Symmetric differentials and the fundamental group

Yohan Brunebarbe, Bruno Klingler, and Burt Totaro

Hélène Esnault asked whether a smooth complex projective variety X with infinite fundamental group must have a nonzero symmetric differential, meaning that $H^0(X, S^i\Omega^1_X) \neq 0$ for some $i > 0$. This was prompted by the second author's work [23]. In fact, Severi had wondered in 1949 about possible relations between symmetric differentials and the fundamental group [34, p. 36]. We know from Hodge theory that the cotangent bundle Ω^1_X has a nonzero section if and only if the abelianization of $\pi_1 X$ is infinite. The geometric meaning of other symmetric differentials is more mysterious, and it is intriguing that they may have such a direct relation to the fundamental group.

In this paper we prove the following result on Esnault's question, in the slightly broader setting of compact Kähler manifolds.

Theorem 0.1. *Let X be a compact Kähler manifold. Suppose that there is a finite-dimensional representation of $\pi_1 X$ over some field with infinite image. Then X has a nonzero symmetric differential.*

All known varieties with infinite fundamental group have a finite-dimensional complex representation with infinite image, and so the theorem applies to them. Depending on what we know about the representation, the proof gives more precise lower bounds on the ring of symmetric differentials.

Remark 0.2. (1) One reason to be interested in symmetric differentials is that they have implications toward Kobayashi hyperbolicity. At one extreme, if Ω^1_X is ample, then X is Kobayashi hyperbolic [25, Theorem 3.6.21]. (Equivalently, every holomorphic map $\mathbf{C} \to X$ is constant.) If X is a surface of general type with $c_1^2 > c_2$, then Bogomolov showed that Ω^1_X is big and deduced that X contains only finitely many rational or elliptic curves, something which remains open for arbitrary surfaces of general type [3, 10]. For any $\alpha \in H^0(X, S^i\Omega^1_X)$ with $i > 0$, the restriction of α to any rational curve in X must be zero (because $\Omega^1_{\mathbf{P}^1}$ is a line bundle of negative degree), and so any symmetric differential gives a first-order algebraic differential equation satisfied by all rational curves in X.

A lot is already known about Kobayashi hyperbolicity in the situation of Theorem 0.1. In particular, Yamanoi showed that for any smooth complex projective variety X such that $\pi_1 X$ has a finite-dimensional complex representation whose image is not virtually abelian, the Zariski closure of any holomorphic map $\mathbf{C} \to X$ is a proper subset of X [41].

(2) Arapura used Simpson's theory of representations of the fundamental group to show that if $\pi_1 X$ has a non-rigid complex representation, then X has a nonzero symmetric differential [1, Proposition 2.4], which we state as Theorem 4.1.

Thus the difficulty for Theorem 0.1 is how to use a rigid representation of the fundamental group. The heart of the proof is a strengthening of Griffiths and

Zuo's results on variations of Hodge structure [16, 44], from weak positivity of the cotangent bundle (analogous to "pseudoeffectivity" in the case of line bundles) to bigness. As a result, we get many symmetric differentials on the base of a variation of Hodge structure.

The strengthening involves two ingredients. First, a new curvature calculation shows that if a compact Kahler manifold X has nonpositive holomorphic bisectional curvature, and the holomorphic sectional curvature is negative at one point, then the cotangent bundle of X is nef and big (Theorem 1.1). Next, we have to relate any variation of Hodge structure to one with discrete monodromy group. For that, we use results of Katzarkov and Zuo which analyze p-adic representations of the fundamental group by harmonic map techniques. This reduction to the case of discrete monodromy group is very much in the spirit of the arguments by Eyssidieux, Katzarkov, and others which prove the Shafarevich conjecture for linear groups [11, 12, 22].

(3) The abundance conjecture in minimal model theory would imply that a smooth complex projective variety X is rationally connected if and only if $H^0(X, (\Omega^1_X)^{\otimes i}) = 0$ for all $i > 0$ [14, Corollary 1.7]. Without abundance, Campana used Gromov's L^2 arguments [18] on the universal cover to show that if X is not simply connected, then $H^0(X, (\Omega^1_X)^{\otimes i}) \neq 0$ for some $i > 0$ [5, Corollary 5.1]. But this conclusion is weaker than that of Theorem 0.1. In particular, finding a section of a general tensor bundle $(\Omega^1_X)^{\otimes i}$ has no direct implication towards Kobayashi hyperbolicity. (There are more subtle implications, however. Demailly has shown that every smooth projective variety X of general type has some algebraic differential equations, typically not first-order, which are satisfied by all holomorphic maps $\mathbf{C} \to X$ [9, Theorem 0.5].)

There are many varieties X of general type (which have many sections of the line bundles $K_X^{\otimes j}$, hence of the bundles $(\Omega^1_X)^{\otimes i}$) which have no symmetric differentials. For example, Schneider showed that a smooth subvariety $X \subset \mathbf{P}^N$ with $\dim(X) > N/2$ has no symmetric differentials [33]. Most such varieties are of general type.

(4) The possible implication from infinite fundamental group to existence of symmetric differentials cannot be reversed. In fact, Bogomolov constructed smooth complex projective varieties which are simply connected but have ample cotangent bundle [7, Proposition 26]. Brotbek recently gave a simpler example of a simply connected variety with ample cotangent bundle: a general complete intersection surface of high multidegree in $\mathbf{P}^N$ for $N \geq 4$ has ample cotangent bundle [4, Corollary 4.8]. The ring of symmetric differentials on such a variety is as big as possible, roughly speaking.

(5) Theorem 0.1 makes it natural to ask whether the fundamental group of a smooth complex projective variety X, if infinite, must have a finite-dimensional complex representation with infinite image. This is not known. We know by Toledo that the fundamental group of a smooth projective variety need not be residually finite [40], in particular need not be linear.

Even if the fundamental group of a smooth projective variety X is infinite and residually finite, it is not known whether $\pi_1 X$ always has a finite-dimensional complex representation with infinite image. Indeed, there are infinite, residually finite, finitely presented groups Γ such that every finite-dimensional complex representation of Γ has finite image. Such a group can be constructed as follows; can it be the fundamental group of a smooth complex projective variety? Let K be a global field of prime characteristic p (for example $K = \mathbf{F}_p(T)$). Let S be a finite set of primes of K and O_S the subring of S-integers of K. Then $\Gamma := SL(n, O_S)$ is finitely presented if $n \geq 3$ and $|S| > 1$ [39], and any finite-dimensional complex representation of Γ has finite image if $n \geq 3$ and $|S| > 0$ [28, Theorem 3.8(c)]. Also, Γ is residually finite.

(6) One cannot strengthen Theorem 0.1 to say that every non-simply-connected variety has a nonzero symmetric differential. For example, Kobayashi showed that every smooth complex projective variety X with torsion first Chern class and finite fundamental group has $H^0(X, S^i\Omega^1_X) = 0$ for all $i > 0$ [24]. This applies to Enriques surfaces, which have fundamental group $\mathbf{Z}/2$.

Acknowledgements: It is a pleasure to thank H. Esnault, who suggested the question leading to Theorem 0.1 after a lecture by the second author on [23]; F. Campana for explaining [5]; and I. Dolgachev, C. Haesemeyer, L. Katzarkov, and K. Zuo for useful questions.

Convention: Throughout the paper, varieties and manifolds are understood to be connected.

1 Negatively curved varieties

Theorem 1.1. *Let X be a compact Kähler manifold with nonpositive holomorphic bisectional curvature. Suppose that the holomorphic sectional curvature is negative at one point of X. Then the cotangent bundle of X is nef and big.*

For a vector bundle E on a compact complex manifold, write $P(E)$ for the projective bundle of hyperplanes in E. We define a vector bundle E to be *ample, nef*, or *big* if the line bundle $O(1)$ on $P(E)$ has the corresponding property [26, Definition 6.1.1, Example 6.1.23]; see Demailly [8, Definition 6.3] for the definition of a nef line bundle on a compact complex manifold. It follows that E is big if and only if there are $c > 0$ and $j_0 \geq 0$ such that

$$h^0(X, S^j E) \geq c j^{\dim(X)+\mathrm{rank}(E)-1}$$

for all $j \geq j_0$. Note that Viehweg and Zuo use "big" for a stronger property of vector bundles, as discussed below in Remark (2).

We give the definition of holomorphic bisectional curvature in the proof of Lemma 1.4. On a Kähler manifold, the holomorphic bisectional curvature $B(x, y)$ for tangent vectors x and y is a positive linear combination of the (Riemannian) sectional curvatures of the real 2-planes $\mathbf{R}\{x, y\}$ and $\mathbf{R}\{x, iy\}$ [13, section 1]. Holomorphic sectional curvature is a special case of holomorphic bisectional curvature; it is also equal to the sectional curvature of a complex line in the tangent space, viewed as a real 2-plane. It follows that a Kähler manifold with negative or nonpositive sectional curvature satisfies the corresponding inequality for holomorphic bisectional curvature, and that in turn implies the corresponding inequality for holomorphic sectional curvature.

For example, Theorem 1.1 applies to the quotient of any bounded symmetric domain by a torsion-free cocompact lattice, or to any smooth subvariety of such a quotient. (This uses that holomorphic bisectional curvature and holomorphic sectional curvature decrease on complex submanifolds [13, section 4].) Thus we have a large class of smooth projective varieties with a lot of symmetric differentials. Theorem 1.1 seems to be new even for these heavily studied varieties. For a quotient X of a symmetric domain of "tube type", it was known that, after passing to some finite covering, $S^n\Omega^1_X$ contains the ample line bundle $K_X = \Omega^n_X$, and so X has some symmetric differentials [6, section 4.2]. But that argument does not show that Ω^1_X is big.

Remark 1.2. (1) If the holomorphic *bisectional* curvature of a compact Kähler manifold is *negative*, then the cotangent bundle is ample. But the cotangent bundle of X need not be ample under the assumptions of Theorem 1.1, even if the holomorphic sectional curvature is everywhere negative. A simple example is the product $C_1 \times C_2$ of two curves of genus at least 2, for which the natural product metric has negative holomorphic sectional curvature and nonpositive holomorphic bisectional curvature. The cotangent bundle is $\pi_1^*\Omega^1_{C_1} \oplus \pi_2^*\Omega^1_{C_2}$, which is not ample because its restriction to a curve $C_1 \times p$ has a trivial summand. A more striking example is the quotient X of the product of two copies of the unit disc by an irreducible torsion-free cocompact lattice. (Some surfaces of this type are known as quaternionic Shimura surfaces.) The curvature conditions of Theorem 1.1 are again satisfied at every point. In this case, Shepherd-Barron showed that the algebra of symmetric differentials $\oplus_{i\geq 0} H^0(X, S^i\Omega^1_X)$ is not finitely generated [35]. It follows that the cotangent bundle of X is nef and big but not semi-ample [26, Example 2.1.30], let alone ample.

(2) The cotangent bundle need not be big in Viehweg's stronger sense under the assumptions of Theorem 1.1. To give the definition, let X be a projective variety over a field with an ample line bundle L. A vector bundle E is *weakly positive* if there is a nonempty open subset U such that for every $a > 0$ there is a $b > 0$ such that the sections of $S^{ab}(E) \otimes L^{\otimes b}$ over X span that bundle over U. A bundle E is *Viehweg big* if there is a $c > 0$ such that $S^c(E) \otimes L^{-1}$ is weakly positive. The assumptions of Theorem 1.1 do not imply that Ω^1_X is Viehweg big, as shown again by X the product of two curves of genus at least 2 [21, Example 1.8].

(3) Following Sakai, we define the *cotangent dimension* $\lambda(X)$ of a compact complex n-fold X to be the smallest number λ such that there is a positive constant C with $\sum_{i=0}^{j} h^0(X, S^i\Omega^1_X) \leq Cj^{\lambda+n}$ for all $j \geq 0$ [32]. Then $\lambda(X)$ is an integer between $-n$ and n, and Ω^1_X is big if and only if $\lambda(X)$ has the maximum value, n.

Proof. (Theorem 1.1) Let $\mathbf{P}(\Omega^1_X) \to X$ be the bundle of hyperplanes in the cotangent bundle Ω^1_X. Since X has nonpositive bisectional curvature, the associated metric on the line bundle $O(1)$ on $\mathbf{P}(\Omega^1_X)$ has nonnegative curvature [15, 2.36]. It follows that Ω^1_X is nef.

The hard part is to show that Ω^1_X is big. Equivalently, we have to show that the line bundle $O(1)$ on $\mathbf{P}(\Omega^1_X)$ is big. By Siu, this holds if the differential form $(c_1O(1))^{2n-1}$, which we know is nonnegative, is positive at some point of the compact complex manifold $\mathbf{P}(\Omega^1_X)$ [38]. The pushforward of the cohomology class $(c_1O(1))^{2n-1}$ to X is the Segré class $s_n(TX)$. In fact, this is true at the level of differential forms, by Guler [20]. (The total Segré class of a vector bundle E is defined as the inverse of the total Chern class, $s(E) = c(E)^{-1}$. For example, $s_1(E) = -c_1(E)$ and $s_2(E) = (c_1^2 - c_2)(E)$.) So we want to show that the Segré number $\int_X s_n(TX)$ is positive (rather than zero).

Lemma 1.3. *Let E be a holomorphic hermitian vector bundle of rank n on a complex manifold X of dimension at least n. Let p be a point in X. Suppose that the curvature Θ_E in $A^{1,1}(End(E))$ is nonnegative at p, and that there is a nonzero vector e in E_p such that the $(1,1)$-form $\Theta_E(e,e)$ at p is positive. Then the Segré form $s_n(E^*)$ is positive at p.*

Proof. Since E has nonnegative curvature, the associated metric on the line bundle $O(1)$ on $\mathbf{P}(E)$ has nonnegative curvature form $c_1O(1)$. The Segré form $s_n(E^*)$ is the integral along the fibers of the differential form $(c_1O(1))^{2n-1}$. So the Segré form is positive at the point p if the $(1,1)$-form $c_1O(1)$ is positive at least at one point e of the fiber over p. Griffiths's formula for the curvature of $O(1)$ [15, 2.36] is:

$$c_1O(1)(y,y) = \frac{\Theta_E(e,e,y_h,y_h)}{|e|^2} + \omega(y_v,y_v),$$

where y is a tangent vector in $\mathbf{P}(E)$ with horizontal and vertical parts y_h and y_v, and ω is a positive $(1,1)$ form on the projective space $\mathbf{P}(E_p)$. So the form $c_1O(1)$ is positive at a point in $\mathbf{P}(E)$ if the corresponding vector e in E_p^* (defined up to $\mathbf{C}^*$) satisfies $\Theta_E(e,e,v,v) > 0$ for all $v \neq 0$ in T_pX. This is exactly our assumption. □

The theorem is now a consequence of the following geometric lemma. □

Lemma 1.4. *Let X be a Kähler manifold. Suppose that at a point p, the holomorphic sectional curvature of X is at most a negative constant $-A$. Then there is a nonzero vector x in T_pX such that the holomorphic bisectional curvature $B(x,y)$ is at most $-A/2$ for all nonzero vectors y in T_pX.*

It seems surprising that the lemma holds without assuming that X has nonpositive holomorphic bisectional curvature. (That is true in our application, however.) The bound $-A/2$ is optimal, as shown by the invariant metric on the complex unit n-ball for $n \geq 2$: if we scale the metric to have holomorphic sectional curvature equal to $-A$, then the holomorphic bisectional curvature $B(x,y)$ varies between $-A$ (when x and y span the same complex line) and $-A/2$ (when the hermitian inner product $\langle x,y\rangle$ is zero).

In the special case where X is a bounded symmetric domain, Mok studied in detail the geometry of the tangent vectors x such that the holomorphic bisectional curvature $B(x,y)$ is zero for some y [29, p. 100 and p. 252]. He called such vectors "higher characteristic vectors".

Proof. The curvature of a holomorphic vector bundle E with hermitian metric over a complex manifold X can be viewed as a form

$$\Theta_E : E_p \times E_p \times T_pX \times T_pX \to \mathbf{C}$$

which is linear in the first and third variables and conjugate linear in the second and fourth variables [42, section 7.5]. It satisfies

$$\Theta_E(y,x,w,z) = \overline{\Theta_E(x,y,z,w)}.$$

Taking the dual bundle changes the sign of the curvature, in the sense that

$$\Theta_E(x,y,z,w) = -\Theta_{E^*}(y^*,x^*,z,w),$$

where x^* and y^* are the dual vectors given by the hermitian form.

We define the curvature $\Theta = \Theta_{TX}$ of a hermitian metric on X to be the curvature of the tangent bundle as a holomorphic vector bundle with hermitian metric. When the metric is Kähler, we also have

$$\Theta(x, y, z, w) = \Theta(z, y, x, w).$$

The holomorphic bisectional curvature is defined by

$$B(x, y) = \frac{\Theta(x, x, y, y)}{|x|^2|y|^2}$$

for nonzero vectors $x, y \in T_pX$. By the identities above, the holomorphic bisectional curvature is real and depends only on the complex lines $\mathbf{C}x$ and $\mathbf{C}y$. The holomorphic sectional curvature is

$$H(x) = B(x, x) = \frac{\Theta(x, x, x, x)}{|x|^4}$$

for a nonzero vector x in T_pX. This depends only on the complex line $\mathbf{C}x$.

Let x be a nonzero vector in T_pX which maximizes the holomorphic sectional curvature. This is possible, because the holomorphic sectional curvature is C^∞ on the complex projective space of lines in T_pX. Write $H(x) = -A < 0$. With this simple choice, we will show that $B(x, y) \leq -A/2$ for all nonzero vectors y in T_pX.

We can scale x and y to have length 1. To first order, for $c \in \mathbf{C}$ near 0, we have

$$\begin{aligned} H(x+cy) &= \frac{1}{|x+cy|^4}\Theta(x+cy, x+cy, x+cy, x+cy) \\ &= (1 - 4 \ \mathrm{Re}\ (\overline{c}\langle x, y\rangle))[H(x) + 4 \ \mathrm{Re}\ (\overline{c}\Theta(x, x, x, y))] + O(|c|^2) \\ &= H(x) + 4 \ \mathrm{Re}\ [\overline{c}(-H(x)\langle x, y\rangle + \Theta(x, x, x, y))] + O(|c|^2), \end{aligned}$$

using that $|x+cy|^2 = 1 + 2 \ \mathrm{Re}\ (\overline{c}\langle x, y\rangle) + |c|^2$. (We take the hermitian metric $\langle x, y\rangle$ on T_pX to be linear in x and conjugate linear in y.) Since the holomorphic sectional curvature is maximized at the vector x, the first-order term in c must be zero for all $c \in \mathbf{C}$, and so $\Theta(x, x, x, y) = H(x)\langle x, y\rangle$.

Next, we compute to second order, for $c \in \mathbf{C}$ near 0. The identities on curvature imply that $B(x, y) = \Theta(x, x, y, y) = \Theta(y, x, x, y) = \Theta(y, y, x, x) = \Theta(x, y, y, x)$, and we know that $\Theta(x, x, x, y) = H(x)\langle x, y\rangle$. Therefore:

$$\begin{aligned} H(x+cy) = {} & \frac{1}{|x+cy|^4}\Theta(x+cy, x+cy, x+cy, x+cy) \\ & = [1 - 4 \ \mathrm{Re}\ (\overline{c}\langle x, y\rangle) - 2|c|^2 + 12 \ \mathrm{Re}\ (\overline{c}\langle x, y\rangle)^2] \\ & \cdot [H(x) + 4H(x) \ \mathrm{Re}\ (\overline{c}\langle x, y\rangle) + 4B(x, y)|c|^2 + 2 \ \mathrm{Re}\ (\overline{c}^2\Theta(x, y, x, y))] + O(|c|^3) \\ & = H(x) - 2H(x)|c|^2 - 4H(x)(\ \mathrm{Re}\ \overline{c}\langle x, y\rangle)^2 + 4B(x, y)|c|^2 + 2 \ \mathrm{Re}\ (\overline{c}^2\Theta(x, y, x, y)) + O(|c|^3). \end{aligned}$$

Since the holomorphic sectional curvature is maximized at x, the quadratic term in c must be ≤ 0 for all $c \in \mathbf{C}$. The term $-4H(x)(\ \mathrm{Re}\ \overline{c}\langle x, y\rangle)^2$ is nonnegative, which works to our advantage. Let c belong to one of the real lines in $\mathbf{C}$ such that $\mathrm{Re}\ (\overline{c}^2\Theta(x, y, x, y)) = 0$; then we must have

$$-2H(x)|c|^2 + 4B(x, y)|c|^2 \leq 0.$$

Therefore, $B(x, y) \leq H(x)/2 = -A/2$, as we want. $\square$

2 More varieties with big cotangent bundle

Corollary 2.1. *Let X be a compact Kähler manifold with nonpositive holomorphic bisectional curvature. Let Y be a compact Kähler manifold with a generically finite meromorphic map $Y \dashrightarrow X$. Suppose that the holomorphic sectional curvature of X is negative at some point in the closure of the image of Y. Then the cotangent bundle of Y is big.*

Proof. By resolution of singularities, there is a compact Kähler manifold Y_2 with a bimeromorphic morphism $Y_2 \to Y$ such that the given map $f : Y \dashrightarrow X$ extends to a morphism $f : Y_2 \to X$. Since the ring of symmetric differentials on a compact complex manifold is a bimeromorphic invariant, it suffices to show that $\Omega^1_{Y_2}$ is big.

Let $W \to X$ be the Grassmannian bundle of subspaces of TX of dimension equal to $n = \dim(Y)$. Since f is generically finite, the derivative of f is injective on an open dense subset U of Y_2. So the derivative of f gives a meromorphic map $g : Y_2 \dashrightarrow W$ lifting f, a morphism over U. Again, there is a compact Kähler manifold Y_3 with a bimeromorphic morphism $Y_3 \to Y_2$, an isomorphism over U, such that g extends to a morphism $g : Y_3 \to W$. It suffices to show that $\Omega^1_{Y_3}$ is big.

There is a natural vector bundle E of rank n on the Grassmannian bundle W, a quotient of the pullback of Ω^1_X to W. The bundle E inherits a hermitian metric from Ω^1_X. Therefore the bundle g^*E on Y_3 has a hermitian metric. Moreover, the restriction of g^*E to $U \subset Y_3$ can be identified with Ω^1_U, with the metric pulled back from the metric on Ω^1_X via the immersion $f : U \to X$. Since X has nonpositive bisectional curvature, and bisectional curvature decreases on complex submanifolds [42, section 7.5], the curvature of g^*E is nonnegative over U, hence over all of X_3. Also, by Lemmas 1.3 and 1.4, the Segré form $s_n((g^*E)^*)$ is positive at some point of U, because X has negative holomorphic sectional curvature at some point in the image of U, and holomorphic sectional curvature decreases on complex submanifolds [42, section 7.5]. The Segré form may not be positive on all of Y_3, but positivity on U implies that the number $\int_X s_n((g^*E)^*)$ is positive. Equivalently, the line bundle $O(1)$ on $\mathbf{P}(g^*E) \to Y_3$ has nonnegative curvature and the number $(c_1 O(1))^{2n-1}$ is positive. So $O(1)$ is nef and big on $\mathbf{P}(g^*E)$. Equivalently, g^*E is nef and big on Y_3.

Because we can pull back 1-forms, we have a natural map $\alpha : f^*\Omega^1_X \to \Omega^1_{Y_3}$ of vector bundles on Y_3, which is surjective over U. Also, we have a natural surjection $\beta : f^*\Omega^1_X \to g^*E$ over Y_3 by definition of E. The map α factors through the surjection β over U, hence over all of Y_3. That is, we have a map $g^*E \to \Omega^1_{Y_3}$ of vector bundles over Y_3, and it is an isomorphism over U.

The resulting map $H^0(Y_3, S^j(g^*E)) \to H^0(Y_3, S^j\Omega^1_{Y_3})$ is injective for all $j > 0$. Since g^*E is big on Y_3, $\Omega^1_{Y_3}$ is big. $\square$

3 Variations of Hodge structure

Let X be a compact Kähler manifold. Consider a complex variation of Hodge structure V over X, and let

$$\varphi : \widetilde{X} \to D$$

be the corresponding period map, where $\widetilde{X}$ is the universal cover of X.

Theorem 3.1. *Suppose that the derivative of φ is injective at some point of X. Then Ω^1_X is big.*

The theorem was inspired by Zuo's result that Ω^1_X is weakly positive under these assumptions [44, Theorem 0.1]. Note that weak positivity (defined in section 1) generalizes the notion of "pseudo-effective" for line bundles. So Zuo's result is similar, but it does not show that $H^0(X, S^i\Omega^1_X)$ is nonzero for some i. We repeat that our notion of a big vector bundle is not the stronger notion "Viehweg big" (section 1).

We recall that a *complex variation of Hodge structure* on a complex manifold X is a complex local system V with an indefinite hermitian form and an orthogonal C^∞ decomposition $V = \oplus_{p\in\mathbf{Z}} V^p$ such that the form is $(-1)^p$-definite on V^p, and such that Griffiths transversality holds: the connection sends V^p into

$$A^{1,0}_X(V^{p-1}) \oplus A^1_X(V^p) \oplus A^{0,1}_X(V^{p+1})$$

[36, section 4]. Let $r_p = \dim(V^p)$; then the corresponding period domain is the complex manifold $D = G/V$ where $G = U(\sum_{p \text{ odd}} r_p, \sum_{p \text{ even}} r_p)$ and $V = \prod_p U(r_p)$. A complex variation of Hodge structure with ranks r_p is equivalent to a representation of $\pi_1 X$ into G and a $\pi_1 X$-equivariant holomorphic map $\widetilde{X} \to D$ which is horizontal with respect to a natural distribution in the tangent bundle of D.

Proof. Griffiths and Schmid defined a G-invariant hermitian metric on a period domain $D = G/V$. The period map $\varphi : \widetilde{X} \to D$ is always tangent to the "horizontal" subbundle of TD. The holomorphic sectional curvatures of D corresponding to horizontal directions are at most a negative constant [17, Theorem 9.1]. Pulling back the metric on D gives a canonical hermitian metric g on the Zariski open subset $U \subset X$ where φ is an immersion. Since holomorphic sectional curvature decreases on submanifolds, g has negative holomorphic sectional curvature on U. Peters showed that g has nonpositive holomorphic bisectional curvature on U [30, Corollary 1.8, Lemma 3.1]. Finally, g is a Kähler metric on U (even though the metric on D is only a hermitian metric) [27, Theorem 1.2].

The metric g may degenerate on X, but we can argue as follows. Let $Y \to D$ be the Grassmannian bundle of subspaces of TD of dimension equal to $n = \dim(X)$. Then the derivative of φ gives a lift of the morphism $\widetilde{X} \to D$ to a $\pi_1 X$-equivariant meromorphic map $f : \widetilde{X} \dashrightarrow Y$ (a morphism over $\widetilde{U}$). Let $\widetilde{X}_2$ be the closure of the graph of f in $\widetilde{X} \times Y$. We have a $\pi_1 X$-equivariant proper bimeromorphic morphism $\widetilde{X}_2 \to \widetilde{X}$, and f extends to a morphism $f : \widetilde{X}_2 \to Y$. Let $X_2 = \widetilde{X}_2/\pi_1 X$, which is a compact analytic space with a proper bimeromorphic morphism $X_2 \to X$. Finally, let $X_3 \to X_2$ be a resolution of singularities; we can assume that X_3 is a compact Kähler manifold since X is a compact Kähler manifold. Then X_3 is a compact Kähler manifold with a bimeromorphic morphism $X_3 \to X$, and $f : \widetilde{X} \dashrightarrow Y$ extends to a $\pi_1 X$-equivariant morphism $f : \widetilde{X}_3 \to Y$.

There is a natural G-equivariant vector bundle E of rank $\dim(X)$ on the Grassmannian bundle Y, a quotient of the pullback of Ω^1_D to Y. The bundle E inherits a hermitian metric from Ω^1_D. Therefore the bundle f^*E on $\widetilde{X}_3$ has a hermitian metric. This bundle is $\pi_1 X_3$-equivariant, and we also write f^*E for the corresponding bundle on X_3. The restriction of f^*E to $U \subset X_3$ can be identified with Ω^1_U with the metric induced from the metric on the dual bundle TU. Because curvature increases for quotient bundles [42, section 7.5], the curvature of f^*E is nonnegative over U, hence over all of X_3. Also, by Lemmas 1.3 and 1.4, the Segré form $s_n((f^*E)^*)$ is positive at each point of U. It may not be positive on all of X_3, but positivity on U implies that the number $\int_X s_n((f^*E)^*)$ is positive. Equivalently, the line bundle $O(1)$ on $\mathbf{P}(f^*E) \to X_3$ has nonnegative curvature and the number $(c_1 O(1))^{2n-1}$ is positive. So $O(1)$ is nef and big on $\mathbf{P}(f^*E)$. Equivalently, f^*E is nef and big on X_3.

Because we can pull back 1-forms, we have a natural map $\alpha : \varphi^*\Omega^1_D \to \Omega^1_{\widetilde{X}_3}$ of vector bundles on $\widetilde{X}_3$, which is surjective over $\widetilde{U}$. Also, we have a natural surjection $\beta : \varphi^*\Omega^1_D \to f^*E$ over $\widetilde{X_3}$ by definition of E. The map α factors through the surjection β over $\widetilde{U}$, hence over all of $\widetilde{X}_3$. That is, we have a map $g^*E \to \Omega^1_{\widetilde{Y}_3}$ of vector bundles over $\widetilde{X}_3$, and it is an isomorphism over U. This map is $\pi_1 X_3$-equivariant, and so we have a corresponding map of vector bundles over X_3.

The resulting map $H^0(X_3, S^j(f^*E)) \to H^0(X_3, S^j\Omega^1_{X_3})$ is injective for all $j \geq 0$. Since f^*E is big on X_3, $\Omega^1_{X_3}$ is big. Since the ring of symmetric differentials on a compact complex manifold is a bimeromorphic invariant, Ω^1_X is big. $\square$

Corollary 3.2. *Let X be a compact Kähler manifold. Consider a complex variation of Hodge structure V over X with discrete monodromy group Γ, and let*

$$\varphi : X \to D/\Gamma$$

be the corresponding period map. After replacing X by a finite étale covering Z, we can assume that Γ is torsion-free. Let Y be a resolution of singularities of the image of $\varphi : Z \to D/\Gamma$. Then Ω^1_Y is big.

For a dominant meromorphic map of compact complex manifolds $Z \dashrightarrow Y$, there is a natural pullback map on the ring of symmetric differentials,

$$\oplus_{i\geq 0} H^0(Y, S^i\Omega^1_Y) \to \oplus_{i\geq 0} H^0(Z, S^i\Omega^1_Z),$$

and this is injective. Therefore, Corollary 3.2 implies that the cotangent dimension $\lambda(X)$ is at least $2\dim(Y) - \dim(X)$, using that $\lambda(X) = \lambda(Z)$ for a finite étale covering $Z \to X$ [32, Theorem 1]. In particular, if X is the base of a variation of Hodge structure with discrete and infinite monodromy, then the image Y of the period map has positive dimension, and so our lower bound for $\lambda(X)$ gives that X has a nonzero symmetric differential.

4 Non-rigid representations

We use the following result which Arapura proved for smooth complex projective varieties [1, Proposition 2.4]. It was extended to compact Kähler manifolds by the second author [23, Theorem 1.6(i)].

Theorem 4.1. *Let X be a compact Kähler manifold. Suppose that $\pi_1 X$ has a complex representation of dimension n which is not rigid. Then $H^0(X, S^i\Omega^1_X) \neq 0$ for some $1 \leq i \leq n$.*

Here we say that a representation of $\pi_1 X$ is rigid if the corresponding point in the moduli space $M_B(X, GL(n))$ (the "Betti moduli space" or "character variety") is isolated. The points of $M_B(X, GL(n))$ are in one-to-one correspondence with the isomorphism classes of n-dimensional semisimple representations (meaning direct sums of irreducibles) of $\pi_1 X$, with a representation being sent to its semisimplification [37, section 7].

We also need the following p-adic analogue, essentially proved by Katzarkov and Zuo using Gromov-Schoen's construction of pluriharmonic maps into the Bruhat-Tits building [22, proof of Theorem 3.2], [43, section 1], [19]. An explicit formulation and proof of Theorem 4.2 can be found in the second author's [23, Theorem 1.6(ii)].

Theorem 4.2. *Let X be a compact Kähler manifold, and let K be a nonarchimedean local field. Suppose that there is a semisimple representation from $\pi_1 X$ to $GL(n, K)$ which is unbounded (equivalently, which is not conjugate to a representation over the ring of integers of K). Then $H^0(X, S^i\Omega^1_X) \neq 0$ for some $1 \leq i \leq n$.*

5 Representations in positive characteristic

We now prove Theorem 0.1 for representations in positive characteristic, which turns out to be easier. That is, we will show that if X is a compact Kähler manifold such that $\pi_1 X$ has a finite-dimensional infinite-image representation over some field k of characteristic $p > 0$, then X has a nonzero symmetric differential.

We can assume that the field k is algebraically closed. By Procesi, for each natural number n, there is an affine scheme $M = M_B(X, GL(n))_{\mathbf{F}_p}$ of finite type over $\mathbf{F}_p$ whose k-points are in one-to-one correspondence with the set of isomorphism classes of semisimple n-dimensional representations of $\pi_1 X$ over k [31, Theorem 4.1]. (To construct this scheme, choose a finite presentation for $\pi_1 X$. The space of homomorphisms $\pi_1 X \to GL(n)$ is a closed subscheme of $GL(n)^r$ over $\mathbf{F}_p$ in a natural way, where r is the number of generators for $\pi_1 X$. We then take the affine GIT quotient by the conjugation action of $GL(n)$.)

We want to show that if X has no symmetric differentials, then every n-dimensional representation of $\pi_1 X$ over k has finite image. Suppose that the Betti moduli space M has positive dimension over $\mathbf{F}_p$. Then M contains an affine curve over $\mathbf{F}_q$ for some power q of p. Therefore, after possibly increasing q, there is a point of $M(\mathbf{F}_q((t)))$ which is not in $M(\mathbf{F}_q[[t]])$. After increasing q again, it follows that there is a semisimple representation of $\pi_1(X)$ over $\mathbf{F}_q((t))$ which is not defined over $\mathbf{F}_q[[t]]$. This contradicts Katzarkov and Zuo's Theorem 4.2, since X has no symmetric differentials. So in fact M has dimension zero over $\mathbf{F}_p$.

It follows that every finite-dimensional semisimple representation of $\pi_1 X$ over k is in fact defined over $\overline{\mathbf{F}_p}$. But every finite-dimensional representation of a finitely generated group over $\overline{\mathbf{F}_p}$ has finite image, since the generators all map to matrices over some finite field. So every finite-dimensional semisimple representation of $\pi_1 X$ over k has finite image.

Finally, we show that any finite-dimensional representation ρ of $\pi_1 X$ over k has finite image. We know that the semisimplification of ρ has finite image. Therefore, a finite-image subgroup H of $\pi_1 X$ maps into the subgroup of strictly upper-triangular matrices in $GL(n, k)$. The latter group is a finite extension of copies of the additive group over k, and so the image of H is a finite extension of abelian groups killed by p. Since H is finitely generated, it follows that the image of H is finite. Therefore the image of $\pi_1 X$ in $GL(n, k)$ is finite, as we want.

6 Representations in characteristic zero

Theorem 6.1. *Let X be a compact Kähler manifold. Suppose that there is a finite-dimensional complex representation of $\pi_1 X$ with infinite image. Then X has a nonzero symmetric differential.*

If a finitely generated group has an infinite-image representation over a field of characteristic zero, then it has an infinite-image representation over $\mathbf{C}$. So Theorem 6.1 will complete the proof of Theorem 0.1.

Proof. By Theorem 4.1, we can assume that the given representation ρ of $\pi_1 X$ is rigid. Therefore, the point associated to ρ in $M_B(X, GL(n))$ is fixed by Simpson's $\mathbf{C}^*$ action, which means that the semisimplification σ of ρ can be made into a complex variation of Hodge structure over X [36, Corollary 4.2].

Suppose that the representation σ has finite image. Then ρ sends a finite-index subgroup H of $\pi_1 X$ into the group U of strictly upper-triangular matrices in $GL(n, \mathbf{C})$. Since U is nilpotent and ρ has infinite image, the abelianization of H must be infinite. By Hodge theory, the finite étale covering $Y \to X$ corresponding to H has a nonzero 1-form $\alpha \in H^0(Y, \Omega^1_Y)$. If H has index r in $\pi_1 X$, the norm of Y is a nonzero element of $H^0(X, S^r\Omega^1_X)$, as we want.

It remains to consider the case where σ is a complex variation of Hodge structure with infinite image. We cannot immediately apply Corollary 3.2 because the image of σ in $GL(n, \mathbf{C})$ need not be discrete. At least σ is conjugate to a representation into $GL(n, F)$ for some number field, because σ is rigid. More precisely, σ is a complex direct factor of a $\mathbf{Q}$-variation of Hodge structure $\tau : \pi_1 X \to GL(m, \mathbf{Q})$ [36, Theorem 5]. Let m be the dimension of τ.

For each prime number p, consider the representation $\tau : \pi_1 X \to GL(m, \mathbf{Q}_p)$. By Katzarkov and Zuo's Theorem 4.2, if this representation is not bounded, then $H^0(X, S^i\Omega^1_X)$ is nonzero for some $1 \le i \le m$.

Therefore, we can assume that τ is p-adically bounded for each prime number p. Then τ is conjugate to a representation into $GL(m, \mathbf{Z})$ [2]. Thus τ is a complex variation of Hodge structure with discrete monodromy group. By Corollary 3.2, X has nonzero symmetric differentials. More precisely, there is a finite étale covering Z of X and a blow-up Z_2 of Z such that the given representation of $\pi_1 Z_2 = \pi_1 Z \subset \pi_1 X$ factors through a surjection $Z_2 \to Y$ with Ω^1_Y big and Y of positive dimension. As a result, the cotangent dimension $\lambda(X)$ is at least $2\dim(Y) - \dim(X)$. □

References

[1] D. Arapura. Higgs bundles, integrability, and holomorphic forms. *Motives, polylogarithms and Hodge theory, Part II* (Irvine, CA, 1998), 605–624. Int. Press (2002).

[2] H. Bass. Groups of integral representation type. *Pacific J. Math.* **86** (1980), 15–51.

[3] F. Bogomolov. Families of curves on a surface of general type. *Dokl. Akad. Nauk* **236** (1977), 1041–1044.

[4] D. Brotbek. Hyperbolicity related problems for complete intersection varieties. *Compos. Math.,* to appear.

[5] F. Campana. Fundamental group and positivity of cotangent bundles of compact Kähler manifolds. *J. Alg. Geom.* **4** (1995), 487–502.

[6] F. Catanese and A. Di Scala. A characterization of varieties whose universal cover is the polydisk or a tube domain. **Math. Ann. 356** (2013), 419–438.

[7] O. Debarre. Varieties with ample cotangent bundle. *Compos. Math.* **141** (2005), 1445–1459. Corrigendum: *Compos. Math.* **149** (2013), 505–506.

[8] J.-P. Demailly. L^2 vanishing theorems for positive line bundles and adjunction theory. *Transcendental methods in algebraic geometry* (Cetraro, 1994), 1–97. Lecture Notes in Mathematics 1646, Springer (1996).

[9] J.-P. Demailly. Holomorphic Morse inequalities and the Green-Griffiths-Lang conjecture. *Pure Appl. Math. Q.* **7** (2011), 1165–1208.

[10] M. Deschamps. Courbes de genre géométrique borné sur une surface de type général [d'après F. A. Bogomolov]. *Séminaire Bourbaki*, 30e année (1977/78), Exp. No. 519, 233–247. Lecture Notes in Mathematics 710, Springer (1979).

[11] P. Eyssidieux. Sur la convexité holomorphe des revêtements linéaires réductifs d'une variété projective algébrique complexe. *Invent. Math.* **156** (2004), 503–564.

[12] P. Eyssidieux, L. Katzarkov, T. Pantev, and M. Ramachandran. Linear Shafarevich conjecture. *Ann. Math.* **176** (2012), 1545–1581.

[13] S. Goldberg and S. Kobayashi. Holomorphic bisectional curvature. *J. Diff. Geom.* **1** (1967), 225–233.

[14] T. Graber, J. Harris, and J. Starr. Families of rationally connected varieties. *J. Amer. Math. Soc.* **16** (2003), 57–67.

[15] P. Griffiths. Hermitian differential geometry, Chern classes, and positive vector bundles. *Global analysis*, 185–251, Univ. Tokyo Press (1969). Also in *Selected works*, v. 1, Amer. Math. Soc. (2003).

[16] P. Griffiths. Periods of integrals on algebraic manifolds III. *Publ. Math. IHES* **38** (1970), 128–180. Also in *Selected works*, v. 3, Amer. Math. Soc. (2003).

[17] P. Griffiths and W. Schmid. Locally homogeneous complex manifolds. *Acta Math.* **123** (1969), 253–302. Also in *Selected works*, v. 1, Amer. Math. Soc. (2003).

[18] M. Gromov. Kähler hyperbolicity and L^2-Hodge theory. *J. Diff. Geom.* **33** (1991), 263–292.

[19] M. Gromov and R. Schoen. Harmonic maps into singular spaces and p-adic superrigidity for lattices in groups of rank one. *Publ. Math. IHES* **76** (1992), 165–246.

[20] D. Guler. On Segré forms of positive vector bundles. *Canad. Math. Bull.* **55** (2012), 108–113.

[21] K. Jabbusch. Positivity of cotangent bundles. *Michigan Math. J.* **58** (2009), 723–744.

[22] L. Katzarkov. On the Shafarevich maps. *Algebraic geometry* (Santa Cruz, 1995), 173–216. Proc. Symp. Pure Math., 62, Part 2, Amer. Math. Soc. (1997).

[23] B. Klingler. Symmetric differentials, Kähler groups and ball quotients. *Invent. Math.*, to appear.

[24] S. Kobayashi. The first Chern class and holomorphic symmetric tensor fields. *J. Math. Soc. Japan* **32** (1980), 325–329.

[25] S. Kobayashi. *Hyperbolic complex spaces.* Springer (1998).

[26] R. Lazarsfeld. *Positivity in algebraic geometry*, 2 vols. Springer (2004).

[27] Z. Lu. On the geometry of classifying spaces and horizontal slices. *Amer. J. Math.* **121** (1999), 177–198.

[28] G. A. Margulis. *Discrete subgroups of semi-simple Lie groups.* Springer (1991).

[29] N. Mok. *Metric rigidity theorems on Hermitian locally symmetric manifolds.* World Scientific (1989).

[30] C. Peters. Rigidity for variations of Hodge structure and Arakelov-type finiteness theorems. *Compos. Math.* **75** (1990), 113–126.

[31] C. Procesi. Finite dimensional representations of algebras. *Israel J. Math.* **19** (1974), 169–182.

[32] F. Sakai. Symmetric powers of the cotangent bundle and classification of algebraic varieties. *Algebraic geometry* (Copenhagen, 1978), 545–563. Lecture Notes in Mathematics 732, Springer (1979).

[33] M. Schneider. Symmetric differential forms as embedding obstructions and vanishing theorems. *J. Alg. Geom.* **1** (1992), 175–181.

[34] F. Severi. La géométrie algébrique italienne: sa rigueur, ses méthodes, ses problèmes. *Colloque de géométrie algébrique* (Liège, 1949), 9–55. Centre belge de recherches mathématiques. Masson, Paris (1950).

[35] N. Shepherd-Barron. Infinite generation for rings of symmetric tensors. *Math. Res. Lett.* **2** (1995), 125–128.

[36] C. Simpson. Higgs bundles and local systems. *Publ. Math. IHES* **75** (1992), 5–95.

[37] C. Simpson. Moduli of representations of the fundamental group of a smooth projective variety II. *Publ. Math. IHES* **80** (1994), 5–79.

[38] Y.-T. Siu. Some recent results in complex manifold theory related to vanishing theorems for the semi-positive case. *Proceedings of the Bonn Arbeitstagung 1984*, 169–192. Lecture Notes in Mathematics 1111, Springer (1985).

[39] S. Splitthoff. Finite presentability of Steinberg groups and related Chevalley groups. *Applications of algebraic K-theory to algebraic geometry and number theory* (Boulder, Colo., 1983), 635–687. Amer. Math. Soc. (1986).

[40] D. Toledo. Projective varieties with non-residually finite fundamental group. *Publ. Math. IHES* **77** (1993), 103–119.

[41] K. Yamanoi. On fundamental groups of algebraic varieties and value distribution theory. *Ann. Inst. Fourier (Grenoble)* **60** (2010), 551–563.

[42] F. Zheng. *Complex differential geometry.* Amer. Math. Soc. (2000).

[43] K. Zuo. Kodaira dimension and Chern hyperbolicity of the Shafarevich maps for representations of π_1 of compact Kähler manifolds. *J. Reine Angew. Math.* **472** (1996), 139–156.

[44] K. Zuo. On the negativity of kernels of Kodaira-Spencer maps on Hodge bundles and applications. *Asian J. Math.* **4** (2000), 279–302.

Institut de Mathématiques de Jussieu, Paris, France
brunebarbe@math.jussieu.fr

Institut de Mathématiques de Jussieu, Paris, France
klingler@math.jussieu.fr

DPMMS, Wilberforce Road, Cambridge CB3 0WB, England
b.totaro@dpmms.cam.ac.uk