_4^2) + 2^3(1/3\, g_5 g_3^3 + 7/3\, g_5^2 d_4),\\
&d_7 d_4^2 + 2(-9 g_{15} - 3 g_5 g_3 d_7),\\
&\ g_9^2 + 25/2187\, d_6 d_4^3 + 2(-1/3\, g_{15} g_3 - 157/1215\, g_3^6 - 31/45\, g_9 g_3 d_6 + 1/405\, g_5 d_7 d_6)\\
&\quad + 2^2(-377/135\, g_9 g_3^3 + 337/1215\, g_5 g_3^2 d_7 + 3473/1215\, g_3^4 d_6 + 5/27\, g_5^2 d_4^2)\\
&\quad + 2^3(1/81\, g_5^3 g_3 + 35/243\, g_5 g_3 d_6 d_4) + 2^4(-1/9\, g_9 g_5 d_4 + 467/729\, g_5 g_3^3 d_4 + 53/6561\, g_3^2 d_4^3),\\
&\ g_5^4 - 1/27\, d_4^5 + 2(-159 g_{15} g_5 - 27 g_9 g_5 d_6 + 9 g_9 d_7 d_4 + 27 g_5^2 d_6 d_4 - 9 g_3 d_7 d_6 d_4)\\
&\quad + 2^2(-387 g_9 g_5 g_3^2 - 35/3\, g_5 g_3^5 + 17 g_5^2 g_3 d_7 + 355 g_5^2 g_3^2 d_4 - 3 g_9 g_3 d_4^2 + 3 g_3^2 d_6 d_4^2 + 67/9\, g_5 g_3 d_4^3)\\
&\quad + 2^3(199 g_5 g_3^3 d_6 - 1/3\, g_3^3 d_7 d_4) + 2^4(1/9\, g_3^4 d_4^2),\\
&\ g_3^8 - 235/1445121\, d_4^6 + 2(24255/5947\, g_{15} g_9 - 3855/5947\, g_9 g_5 d_6 d_4 + 566695/1445121\, g_5 g_3 d_4^4)\\
&\quad + 2^2(68499/5947\, g_9 g_3^5 - 69969/5947\, g_3^6 d_6 - 12991/17841\, g_5 g_3^2 d_7 d_6 - 105395/53523\, g_{15} g_5 d_4\\
&\quad + 55645/160569\, g_5^2 g_3 d_7 d_4 + 1342735/160569\, g_5^2 g_3^2 d_4^2 - 47675/481707\, g_3^4 d_4^3)\\
&\quad + 2^3(-335/939\, g_{15} g_3^3 + 5155/160569\, g_5^3 g_3^3 + 2145/5947\, g_9 g_5 g_3 d_7 - 39167/53523\, g_5 g_3^4 d_7\\
&\quad - 35/17841\, g_5^3 g_3 d_6 + 2875/17841\, g_5^2 d_6 d_4^2 - 805/17841\, g_9 g_3 d_4^3 + 7895/160569\, g_3^2 d_6 d_4^3)\\
&\quad + 2^4(-3085/5947\, g_{15} g_3 d_6 + 567/5947\, g_9 g_3^3 d_6 - 13615/5947\, g_9 g_5 g_3^2 d_4)\\
&\quad + 2^5(5/5947\, g_9 g_5^3 - 220885/160569\, g_5 g_3^5 d_4) + 2^7(15115/53523\, g_5 g_3^3 d_6 d_4),\\
&\ g_{15}^2 - 4423646851/37005741\, g_3^2 d_4^6 + 2(-6688025/152287\, g_{15} g_5^3 + 42564773535/152287\, g_{15} g_9 g_3^2\\
&\quad - 664821/152287\, g_{15} g_9 d_6 - 1971/152287\, g_9 g_5^3 d_6 + 371445/152287\, g_9 g_5^2 d_7 d_4\\
&\quad + 1957715/456861\, g_{15} g_5 d_6 d_4 - 7703714311/152287\, g_9 g_5 g_3^2 d_6 d_4 + 1047393557/1370583\, g_5^2 g_3 d_7 d_6 d_4\\
&\quad + 2939006257/456861\, g_9 g_5^2 g_3 d_4^2 + 1831149342661/1370583\, g_5^2 g_3^4 d_4^2 - 34405219/4111749\, g_{15} g_3 d_4^3\\
&\quad - 80699929051/12335247\, g_5^3 g_3 d_4^3 + 59584614437/37005741\, g_3^6 d_4^3 + 31947038297/4111749\, g_3^4 d_6 d_4^3\\
&\quad + 270247/1370583\, g_9 g_5 d_4^4 + 916241713331/37005741\, g_5 g_3^3 d_4^4) + 2^2(30654145/152287\, g_9 g_5^3 g_3^2\\
&\quad - 122749825/4111749\, g_5^3 g_3^5 - 575158605/152287\, g_{15} g_5 g_3 d_7 - 29792441557/4111749\, g_5 g_3^6 d_7\\
&\quad - 60140331/152287\, g_9 g_5 g_3 d_7 d_6 - 332646887695/456861\, g_9 g_5 g_3^4 d_4 - 265228485653/12335247\, g_5 g_3^7 d_4\\
&\quad + 187962435235/4111749\, g_5^2 g_3^3 d_7 d_4 + 3077559069377/4111749\, g_5 g_3^5 d_6 d_4\\
&\quad + 90822041303/4111749\, g_5^2 g_3^2 d_6 d_4^2 - 5373504349/1370583\, g_9 g_3^3 d_4^3) + 2^3(2686640435/152287\, g_9 g_5 g_3^3 d_7\\
&\quad - 15326689/152287\, g_5^3 g_3^3 d_6 - 94853486069/1370583\, g_{15} g_5 g_3^2 d_4 + 3340711/152287\, g_9 g_3 d_6 d_4^3\\
&\quad - 1815079/37005741\, g_5^2 d_4^5) + 2^5(-3548969723/1370583\, g_{15} g_3^5) + 2^6(-1330132295/152287\, g_{15} g_3^3 d_6\\
&\quad - 998302129/456861\, g_5 g_3^4 d_7 d_6)).
\end{aligned}
$$

In Lemma 4.4, we have written all the relations as $\mathbf{Z}_{(2)}$-polynomials in the elements $g_{15}, g_9, g_5, g_3, d_7, d_6, d_4$. Since $H^*(E_8/A_8, \mathbf{Z}_{(2)})$ is torsion-free, these relations also hold in $H^*(E_8/A_8, \mathbf{Z}_{(2)})$. Let R be the quotient of $\mathbf{Z}_{(2)}[g_{15}, g_9, g_5, g_3, d_7, d_6, d_4]$ by the relations listed in Lemma 4.4. Then we have a homomorphism from R to $H^*(E_8/A_8, \mathbf{Z}_{(2)})$, which is surjective by Toda's theorem.

By reducing these relations modulo 2, we find that the ring $R/2$ has a very short description:

$$R/2 = \mathbf{Z}/2[g_{15}, g_9, g_5, g_3, d_7, d_6, d_4]/ \\ (d_6^2 + d_4^3, d_7^2 + d_6 d_4^2, d_7 d_4^2, g_9^2 + d_6 d_4^3, g_5^4 + d_4^5, g_3^8 + d_4^6, g_{15}^2 + g_3^2 d_4^6).$$

In particular, it is easy to check that $R/2$ is a complete intersection ring, with generators in degrees $3, 4, 5, 6, 7, 9, 15$ and relations in degrees $12, 14, 15, 18, 20, 24, 30$. So we know the Hilbert series of $R/2$. Also we know the Hilbert series of $H^*(E_8/A_8, \mathbf{Z}/2)$. Indeed, E_8/A_8 has no 2-torsion in its cohomology, and so this Hilbert series is the same as the Hilbert series of $H^*(E_8/A_8, \mathbf{Q})$. The latter ring is a complete intersection ring with generators in degrees $2, 3, 4, 5, 6, 7, 8, 9$ and relations in degrees $2, 8, 12, 14, 18, 20, 24, 30$, these being the fundamental degrees of E_8. As a result, $R/2$ has the same Hilbert series as $H^*(E_8/A_8, \mathbf{Z}/2)$. Since we have a surjective homomorphism from $R/2$ to $H^*(E_8/A_8, \mathbf{Z}/2)$, this homomorphism must be an isomorphism. Thus we have a complete calculation of $H^*(E_8/A_8, \mathbf{Z}/2)$, as follows. Earlier, Kono and Ishitoya computed this ring in degrees at most 23 (meaning H^i for i at most 46) ([12], 5.5).

Theorem 4.5

$$H^*(E_8/A_8, \mathbf{Z}/2) = \mathbf{Z}/2[g_{15}, g_9, g_5, g_3, d_7, d_6, d_4]/ \\ (d_6^2 + d_4^3, d_7^2 + d_6 d_4^2, d_7 d_4^2, g_9^2 + d_6 d_4^3, g_5^4 + d_4^5, g_3^8 + d_4^6, g_{15}^2 + g_3^2 d_4^6).$$

Moreover, we have a surjection from the above ring R to $H^*(E_8/A_8, \mathbf{Z}_{(2)})$ which becomes an isomorphism after reducing modulo 2. Since $H^*(E_8/A_8, \mathbf{Z}_{(2)})$ is torsion-free, it follows that R maps isomorphically to $H^*(E_8/A_8, \mathbf{Z}_{(2)})$. That is, we have shown:

Theorem 4.6 *The $\mathbf{Z}_{(2)}$-cohomology ring of E_8/A_8 is the quotient of the polynomial ring $\mathbf{Z}_{(2)}[g_{15}, g_9, g_5, g_3, d_7, d_6, d_4]$ by the relations listed in Lemma 4.4.*

Fortunately, only a small part of the information in these relations will be needed for our calculation of the torsion index of E_8. We will only need to know these relations modulo 4, at most.

The simple description of the ring $H^*(E_8/A_8, \mathbf{Z}/2)$ in Theorem 4.5 determines a basis for this ring as a $\mathbf{Z}/2$-vector space, in the following way. Consider the "reverse lexicographic" ordering of the monomials of a given degree, based on the order of variables $g_{15}, g_9, g_5, g_3, d_7, d_6, d_4$. This means that we define one monomial to be smaller than another if it has a higher power of d_4, or the same power of d_4 and a higher power of d_6, and so on. We say that a monomial is "reduced" if it is not equal in $H^*(E_8/A_8, \mathbf{Z}/2)$ to a linear combination of smaller monomials. Then it is clear that the reduced monomials form a basis for $H^*(E_8/A_8, \mathbf{Z}/2)$.

Buchberger gave an algorithm to determine which monomials are reduced, starting from any set of defining relations for a commutative algebra over a field. Buchberger's algorithm is the basis of the theory of Gröbner bases [18]. Our choice of monomial ordering works well: unusually, the algorithm can be carried out quickly by hand, as follows. First, from the relations which define $H^*(E_8/A_8, \mathbf{Z}/2)$, we know that the monomials d_6^2, d_7^2, $d_7 d_4^2$, g_9^2, g_5^4, g_3^8, g_{15}^2 are non-reduced. Next, Buchberger's algorithm says to consider any "overlaps" between these monomials. We find that in $H^*(E_8/A_8, \mathbf{Z}/2)$, $d_7^2 d_4^2$ is equal both to $d_4^2(d_6 d_4^2) = d_6 d_4^4$ and to $d_7(0) = 0$. So $d_6 d_4^4 = 0$, and hence the monomial $d_6 d_4^4$ is also non-reduced. Next, $d_6^2 d_4^4$ is equal both to $d_4^4(d_4^3) = d_4^7$ and to $d_6(0) = 0$. So $d_4^7 = 0$, and hence the monomial d_4^7 is also non-reduced. The remaining overlaps between these monomials just yield that $f = f$ for some polynomials f. That ends Buchberger's algorithm. The result is that a monomial is non-reduced if and only if it is a multiple of d_6^2, d_7^2, $d_7 d_4^2$, g_9^2, g_5^4, g_3^8, g_{15}^2, $d_6 d_4^4$, or d_4^7. We have also found reduced expressions for each of these monomials; together, these relations are called a "Gröbner basis" for the ring $H^*(E_8/A_8, \mathbf{Z}/2)$.

$$\begin{aligned}
d_6^2 &= d_4^3 \\
d_7^2 &= d_6 d_4^2 \\
d_7 d_4^2 &= 0 \\
g_9^2 &= d_6 d_4^3 \\
g_5^4 &= d_4^5 \\
g_3^8 &= d_4^6 \\
g_{15}^2 &= g_3^2 d_4^6 \\
d_6 d_4^4 &= 0 \\
d_4^7 &= 0.
\end{aligned}$$

Since $H^*(E_8/A_8, \mathbf{Z}_{(2)})$ is a torsion-free $\mathbf{Z}_{(2)}$-algebra, and the reduced monomials in $g_{15}, g_9, g_5, g_3, d_7, d_6, d_4$ form a basis for this ring modulo 2, it follows that the same reduced monomials form a basis for this ring as a free $\mathbf{Z}_{(2)}$-module. In order to be able to express any polynomial in the generators in reduced form, we should lift each of the above Gröbner relations in $H^*(E_8/A_8, \mathbf{Z}/2)$ to a relation in $H^*(E_8/A_8, \mathbf{Z}_{(2)})$. In fact, we can now say how we chose a particular way to write the relations defining this ring in Theorem 4.6 and Lemma 4.4: these relations are the *unique* relations of the form $m - r$ such that m is one of the monomials $d_6^2, d_7^2, d_7 d_4^2, g_9^2, g_5^4, g_3^8, g_{15}^2$ and r is a $\mathbf{Z}_{(2)}$-linear combination of reduced monomials. Using these, we can deduce the corresponding relations involving $d_6 d_4^4$ and d_4^7. In fact, for our purpose, we only need to know the relations in Lemma 4.4 modulo 4. Namely, we compute that the following relations hold in $H^*(E_8/A_8, \mathbf{Z}_{(2)})$.

$$\begin{aligned}
d_6^2 &= d_4^3 \pmod 2 \\
d_7^2 &= 3 d_6 d_4^2 + 2(g_9 g_5 + g_5 g_3 d_6 + g_3 d_7 d_4) \pmod{2^2} \\
d_7 d_4^2 &= 2(g_{15} + g_5 g_3 d_7) \pmod{2^2} \\
g_9^2 &= d_6 d_4^3 \pmod 2 \\
g_5^4 &= d_4^5 \pmod 2 \\
g_3^8 &= d_4^6 \pmod 2 \\
g_{15}^2 &= 3 g_3^2 d_4^6 + 2(g_{15} g_5^3 + g_{15} g_9 g_3^2 + g_{15} g_9 d_6 + g_9 g_5^3 d_6 + g_9 g_5^2 d_7 d_4 + g_{15} g_5 d_6 d_4 + g_9 g_5 g_3^2 d_6 d_4 + g_5^2 g_3 d_7 d_6 d_4 \\
&\quad + g_9 g_5^2 g_3 d_4^2 + g_5^2 g_3^4 d_4^2 + g_{15} g_3 d_4^3 + g_5^3 g_3 d_4^3 + g_3^6 d_4^3 + g_3^4 d_6 d_4^3 + g_9 g_5 d_4^4 + g_5 g_3^3 d_4^4) \pmod{2^2} \\
d_6 d_4^4 &= 2(g_{15} d_7 + g_5 g_3 d_7^2 + g_9 g_5 d_4^2 + g_5 g_3 d_6 d_4^2 + g_3 d_7 d_4^3) \mod (2^2 d_7, 2^2 d_4^2) \\
d_4^7 &= 2(g_{15} d_7 d_6 + g_5 g_3 d_7^2 d_6 + g_9 g_5 d_6 d_4^2 + g_5 g_3 d_6^2 d_4^2 + g_3 d_7 d_6 d_4^3) \mod (2 d_4^4, 2^2 d_7 d_6, 2^2 d_6 d_4^2)
\end{aligned}$$

The formulas for $d_6 d_4^4$ and d_4^7 could be simplified further, but we have written them in the form which follows immediately from the first three formulas.

These formulas contain just enough information to prove that the torsion index of E_8 is a multiple of 2^6, as we want. To do this, we will define a valuation on the ring $H^*(E_8/A_8, \mathbf{Z}_{(2)})$, meaning a function v from the ring to $\mathbf{R} \cup \{\infty\}$ with the properties:

$v(x) = \infty \iff x = 0$.
$v(ax) = \text{ord}_2(a) + v(x)$ for $a \in \mathbf{Z}_{(2)}$, $x \in H^*(E_8/A_8, \mathbf{Z}_{(2)})$.
$v(x + y) \geq \min(v(x), v(y))$.
$v(xy) \geq v(x) + v(y)$.

Namely, define the valuation on the generators of this algebra by

$$\begin{aligned} v(d_4) &= 0 \\ v(d_6) &= 0 \\ v(d_7) &= 0 \\ v(g_3) &= -1/4 \\ v(g_5) &= -2/3 \\ v(g_9) &= -1/3 \\ v(g_{15}) &= -1. \end{aligned}$$

Define the valuation of any reduced monomial in the generators to be the sum of the valuations of the corresponding generators, with multiplicities. Finally, define the valuation of any element of $H^*(E_8/A_8, \mathbf{Z}_{(2)})$ by writing it as a sum of reduced monomials m with coefficients a_m in $\mathbf{Z}_{(2)}$, and defining

$$v\left(\sum_m a_m m\right) = \min_m(\text{ord}_2(a_m) + v(m)).$$

It is clear that v satisfies the first three properties of a valuation, as listed above. Let us show that v also satisfies the last property, $v(xy) \geq v(x) + v(y)$. It suffices to show that for each of the 9 "basic non-reduced monomials" $d_6^2, \ldots, d_4^7$, the valuation of the reduced expression for the monomial is at least the sum of the valuations of the generators in the monomial. We check this for each of the basic non-reduced monomials, using its reduced expression modulo a small power of 2 given above. For example, we know the relation $d_6^2 = d_4^3 \pmod 2$. Here $2v(d_6) = 0$, and so we need to prove that the reduced expression for d_6^2 has valuation at least 0. This is true for the monomial d_4^3, and it is also true for all monomials of degree 12 multiplied by 2. The latter fact holds because, as we check, all monomials of degree 12 have valuation at least -1; the worst case is g_3^4, which has valuation exactly -1. A similar analysis works for each of the above 9 formulas expressing a basic non-reduced monomial modulo small powers of 2. Thus v is a valuation.

To show that the torsion index of E_8 is a multiple of 2^6, it is equivalent to show that any element of the top degree, 84, in the subring generated by $d_2, \ldots, d_9$ in $H^*(E_8/A_8, \mathbf{Z}_{(2)})$ is zero modulo 2^6, by Lemma 4.2. We observe that $v(d_i)$ is nonnegative for $i = 2, \ldots, 9$. This is clear for d_4, d_6, and d_7. For the rest, it follows from the relations we know. Indeed, d_2 is 0, $v(d_3) = v(2g_3) = 3/4$, and $v(d_5) = v(2g_5) = 1/3$. Next, we showed earlier that

$$d_8 = 1/9\, d_4^2 - 2/3\, g_3 g_5$$

in $H^*(E_8/A_8, \mathbf{Z}_{(2)})$. So

$$\begin{aligned} v(d_8) &= \min(0, 1 - 1/4 - 2/3) \\ &= 0, \end{aligned}$$

which is nonnegative, as we want. Finally, by the definition of g_9, we have

$$d_9 = 2(g_9 - g_5 d_4 - g_3 d_6)$$

and so

$$\begin{aligned} v(d_9) &= 1 + \min(-1/3, -2/3, -1/4) \\ &= 1/3. \end{aligned}$$

Thus, $v(d_i) \geq 0$ for $i = 2, \ldots, 9$, and so any element of the subring generated by $d_2, \ldots, d_9$ in $H^*(E_8/A_8, \mathbf{Z}_{(2)})$ has valuation at least 0.

On the other hand, our description of the reduced monomials shows that the ring $H^*(E_8/A_8, \mathbf{Z}_{(2)})$ is generated in its top degree, 84, by the reduced monomial $g_{15} g_9 g_5^3 g_3^7 d_4^6$. This monomial has valuation

$$\begin{aligned} v(g_{15} g_9 g_5^3 g_3^7 d_4^6) &= -1 - 1/3 - 3(2/3) - 7(1/4) \\ &= -61/12. \end{aligned}$$

Since this is (barely) less than -5, it follows that any top-degree element of the subring generated by $d_2, \ldots, d_9$ in $H^*(E_8/A_8, \mathbf{Z}_{(2)})$ must be a multiple of 2^6 times the basis element $g_{15} g_9 g_5^3 g_3^7 d_4^6$. Thus we have shown that the torsion index of E_8 is a multiple of 2^6. This completes the proof of Theorem 4.1, that the torsion index of E_8 is equal to $2^6 3^2 5 = 2880$. QED

5 The torsion index of the semispin group $Ss(12)$

Theorem 5.1 *The torsion index of the semispin group $Ss(12)$ divides 2^2.*

By definition, the semispin group $Ss(4n)$ is the quotient of the spin group $Spin(4n)$ by a central subgroup of order 2 other than the one which gives $SO(4n)$. (The center of $Spin(4n)$ is isomorphic to $(\mathbf{Z}/2)^2$.) We will not try to compute the torsion index of the semispin groups in general; in fact, that is the main calculation of torsion indices left open by this paper and [25]. We do this partial calculation for $Ss(12)$ because it will be used in our calculation for the adjoint group $E_7/(\mathbf{Z}/2)$ (Theorem 6.1). The torsion index of $Ss(12)$ is in fact equal to 2^2, but we do not include a proof since it is not needed for the application to $E_7/(\mathbf{Z}/2)$. Note that $Ss(12)$ is the first "nontrivial" semispin group. Indeed, $Ss(4)$ is isomorphic to $SU(2) \times SO(3)$, so has torsion index 2^1, and $Ss(8)$ is isomorphic to $SO(8)$ by triality, so has torsion index 2^3 (Theorem 3.2 in [25], which follows from results of Marlin and Merkurjev).

Proof. We use that the semispin group $Ss(12)$ has a Levi subgroup which is isogenous to the subgroup $U(6)$ in $SO(12)$. More precisely, there are two conjugacy classes of such subgroups in $SO(12)$ which are switched by the outer automorphism of $SO(12)$, and these correspond to two conjugacy classes of subgroups in $Ss(12)$

which are not isomorphic; one has derived subgroup isomorphic to $SU(6)$ while the other has derived subgroup $SU(6)/(\mathbf{Z}/2)$. Let H be the Levi subgroup of $Ss(12)$ whose derived subgroup is $SU(6)$.

By Lemma 2.1, the group H has torsion index 1. By Lemma 4.2, the $\mathbf{Z}_{(2)}$-cohomology of $Ss(12)/H = SO(12)/U(6)$ is torsion-free, and the 2-part of the index of $Ss(12)$ is equal to the index in the top degree of the image of the $\mathbf{Z}_{(2)}$-cohomology of BH in that of $Ss(12)/H$.

The cohomology of $Ss(12)/H = SO(12)/U(6)$ is well known ([17], III.6.9):

$$H^*(SO(12)/U(6), \mathbf{Z}) = \mathbf{Z}[e_1, \ldots, e_5]/(e_{2i} - 2e_1 e_{2i-1} + 2e_2 e_{2i-2} - \cdots + (-1)^i e_i^2 = 0,\ i \geq 1),$$

where the Chern classes c_j in $H^*BU(6)$ restrict to $2e_j$ for $j > 0$. Here we understand e_j to mean 0 for $j \geq 6$. The element e_j is in H^{2j}.

The group $H \cong (SU(6) \times S^1)/(\mathbf{Z}/3)$ has a 1-dimensional complex representation which in terms of the isogenous group $U(6)$ is $\det V$, where V denotes the standard representation of $U(6)$. The group H also has a complex representation of dimension 6 which in terms of the isogenous group $U(6)$ is $V \otimes (\det V)^{1/2}$. The Chern classes of these two representations of H give us some integral cohomology classes in BH, and we can compute their images in the cohomology of $Ss(12)/H$. First, the 1-dimensional representation of H shows that $c_1(V) = 2e_1$ is in the image of the integral cohomology of BH. Next, we compute the Chern classes of the above 6-dimensional representation of H:

$$\begin{aligned} c(V \otimes (\det V)^{1/2}) &= \sum_{i=0}^{6} (1 + c_1(\det V)^{1/2})^{6-i} c_i V \\ &= (1+e_1)^6 + \sum_{i=1}^{6} (1+e_1)^{6-i} 2e_i \\ &= 1 + (8e_1) + (25e_1^2 + 2e_2) + \cdots . \end{aligned}$$

Therefore, the image of the integral cohomology ring of BH in that of $Ss(12)/H = SO(12)/H$ contains the elements $2e_1$ and $25e_1^2 + 2e_2 = 27e_1^2$. Working 2-locally, we can say that this image ring contains $2e_1$ and e_1^2.

Therefore, the image of the $\mathbf{Z}_{(2)}$-cohomology ring of BH in that of $Ss(12)/H = SO(12)/H$ contains the top-degree element

$$(2e_1)(e_1^2)^7 = 2e_1^{15}.$$

From our knowledge of the cohomology ring of $SO(12)/U(6)$, we compute that e_1^{15} is 2 times an odd multiple of a generator of the top-degree cohomology group of $SO(12)/U(6)$. (This calculation is done in the proof of Lemma 3.3 in [25], for example.) Therefore the image of the $\mathbf{Z}_{(2)}$-cohomology ring of BH in that of $Ss(12)/H$ contains 2^2 times a top-degree generator. By Lemma 4.2, as discussed earlier, it follows that the torsion index of $Ss(12)$ divides 2^2. QED

6 The torsion index of $E_6/(\mathbf{Z}/3)$ and $E_7/(\mathbf{Z}/2)$

Theorem 6.1 *The adjoint group $E_6/(\mathbf{Z}/3)$ has torsion index $2 \cdot 3^3 = 54$. The adjoint group $E_7/(\mathbf{Z}/2)$ has torsion index $2^3 3 = 24$.*

This completes the calculation of the torsion index for all simple groups of exceptional type, in view of the calculation that $t(G_2) = 2$, Tits's calculations that $t(F_4) = 2 \cdot 3$, $t(E_6) = 2 \cdot 3$, $t(E_7) = 2^2 3$, and Theorem 4.1 which says that $t(E_8) = 2^6 3^2 5$.

Proof. We consider $E_6/(\mathbf{Z}/3)$ first. From the definition of the torsion index, it is clear that the p-part of the torsion index of $E_6/(\mathbf{Z}/3)$ is the same as that of the simply connected group E_6 for $p \neq 3$, so it remains to compute the 3-part. By Merkurjev [16], 4.5.2, the torsion index of $E_6/(\mathbf{Z}/3)$ is a multiple of 3^3. So we need to show that the 3-adic order of the torsion index of $E_6/(\mathbf{Z}/3)$ is at most 3.

We use that $E_6/(\mathbf{Z}/3)$ has a Levi subgroup H which is isogenous to $A_5 \times S^1$. The homogeneous space $(E_6/(\mathbf{Z}/3))/H$ has Euler characteristic equal to $|W(E_6)|/|W(A_5)| = 2^7 3^4 5/6! = 2^3 3^2$. Therefore, by Lemmas 2.2 and 2.1,

$$\begin{aligned}
\mathrm{ord}_3 t(E_6/(\mathbf{Z}/3)) &\leq 2 + \mathrm{ord}_3 t(H) \\
&\leq 2 + \mathrm{ord}_3 t(PSU(6)) \\
&\leq 2 + \mathrm{ord}_3(6) \\
&= 3.
\end{aligned}$$

Thus, $t(E_6/(\mathbf{Z}/3)) = 2 \cdot 3^3$, as we want.

Similarly, we know that the p-part of the torsion index of $E_7/(\mathbf{Z}/2)$ is the same as that of the simply connected group E_7 for $p \neq 2$, so it remains to compute the 2-part. By Merkurjev [16], 4.5.3, the torsion index of $E_7/(\mathbf{Z}/2)$ is a multiple of 2^3. So we need to show that the 2-adic order of the torsion index of $E_7/(\mathbf{Z}/2)$ is at most 2.

We use that $E_7/(\mathbf{Z}/2)$ has a Levi subgroup H which is isogenous to $D_6 \times S^1$. More precisely, the derived subgroup of H is isomorphic to the semispin group $Ss(12)$. To check this, one can use Adams's book [2], p. 50 and p. 82: we see that $Spin(12)$ injects into E_7, with centralizer $SU(2)$, and that $Spin(12)$ acts on the Lie algebra $L(E_7)$ by

$$L(E_7) = L(Spin(12)) \oplus L(SU(2)) \oplus \Delta^+$$

where Δ^+ is one of the spin representations of $Spin(12)$. Thus, the image of $Spin(12)$ in the adjoint group $E_7/(\mathbf{Z}/2)$ is isomorphic to its image acting on Δ^+, which is the semispin group $Ss(12)$.

The Euler characteristic of the homogeneous space $(E_7/(\mathbf{Z}/2))/H$ is

$$\begin{aligned}
|W(E_7)|/|W(D_6)| &= 2^{10} 3^4 5 \cdot 7/2^9 3^2 5 \\
&= 2 \cdot 3^2 7.
\end{aligned}$$

Therefore, by Lemmas 2.2 and 2.1, together with Lemma 5.1, we have

$$\begin{aligned}
\mathrm{ord}_2 t(E_7/(\mathbf{Z}/2)) &\leq 1 + \mathrm{ord}_2 t(H) \\
&= 1 + \mathrm{ord}_2 t(Ss(12)) \\
&\leq 1 + 2 \\
&= 3.
\end{aligned}$$

Thus, $t(E_7/(\mathbf{Z}/2)) = 2^3 3$, as we want. QED

7 The torsion index of $PSO(2n)$

We now complete the calculation of the torsion index for all compact Lie groups of adjoint type. The remaining case is $PSO(2n) = SO(2n)/\{\pm 1\}$. Let us sum up the other cases first. The torsion index of $SU(n)/(\mathbf{Z}/n)$ is n, and the torsion index of the group $Sp(2n)/(\mathbf{Z}/2)$ of type C_n is $2^{\text{ord}_2(n)+1}$, by Merkurjev [16] as discussed in the introduction. The group $SO(2n+1)$ is of adjoint type, and its torsion index is 2^n, by the references in section 2. The simply connected groups G_2, F_4, and E_8 are also of adjoint type, and so we know their torsion indices: 2, $2 \cdot 3$, and (by Theorem 4.1) $2^6 3^2 5$. Finally, Theorem 6.1 gives that the adjoint groups $E_6/(\mathbf{Z}/3)$ and $E_7/(\mathbf{Z}/2)$ have torsion indices $2 \cdot 3^3$ and $2^3 3$.

Theorem 7.1 *For all $n \geq 2$, the torsion index of $PSO(2n)$ is 2^{n-1} if n is not a power of 2, and 2^n if n is a power of 2.*

This is a surprisingly small answer. In particular, it is clear from the definitions that the torsion index of $SO(2n)$ divides that of $PSO(2n)$, but the theorem says that these two torsion indices are actually equal for all n not a power of 2. (Here $SO(2n)$ was known to have torsion index 2^{n-1} for all n; see the references in section 2.)

Proof. It is immediate from the definition of the torsion index that the odd part of the torsion index of $PSO(2n)$ is the same as that of $SO(2n)$, that is, 1. So the torsion index of $PSO(2n)$ is a power of 2.

We will use Lemma 4.2 to compute the torsion index of $PSO(2n)$ using a torsion-free subgroup K of maximal rank. To simplify the calculation, we want a big subgroup K with these properties. So let K be the Levi subgroup

$$K = (U(n-1) \times U(1))/(\mathbf{Z}/2) \subset U(n)/(\mathbf{Z}/2) \subset SO(2n)/(\mathbf{Z}/2) = PSO(2n).$$

We know that K is torsion-free because its derived subgroup is $SU(n-1)$, using Lemma 2.1. Thus, by Lemma 4.2, the torsion index of $PSO(2n)$ is the index in the top degree of the image of $H^*(BK, \mathbf{Z})$ in $H^*(PSO(2n)/K, \mathbf{Z}) = H^*(SO(2n)/U(n-1) \times U(1), \mathbf{Z})$.

This homogeneous space $SO(2n)/U(n-1) \times U(1)$ is the $\mathbf{CP}^{n-1}$-bundle associated to the natural vector bundle over the isotropic Grassmannian $SO(2n)/U(n)$. The cohomology of the isotropic Grassmannian is well known: ([17], III.6.9):

$$H^*(SO(2n)/U(n), \mathbf{Z}) = \mathbf{Z}[e_1, \ldots, e_{n-1}]/(e_{2i} - 2e_1 e_{2i-1} + 2e_2 e_{2i-2} - \cdots + (-1)^i e_i^2,\ i \geq 1),$$

where the Chern classes c_j in $H^*BU(l)$ restrict to $2e_j$ for $j > 0$. Here we understand e_j to mean 0 for $j \geq n$. The element e_j is in H^{2j}. From this, we deduce the cohomology of the homogeneous space we need to consider:

$$H^*(SO(2n)/U(n-1) \times U(1), \mathbf{Z}) = \mathbf{Z}[t, e_1, \ldots, e_{n-1}]/$$
$$(t^n - (2e_1)t^{n-1} + \cdots + (-1)^{n-1}(2e_{n-1})t,\ e_i^2 - 2e_{i-1}e_{i+1} + 2e_{i-2}e_{i+2} - \cdots + (-1)^i e_{2i}).$$

Next, we need to know the integral cohomology ring of $B(U(n-1) \times U(1))/(\mathbf{Z}/2)$. Clearly the integral cohomology of $B(U(n-1) \times U(1))$ is the polynomial ring generated by the Chern classes $c_1 V_{n-1}, \ldots, c_{n-1} V_{n-1}$ and $c_1 L$, where V_{n-1} is the standard complex representation of $U(n-1)$ and L is the standard complex representation of $U(1)$. To compute the cohomology of $B(U(n-1)\times U(1))/(\mathbf{Z}/2)$, we can use the spectral sequence of the fibration

$$BU(n-1) \to B(U(n-1)\times U(1))/(\mathbf{Z}/2) \to BU(1)/(\mathbf{Z}/2),$$

which degenerates since the base and fiber have cohomology concentrated in even degrees. Using this, it is straightforward to check that the integral cohomology of $B(U(n-1)\times U(1))/(\mathbf{Z}/2)$ is the polynomial ring generated by $v := c_1(L\otimes L) = 2c_1 L$ and $u_i := c_i(V_{n-1}\otimes L^*)$ for $i = 1,\ldots,n-1$.

We then have to consider the image of this polynomial ring in the cohomology of $SO(2n)/U(n-1)\times U(1)$. We can simplify the calculation by translating the problem from the orthogonal groups to the symplectic groups, as I also did when computing the torsion index of the spin groups ([25], section 4). Let $Sp(2n)$ denote the simply connected group of type C_n, which topologists usually call $Sp(n)$. The homogeneous space $Sp(2n)/U(n-1)\times U(1)$ has the following cohomology ring, similar to that of $SO(2n)/U(n-1)\times U(1)$:

$$H^*(Sp(2n)/U(n-1)\times U(1),\mathbf{Z}) = \mathbf{Z}[t, f_1,\ldots,f_n]/$$
$$(t^n - f_1 t^{n-1} + \cdots + (-1)^{n-1} f_{n-1} t + (-1)^n f_n,\ f_i^2 - 2f_{i-1}f_{i+1} + 2f_{i-2}f_{i+2} - \cdots + (-1)^i 2f_{2i}),$$

where $f_i = 0$ for $i > n$. Here, if we write V_{n-1} and L for the standard representations of $U(n-1)$ and $U(1)$, then t denotes $c_1 L$ and f_i denotes $c_i(V_{n-1}\oplus L)$. We can define a homomorphism from this cohomology ring to the integral cohomology ring of $SO(2n)/U(n-1)\times U(1)$ which takes t to t and f_i to $2e_i$, which means in particular that f_n maps to 0. (This homomorphism is not claimed to come from a map of spaces.) By inspection of the relations, we can see that the quotient ring $H^*(Sp(2n)/U(n-1)\times U(1),\mathbf{Z})/(f_n)$ is torsion-free. Also, the map from this quotient ring to the cohomology of $SO(2n)/U(n-1)\times U(1)$ is rationally an isomorphism. Therefore the map from the quotient ring $H^*(Sp(2n)/U(n-1)\times U(1),\mathbf{Z})/(f_n)$ to the integral cohomology of $SO(2n)/U(n-1)\times U(1)$ is injective.

It is clear that the homomorphism from the cohomology of $B(U(n-1)\times U(1))/(\mathbf{Z}/2)$ to the cohomology of $SO(2n)/U(n-1)\times U(1)$ factors through the quotient ring $H^*(Sp(2n)/U(n-1)\times U(1),\mathbf{Z})/(f_n)$. We have shown that the torsion index of $PSO(2n)$ is the index of the first homomorphism in the top degree. So the torsion index of $PSO(2n)$ is the product of two numbers: the index of the image of $H^*(BU(n-1)\times U(1)/(\mathbf{Z}/2),\mathbf{Z})$ in the top degree of $H^*(Sp(2n)/U(n-1)\times U(1),\mathbf{Z})/(f_n)$, and the index of the image of $H^*(Sp(2n)/U(n-1)\times U(1),\mathbf{Z})/(f_n)$ in the top degree of $H^*(SO(2n)/U(n-1)\times U(1),\mathbf{Z})$. Moreover, the second of these two numbers is easy to compute. The element $t^{n-1}f_1\cdots f_{n-1}$ is a basis element for $H^*(Sp(2n)/U(n-1)\times U(1),\mathbf{Z})/(f_n)$ in the top degree, and it maps to 2^{n-1} times the basis element $t^{n-1}e_1\cdots e_{n-1}$ of $H^*(SO(2n)/U(n-1)\times U(1),\mathbf{Z})$.

Therefore, the theorem will follow if we can show that the image of $H^*(BU(n-1)\times U(1)/(\mathbf{Z}/2),\mathbf{Z})$ in $H^*(Sp(2n)/U(n-1)\times U(1),\mathbf{Z})/(f_n)$ contains the top-degree basis element $t^{n-1}f_1\cdots f_{n-1}$ whenever n is not a power of 2, and contains 2 times that element but not the element itself when n is a power of 2. In other words, we have reduced to a calculation modulo 2 for most values of n, and to a calculation modulo 4 in the worst case, when n is a power of 2. The image of $H^*(BU(n-$

$1) \times U(1)/(\mathbf{Z}/2), \mathbf{Z})$ is the subring generated by $v := 2t$ and $u_i = c_i(V_{n-1} \otimes L^*)$ for $i = 1, \dots, n-1$. It is convenient to let $V_n = V_{n-1} \oplus L$; then we can also write

$$\begin{aligned} u_i &= c_i(V_{n-1} \otimes L^*) \\ &= c_i((V_n - L) \otimes L^*) \\ &= c_i(V_n \otimes L^* - 1) \\ &= c_i(V_n \otimes L^*) \end{aligned}$$

for $u = 1, \dots, n-1$. This is also true for $i = n$, with the convention that $u_n = 0$.

To analyze the above subring, it is helpful to note that the whole quotient ring $H^*(Sp(2n)/U(n-1) \times U(1), \mathbf{Z})/(f_n)$ is generated by $t, u_1, \dots, u_{n-1}$, as well as by $t, f_1, \dots, f_{n-1}$. This is clear from the formulas $t = c_1 L$, $u_i = c_i(V_n \otimes L^*)$, and $f_i = c_i V_n$. It will also be useful to work out the relations among the generators $t, u_1, \dots, u_{n-1}$. The relations defining $H^*(Sp(2n)/U(n-1) \times U(1), \mathbf{Z})$ say, in particular, that the Chern classes of $V_n \oplus V_n^*$ are zero in positive degrees. It follows that, for $i \geq 1$,

$$\begin{aligned} c_i((V_n \oplus V_n^*) \otimes L) &= c_i(L^{\oplus 2n}) \\ &= \binom{2n}{i} t^i. \end{aligned}$$

Here we can write $(V_n \oplus V_n^*) \otimes L$ as $(V_n \otimes L^*) \otimes L^{\otimes 2} \oplus (V_n \otimes L^*)^*$; so the above formula shows that the $\binom{2n}{i} t^{2i}$ belongs to the subring of $H^*(Sp(2n)/U(n-1) \times U(1), \mathbf{Z})/(f_n)$ generated by the elements u_i and v. This becomes particularly simple modulo 2, so that a bundle and its dual have the same Chern classes. We use that $u_i = c_i(V_n \otimes L^*)$. The result is that, in $H^*(Sp(2n)/U(n-1) \times U(1), \mathbf{Z}/2)/(f_n)$, we have

$$\begin{aligned} u_i^2 &= \binom{2n}{2i} t^{2i} \\ &= \binom{n}{i} t^{2i} \end{aligned}$$

for $i = 1, \dots, n-1$.

This is enough to see that the torsion index of $PSO(2n)$ is 2^a for some $a \geq n$ when n is a power of 2. (Note that we assume $n \geq 2$ throughout.) Namely, from what we have shown, the torsion index of $PSO(2n)$ is 2^{n-1} if and only if the subring of $H^*(Sp(2n)/U(n-1) \times U(1), \mathbf{Z}/2)/(f_n)$ generated by the elements u_i is nonzero in the top degree; the element v is irrelevant since it is 0 modulo 2. (A top-degree basis element is $t^{n-1} f_1 \cdots f_{n-1}$, in degree $\binom{n+1}{2} - 1$.) Now, if n is a power of 2, then the relation $u_i^2 = \binom{n}{i} t^{2i}$ implies that $u_i^2 = 0$ for $i = 1, \dots, n-1$. So the subring of $H^*(Sp(2n)/U(n-1) \times U(1), \mathbf{Z}/2)/(f_n)$ generated by the elements u_i is zero in degrees greater than $\binom{n}{2}$, the degree of $u_1 \cdots u_{n-1}$. Since $n \geq 2$, it follows that the subring generated by the elements u_i is 0 in the top degree, degree $\binom{n+1}{2} - 1$. Thus we have shown that the torsion index of $PSO(2n)$ is 2^a for some $a \geq n$ when n is a power of 2.

We now find another relation among t and the elements u_i, this time in integral cohomology (not just modulo 2). We start with the fact that $c_n(V_n) = 0$ in $H^*(Sp(2n)/U(n-1) \times U(1), \mathbf{Z})/(f_n)$; this is what it means to set f_n to 0. It follows that the only nonzero Chern classes in $H^*(Sp(2n)/U(n-1) \times U(1), \mathbf{Z})/(f_n)$

of the virtual bundle $V_n - 1$ of rank $n-1$ are c_i with $i \leq n-1$. So the virtual bundle $(V_n - 1) \otimes L^* = V_n \otimes L^* - L^*$ has the same property. In particular, $c_n(V_n \otimes L^* - L^*) = 0$. Since $u_i = c_i(V_n \otimes L^*)$, with $u_n = 0$ as mentioned earlier, this gives the following relation in $H^*(Sp(2n)/U(n-1) \times U(1), \mathbf{Z})/(f_n)$:

$$t^n + t^{n-1}u_1 + \cdots + tu_{n-1} = 0.$$

It seems helpful to find a Gröbner basis for the ring $H^*(Sp(2n)/U(n-1) \times U(1), \mathbf{Z}/2)/(f_n)$, using the generators $u_{n-1}, \ldots, u_1, t$ in that order, and the reverse lexicographic order of monomials. We use Buchberger's algorithm, as in section 4. We start with the relations we have found already:

$$u_i^2 = \binom{n}{i} t^{2i}$$

and, using that $n \geq 2$,

$$u_{n-1}t = u_{n-2}t^2 + \cdots + u_1 t^{n-1} + t^n.$$

The first overlap between non-reduced monomials is $u_{n-1}^2 t$, which is equal both to $\binom{n}{n-1} t^{2n-1} = nt^{2n-1}$ and to

$$\begin{aligned} u_{n-1}^2 t &= u_{n-1}(u_{n-2}t^2 + \cdots + u_1 t^{n-1} + t^n) \\ &= (u_{n-1}t)(u_{n-2}t + \cdots + u_1 t^{n-2} + t^{n-1}) \\ &= (u_{n-2}t + \cdots + t^{n-1})^2 t \\ &= u_{n-2}^2 t^3 + u_{n-3}^2 t^5 + \cdots + t^{2n-1} \\ &= \left(\binom{n}{n-2} + \binom{n}{n-3} + \cdots + \binom{n}{0} \right) t^{2n-1} \\ &= (n+1)t^{2n-1}. \end{aligned}$$

Thus $nt^{2n-1} = (n+1)t^{2n-1}$, and so $t^{2n-1} = 0$. As a result, since we know the ring $H^*(Sp(2n)/U(n-1) \times U(1), \mathbf{Z}/2)/(f_n)$ is generated by $u_{n-1}, \ldots, u_1, t$, this ring is spanned as a $\mathbf{Z}/2$-vector space by all monomials which are not multiples of u_i^2, $u_{n-1}t$, or t^{2n-1}. We also know the Hilbert series of this ring, from which we deduce that these monomials actually form a basis for $H^*(Sp(2n)/U(n-1) \times U(1), \mathbf{Z}/2)/(f_n)$. Thus we have found the whole Gröbner basis for this ring. In particular, it follows that the ring is spanned in the top degree by the monomial $u_{n-2} \cdots u_1 t^{2n-2}$.

To prove that $PSO(2n)$ has torsion index only 2^{n-1} for n not a power of 2, it is equivalent to show that the subring of $H^*(Sp(2n)/U(n-1) \times U(1), \mathbf{Z}/2)/(f_n)$ generated by $u_1, \ldots, u_{n-1}$ contains the top-degree basis element, $u_{n-2} \cdots u_1 t^{2n-2}$. We now prove this, by making a good choice of a monomial in $u_1, \ldots, u_{n-1}$. Let $a = \text{ord}_2 n$. Then the binomial coefficient $\binom{n}{2^a}$ is nonzero modulo 2, and $1 \leq 2^a \leq n-1$, since n is not a power of 2. So the relations above imply that

$$u_{2^a}^2 = t^{2^{a+1}}$$

in $H^*(Sp(2n)/U(n-1) \times U(1), \mathbf{Z}/2)/(f_n)$. Let $b = n/2^a$, which is an odd integer. Then we consider the following top-degree monomial in $u_1, \ldots, u_{n-1}$:

$$u_{n-1} \cdots u_{2^a+2}(u_{2^a})^{b+2} u_{2^a-1} \cdots u_1 = u_{n-1} \cdots u_{2^a+2} u_{2^a} \cdots u_1 t^{(b+1)2^a},$$

using that $b+1$ is even and that $u_{2^a}^2 = t^{2^{a+1}}$. Using the above Gröbner relations, this expression reduces to:

$$= u_{n-2} \cdots u_{2^a+2} u_{2^a} \cdots u_1 t^{(b+1)2^a - 1}(u_{n-2}t^2 + \cdots + u_1 t^{n-1} + t^n).$$

Using the relations $u_i^2 = \binom{n}{i} t^{2i}$ and $t^{2n-1} = 0$, where $n = 2^a b$, all the terms here reduce to 0 except one:

$$= u_{n-2} \cdots u_1 t^{2n-2}.$$

As we have said, this is the nonzero element in the top degree of $H^*(Sp(2n)/U(n-1) \times U(1), \mathbf{Z}/2)/(f_n)$. Thus the subring generated by the elements u_i is nonzero in the top degree. This shows that $PSO(2n)$ has torsion index only 2^{n-1} for n not a power of 2.

It remains only to show that the torsion index of $PSO(2n)$ divides 2^n when n is a power of 2. Equivalently, we have to show that the subring of $H^*(Sp(2n)/U(n-1) \times U(1), \mathbf{Z}/4)/(f_n)$ generated by the elements $u_1, \ldots, u_{n-1}$ and $v = 2t$ is nonzero in the top degree. As mentioned earlier, we know that this subring contains $\binom{2n}{i} t^i$ for all i. We apply this with $i = n$; in that case, $\binom{2n}{n} \equiv 2 \pmod 4$, using that n is a power of 2. So the subring of $H^*(Sp(2n)/U(n-1) \times U(1), \mathbf{Z}/4)/(f_n)$ generated by $u_1, \ldots, u_{n-1}$ and v contains $2t^n$. We consider the following top-degree element of that subring:

$$u_{n-1} \cdots u_2 (2t^n).$$

To show that this is nonzero modulo 4, as we want, it suffices to show that $u_{n-1} \cdots u_2(t^n)$ is nonzero modulo 2. So we can use the relations we worked out in $H^*(Sp(2n)/U(n-1) \times U(1), \mathbf{Z}/2)/(f_n)$:

$$u_{n-1} \cdots u_2(t^n) = u_{n-2} \cdots u_2 t^{n-1}(u_{n-2}t^2 + \cdots + u_1 t^{n-1} + t^n).$$

Here all but one of the terms are 0, using the relations $u_i^2 = \binom{n}{i} t^{2i}$ and $t^{2n-1} = 0$. What remains is:

$$= u_{n-2} \cdots u_1 t^{2n-2},$$

which is a nonzero element in the top degree of $H^*(Sp(2n)/U(n-1) \times U(1), \mathbf{Z}/2)/(f_n)$. Therefore the torsion index of $PSO(2n)$ divides 2^n when n is a power of 2, and hence is equal to 2^n. QED

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