# Hodge structures of type $(n, 0, \ldots, 0, n)$

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Completing earlier work by Albert, Shimura found all the possible endomorphism algebras (tensored with the rationals) for complex abelian varieties of a given dimension [12, Theorem 5]. In five exceptional cases, every abelian variety on which a certain algebra acts has "extra endomorphisms", so that the full endomorphism algebra is bigger than expected.

Complex abelian varieties $X$ up to isogeny are equivalent to polarizable $\mathbf{Q}$-Hodge structures of weight 1, with Hodge numbers $(n, n)$ (where $n$ is the dimension of $X$). In this paper, we generalize Shimura's classification to determine all the possible endomorphism algebras for polarizable $\mathbf{Q}$-Hodge structures with Hodge numbers $(n, 0, \ldots, 0, n)$. For Hodge structures of odd weight, the answer is the same as for abelian varieties. For Hodge structures of even weight, the answer is similar but not identical. The proof combines ideas from Shimura with Green-Griffiths-Kerr's approach to computing Mumford-Tate groups [4, Proposition VI.A.5].

As with abelian varieties, the most interesting feature of the classification is that in certain cases, every Hodge structure on which a given algebra acts must have extra endomorphisms. (Throughout this discussion, we only consider polarizable Hodge structures.) One known case (pointed out to me by Beauville) is that every $\mathbf{Q}$-Hodge structure with Hodge numbers $(1, 0, 1)$ has endomorphisms by an imaginary quadratic field and hence is of complex multiplication (CM) type, meaning that its Mumford-Tate group is commutative. More generally, every $\mathbf{Q}$-Hodge structure with Hodge numbers $(n, 0, n)$ that has endomorphisms by a totally real field $F$ of degree $n$ has endomorphisms by a totally imaginary quadratic extension field of $F$, and hence is of CM type. Another case, which seems to be new, is that a $\mathbf{Q}$-Hodge structure $V$ with Hodge numbers $(2, 0, 2)$ that has endomorphisms by an imaginary quadratic field $F_0$ must have endomorphisms by a quaternion algebra over $\mathbf{Q}$. In this case, $V$ need not be of CM type; there is a period space isomorphic to $\mathbf{CP}^1$ of Hodge structures of this type, whereas there are only countably many Hodge structures of CM type.

To motivate the results of this paper on endomorphism algebras, consider the geometric origin of Hodge structures. A Hodge structure *comes from geometry* if it is a summand of the cohomology of a smooth complex projective variety defined by an algebraic correspondence. Griffiths found ("Griffiths transversality") that a family of Hodge structures coming from geometry can vary only in certain directions, expressed by the notion of a variation of Hodge structures [15, Theorem 10.2]. In particular, any variation of Hodge structures of weight at least 2 with Hodge numbers $(n, 0, \ldots, 0, n)$ (so there is at least one 0) is locally constant; more generally, this holds whenever there are no two adjacent nonzero Hodge numbers. This has the remarkable consequence that only countably many Hodge structures of weight at least 2 with Hodge numbers $(n, 0, \ldots, 0, n)$ come from geometry. Very little is known about this countable subset of the period domain of all Hodge structures.

One way to produce a Hodge structure with Hodge numbers $(n, 0, n)$ is from a smooth complex projective surface $X$ with maximal Picard number, meaning that the Picard number is equal to $h^{1,1}(X)$. (Then $H^2(X, \mathbf{Q})$ modulo the subspace of Hodge classes, or equivalently of divisors, is a Hodge structure with Hodge numbers $(p_g(X), 0, p_g(X))$ that comes from geometry.) A recent survey of surfaces with maximal Picard number is Beauville [2]. Many of these examples give Hodge structures of CM type. For example, since all $\mathbf{Q}$-Hodge structures with Hodge numbers $(1, 0, 1)$ are of CM type, all complex K3 surfaces with Picard number 20 give Hodge structures (with Hodge numbers $(1, 0, 1)$) that are of CM type.

Thus, one might ask whether all Hodge structures with Hodge numbers $(n, 0, n)$ that come from geometry are of CM type. The answer is almost certainly no. Indeed, a classical modular form $f$ (more precisely, a normalized eigenform) of weight $k \geq 2$ and level $N$ determines a motive over $\mathbf{Q}$ with coefficients in the field $E = \mathbf{Q}(f)$ of coefficients of $f$ [11]. This motive has weight $k - 1$ and Hodge numbers $(1, 0, \ldots, 0, 1)$. In particular, a modular form of weight 3 and some level $N$ determines an $E$-Hodge structure with Hodge numbers $(1, 0, 1)$, and hence a $\mathbf{Q}$-Hodge structure with Hodge numbers $(n, 0, n)$, where $n = [E : \mathbf{Q}]$. (Explicitly, this motive occurs in $H^2$ of the elliptic modular surface of level $N$.)

Ribet explained how to check from the coefficients of a modular form whether the associated Galois representation $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to GL(2, E \otimes_{\mathbf{Q}} \mathbf{Q}_l)$ is of CM type, meaning that the image of the representation has an open abelian subgroup [9]. From Stein's tables of modular forms, one can read off many forms which are not of CM type, such as the unique newform of weight 3 and level 9, with $E = \mathbf{Q}(\sqrt{-3})$ [14]. It would follow from the Hodge conjecture, or from the weaker conjecture that every Hodge cycle is absolute Hodge, that the associated $E$-Hodge structure with Hodge numbers $(1, 0, 1)$ is not of CM type. Without conjectures, it is an attractive problem, not addressed here, to show that this Hodge structure (which comes from geometry) is not of CM type. The problem amounts to proving the irrationality of a suitable period of the given modular form.

All this motivates the topic of this paper: the unexpected symmetries of Hodge structures with Hodge numbers $(n, 0, \ldots, 0, n)$. Several examples of "extra" endomorphisms in our classification were suggested by Ribet's analysis of the Galois representation associated to a modular form; in those cases, the extra endomorphisms come from algebraic cycles [9]. In particular, the quaternionic structure on the motive of a form of odd weight comes from the outer automorphism of the group $\Gamma_1(N)$ given by the "$W$-operator" $W = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$. It would be interesting to show that the extra endomorphisms of Hodge structures which we construct come from algebraic cycles for other Hodge structures with Hodge numbers $(n, 0, \ldots, 0, n)$ that come from geometry, as the Hodge conjecture would predict.

The results of this paper have some of the flavor of the Kuga-Satake construction, which shows that all polarizable $\mathbf{Q}$-Hodge structures with Hodge numbers $(1, b, 1)$ are in the tensor category generated by the cohomology of curves (or, equivalently, the cohomology of abelian varieties). But in fact the situation of this paper is very different. Namely, a $\mathbf{Q}$-Hodge structure of weight at least 2 with Hodge numbers $(n, 0, \ldots, 0, n)$ which is not of CM type is not in the tensor category generated by the cohomology of curves (Corollary 4.2).

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# 1 Notation

A $\mathbf{Q}$-*Hodge structure* $V$ is a rational vector space of finite dimension together with a decomposition of $V_{\mathbf{C}} := V \otimes_{\mathbf{Q}} \mathbf{C}$ as a direct sum of complex linear subspaces $V^{a,b}$ for integers $a, b$ such that $\overline{V^{a,b}} = V^{b,a}$ and such that the grading by $a+b$, called the weight grading, is defined over $\mathbf{Q}$. A reference on Hodge structures is Voisin [15, Chapter 7]. A Hodge structure is pure of *weight* $m$ if $V^{a,b} = 0$ for $a+b \neq m$. Hodge structures can also be defined in terms of the *Hodge filtration* $F^j(V_{\mathbf{C}}) = \oplus_{a \geq j, b \in \mathbf{Z}} V^{a,b}$. A smooth complex projective variety $X$ has a Hodge structure of weight $m$ on $H^m(X, \mathbf{Q})$, for any $m$. The *Tate Hodge structure* $\mathbf{Q}(j)$ for an integer $j$ is $V = \mathbf{Q}$ with $V_{\mathbf{C}} = V^{-j,-j}$.

A *polarization* of a $\mathbf{Q}$-Hodge structure $V$ of weight $m$ is a bilinear form $\langle , \rangle \colon V \times V \to \mathbf{Q}$ which is symmetric if $m$ is even, alternating if $m$ is odd, and which satisfies the properties [15, section 7.1.2]:
(i) $\langle V^{a,b}, V^{a',b'} \rangle = 0$ for $a' \neq m - a$;
(ii) $i^{a-b}(-1)^{m(m-1)/2} \langle x, \overline{x} \rangle > 0$ for all nonzero elements $x$ of $V^{a,b}$.
Here we write $\langle , \rangle$ for the complex bilinear form on $V \otimes_{\mathbf{Q}} \mathbf{C}$ associated to the given form on $V$. For example, an ample line bundle on a smooth complex projective variety $X$ determines a polarization of $H^m(X, \mathbf{Q})$ for all $m$. The polarizable $\mathbf{Q}$-Hodge structures form a semisimple abelian category. In this paper, all the $\mathbf{Q}$-Hodge structures we consider will be polarizable, unless stated otherwise.

A $\mathbf{Q}$-Hodge structure $V$ (ignoring the polarization) can also be described as a $\mathbf{Q}$-vector space with a homomorphism of $\mathbf{R}$-groups $R_{\mathbf{C}/\mathbf{R}} G_m \to GL(V_{\mathbf{R}})$ [7, section 1.3]. The *Mumford-Tate group* of a Hodge structure $V$ is the $\mathbf{Q}$-Zariski closure of the image of this homomorphism. The book [4] uses "Mumford-Tate group" for a slightly smaller group which we call the *Hodge group*: the $\mathbf{Q}$-Zariski closure of the circle group $\ker(N \colon R_{\mathbf{C}/\mathbf{R}} G_m \to G_m) \to GL(V_{\mathbf{R}})$ [7, section 1.11]. For example, if $V$ is pure of nonzero weight, then the Mumford-Tate group is the product in $GL(V)$ of the Hodge group with the group $G_m$ of scalars. The Mumford-Tate group of a polarizable Hodge structure $V$ is a connected reductive group over $\mathbf{Q}$; in a sense, it describes the complexity of a Hodge structure.

For a polarized $\mathbf{Q}$-Hodge structure $V$, the endomorphism algebra $L = \mathrm{End}_{\mathbf{Q}\text{-HS}}(V)$ is a semisimple $\mathbf{Q}$-algebra with an involution $f \mapsto \overline{f}$ given by

$$\langle fx, y \rangle = \langle x, \overline{f} y \rangle.$$

This is called the *Rosati involution*. The Rosati involution is *positive* in the sense that $L$ has finite dimension as a $\mathbf{Q}$-vector space and the reduced trace $\mathrm{tr}_{L/\mathbf{Q}}(x\overline{x})$ is positive for all nonzero $x$ in $V$ [7, Remark 1.20]. It follows, for example, that if $L$ is a field, then it must be either totally real or else a CM field (a totally imaginary quadratic extension of a totally real number field), and the involution must be complex conjugation. A convenient reference on algebras with positive involution, in connection with endomorphisms of abelian varieties, is Mumford [8, section 21].

For a central simple algebra $A$ over a field $F_0$ with involution $^-$, define

$$\begin{aligned}\mathrm{Sym}(A,{}^-) &= \{x \in A : \overline{x} = x\}\\ \mathrm{Alt}(A,{}^-) &= \{x \in A : \overline{x} = -x\}.\end{aligned}$$

Let $q$ be the degree of $A$ over $F_0$, meaning that $A$ has dimension $q^2$ as an $F_0$-vector space. Let $F$ be the subfield of $F_0$ fixed by the involution. Following the Book of Involutions, the involution on $A$ is said to be *orthogonal* if $F_0 = F$ and $\dim_F \mathrm{Sym}(A,{}^-) = q(q+1)/2$, *symplectic* if $F_0 = F$ and $\dim_F \mathrm{Sym}(A,{}^-) = q(q-1)/2$, and *unitary* if $F_0 \neq F$ [5, Proposition I.2.6]. These are the only possibilities.

By Albert, every division algebra $L$ with positive involution falls into one of four types [8, section 21]. Type I: $L$ is equal to $F$, a totally real field. Type II: $L$ is a totally indefinite quaternion algebra over a totally real field $F$, with an orthogonal involution. ("Totally indefinite" means that $L$ is split at every real place of $F$.) Type III: $L$ is a totally definite quaternion algebra over a totally real field $F$, with a symplectic involution. Type IV: $L$ is a central simple algebra of degree $q$ over a CM field $F_0$, and the involution on $F_0$ is complex conjugation.

Let $(,)$ be a symmetric bilinear form on a vector space $V$ of dimension $n$ over a field $F$. The *determinant* of the form is

$$\det(V) := \det((e_i, e_j))_{1\leq i,j\leq n} \in F^*/(F^*)^2,$$

for any basis $e_1, \ldots, e_n$ for $V$. The *discriminant* of the form is the signed determinant:

$$\mathrm{disc}(V) = (-1)^{n(n-1)/2} \det(V) \in F^*/(F^*)^2.$$

For a central simple algebra $A$ of even degree $n = 2m$ over a field $F$ with orthogonal involution, the Book of Involutions defines the *determinant* as the reduced norm of any alternating unit, $\det(A,{}^-) = \mathrm{Nrd}_F^A(a) \in F^*/(F^*)^2$ for $a \in \mathrm{Alt}(A,{}^-) \cap A^*$. The *discriminant* is the signed determinant:

$$\mathrm{disc}(A,{}^-) = (-1)^m \det(A,{}^-) \in F^*/(F^*)^2.$$

For a vector space $V$ of even dimension over a field $F$ with a symmetric bilinear form, the discriminant of $V$ is equal to the discriminant of the adjoint involution on $\mathrm{End}_F(V)$.

Let $E$ be a number field. We define an *$E$-Hodge structure* $V$ to be a $\mathbf{Q}$-Hodge structure together with a homomorphism $E \to \mathrm{End}_{\mathbf{Q}\text{-HS}}(V)$ of $\mathbf{Q}$-algebras.

Let $K$ be a number field which is either totally real or a CM field. Write $a \mapsto \overline{a}$ for the involution of $K$ given by complex conjugation, which is the identity if $K$ is totally real. A *polarized $K$-Hodge structure* $V$ means a polarized $\mathbf{Q}$-Hodge structure $V$ together with a homomorphism $K \to \mathrm{End}_{\mathbf{Q}\text{-HS}}(V)$ of $\mathbf{Q}$-algebras with involution. That is, the form $\langle,\rangle \colon V \times V \to \mathbf{Q}$ satisfies

$$\langle ax, y\rangle = \langle x, \overline{a}y\rangle$$

for all $a$ in $K$ and $x, y$ in $V$. There does not seem to be a reasonable notion of a polarized $E$-Hodge structure for a number field $E$ which is not totally real or a CM field, although one could require the underlying $\mathbf{Q}$-Hodge structure to be polarizable.

Let $E$ be a totally real or CM field, and let $V$ be a polarized $E$-Hodge structure of weight $m$. Then there is a unique $(-1)^m$-hermitian form $(,)\colon V \times V \to E$ such that $\langle x, y\rangle = \mathrm{tr}^E_{\mathbf{Q}}(x, y)$. By a $(-1)^m$-hermitian form, we mean that $(ax, y) = a(x, y)$, $(x, ay) = \overline{a}(x, y)$, and $(x, y) = (-1)^m\overline{(y, x)}$ for $x, y \in V$ and $a \in E$; thus $(,)$ is a bilinear form if $E$ is totally real. The existence and uniqueness of $(,)$ follow by observing that for $x, y$ in $V$, $(x, y)$ must be the unique element $u \in E$ such that $\langle ax, y\rangle = \mathrm{tr}^E_{\mathbf{Q}}(au)$ for all $a \in E$. This uniquely determines $u$, because $a, b \mapsto \mathrm{tr}^E_{\mathbf{Q}}(ab)$ is a nondegenerate bilinear form on $E$ as a $\mathbf{Q}$-vector space.

A $\mathbf{Q}$-Hodge structure $V$ is of *CM type* if it is polarizable and its Mumford-Tate group is commutative. In particular, if there is a CM field $K$ such that $V$ is a $K$-Hodge structure and $\dim_K V = 1$, then $V$ is of CM type [4, Proposition V.3]. There are only countably many isomorphism classes of Hodge structures of CM type. They all come from geometry, in fact (up to Tate twists) from the rational cohomology of abelian varieties with complex multiplication, by Serre [6, section 1.7]. More strongly, by Abdulali, every effective Hodge structure of CM type occurs in the cohomology of some abelian variety with complex multiplication, with no Tate twist needed [1].

We say that a $\mathbf{Q}$-Hodge structure $V$ has *Hodge numbers* $(a_0, \ldots, a_m)$ if $V$ has weight $m$, $\dim_{\mathbf{C}} V^{j,m-j} = a_j$ for $0 \leq j \leq m$, and all other subspaces $V^{a,b}$ are zero. Let $V$ be a $\mathbf{Q}$-Hodge structure of weight 2 with Hodge numbers $(n, 0, n)$, the main situation considered in this paper. Then the bilinear form $\langle,\rangle$ on $V$ is positive definite. Conversely, for a $\mathbf{Q}$-vector space $V$ of dimension $2n$ with a positive definite symmetric bilinear form $\langle,\rangle$, a Hodge structure with Hodge numbers $(n, 0, n)$ on $(V, \langle,\rangle)$ is equivalent to an isotropic $\mathbf{C}$-linear subspace $V^{2,0} \subset V \otimes_{\mathbf{Q}} \mathbf{C}$ of dimension $n$. (The positivity property $\langle x, \overline{x}\rangle > 0$ for nonzero $x$ in $V^{2,0}$ is automatic; in fact, $\langle x, \overline{x}\rangle > 0$ for all nonzero $x$ in $V \otimes_{\mathbf{Q}} \mathbf{C}$.) Therefore, the period domain of Hodge structures with Hodge numbers $(n, 0, n)$ on $(V, \langle,\rangle)$ is the isotropic Grassmannian $\mathrm{Gr}_{\mathrm{isot}}(n, 2n)$ over $\mathbf{C}$.

Let $E$ be a number field which is totally real or a CM field, and let $r = [E : \mathbf{Q}]$. Let $V$ be an $E$-Hodge structure. Then $V^{a,b}$ is an $E \otimes_{\mathbf{Q}} \mathbf{C}$-module for each $a, b \in \mathbf{Z}$. The ring $E \otimes_{\mathbf{Q}} \mathbf{C}$ is isomorphic to a product of copies of $\mathbf{C}$, indexed by the embeddings $\sigma_1, \ldots, \sigma_r$ of $E$ into $\mathbf{C}$. Therefore, the complex vector space $V^{a,b}$ splits as a direct sum indexed by the embeddings $\sigma_l$ (the subspace where $E$ acts through its embedding $\sigma_l$ in $\mathbf{C}$). We say that an $E$-Hodge structure $V$ has *Hodge numbers* $(a_0, \ldots, a_n)$ if, for each embedding $\sigma_l\colon E \hookrightarrow \mathbf{C}$, the summand of $V^{j,n-j}$ corresponding to $\sigma_l$ has complex dimension $a_j$. It follows that, as a $\mathbf{Q}$-Hodge structure, $V$ has Hodge numbers $(ra_0, \ldots, ra_n)$.

In general, an $E$-Hodge structure need not have a single set of Hodge numbers $(a_0, \ldots, a_n)$ in this sense. For example, if $X$ is an elliptic curve with complex multiplication by an imaginary quadratic field $K$, then $V = H^1(X, \mathbf{Q})$ can be viewed as a $K$-Hodge structure, of dimension 1 as a $K$-vector space. This $K$-Hodge structure has Hodge numbers $(1, 0)$ under one complex embedding of $K$ and Hodge numbers $(0, 1)$ under the conjugate embedding.

## 2 Polarizations

**Lemma 2.1.** *Let $K$ be a number field which is either totally real or a CM field. Let $V$ be a $K$-Hodge structure such that the underlying $\mathbf{Q}$-Hodge structure is polarizable. Then $V$ is polarizable as a $K$-Hodge structure.*

*Proof.* We can assume that $V$ is pure of some weight $m$. Let $\langle,\rangle$ be a polarization of $V$ as a $\mathbf{Q}$-Hodge structure. We have to produce another polarization $\langle,\rangle_2 \colon V \times V \to \mathbf{Q}$ such that $\langle ax, y\rangle_2 = \langle x, \overline{a}y\rangle_2$ for all $a$ in $K$ and $x, y$ in $V$. Here $\overline{a}$ denotes complex conjugation on $K$, which is the identity if $K$ is totally real.

For $a, b$ in $K$, define $\langle a, b\rangle = \mathrm{tr}^K_{\mathbf{Q}}(a\overline{b})$. This is a positive definite symmetric bilinear form on $K$ as a $\mathbf{Q}$-vector space. In what follows, write $K^*$ for the dual of $K$ as a $\mathbf{Q}$-vector space. Then the form we defined on $K$ gives an identification of $K$ with $K^*$. As a result, the identity map $1_K \in K^* \otimes_{\mathbf{Q}} K$ corresponds to a canonical element $B \in K \otimes_{\mathbf{Q}} K$. We can write $B$ explicitly in terms of a basis $e_1, \ldots, e_r$ for $K$ as a $\mathbf{Q}$-vector space. Let $f_1, \ldots, f_r$ be the dual basis for $K$, meaning that $\mathrm{tr}^K_{\mathbf{Q}}(e_i\overline{f_j}) = \delta_{i,j}$ for all $i, j$. Then $B = \sum_j f_j \otimes e_j$.

We use $B$ to "average" the given polarization on $V$. Explicitly, define

$$\langle x, y\rangle_2 = \sum_{j=1}^{r} \langle f_j x, e_j y\rangle.$$

We want to show that $\langle ux, y\rangle_2 = \langle x, \overline{u}y\rangle_2$ for all $u$ in $K$ and $x, y$ in $K$. That is, we have to show that $\sum_j \langle f_j ux, e_j y\rangle = \sum_j \langle f_j x, e_j \overline{u}y\rangle$. It suffices to show that $\sum_j f_j u \otimes e_j = \sum_j f_j \otimes \overline{u}e_j$ in $K \otimes_{\mathbf{Q}} K$. We can identify $K \otimes_{\mathbf{Q}} K$ with $\mathbf{Q}^{r^2}$ as a $\mathbf{Q}$-vector space by pairing the first variable with $e_j$ and the second variable with $f_k$, for any given $j, k \in \{1, \ldots, r\}$, using the bilinear form on $K$. Thus it suffices to show that

$$\langle e_j, f_k\overline{u}\rangle = \langle \overline{u}e_j, f_k\rangle \in \mathbf{Q}$$

for all $j$ and $k$. This is true, since the left side is $\mathrm{tr}^K_{\mathbf{Q}}(e_j\overline{f_k u})$ and the right side is $\mathrm{tr}^K_{\mathbf{Q}}(\overline{u}e_j\overline{f_k})$.

It remains to check that $\langle,\rangle_2$ is a polarization of $V$ as a $\mathbf{Q}$-Hodge structure, using that $\langle,\rangle$ is a polarization. First, the formula $B = \sum_j f_j \otimes e_j$ for the tensor $B$ above works using any basis for $K$ as a $\mathbf{Q}$-vector space in place of $e_1, \ldots, e_r$ and the dual basis in place of $f_1, \ldots, f_r$. In particular, $B$ can also be written as $B = \sum_j e_j \otimes f_j$. From that it is clear that $\langle,\rangle_2$ is $(-1)^m$-symmetric, since $\langle,\rangle$ is $(-1)^m$-symmetric.

Since the action of $K$ on $V$ sends each subspace $V^{a,b}$ of $V \otimes_{\mathbf{Q}} \mathbf{C}$ into itself, the definition of $\langle,\rangle_2$ shows that we have $\langle V^{a,b}, V^{a',b'}\rangle_2 = 0$ for $a' \neq m - a$. To prove the positivity property of $\langle,\rangle_2$, it is convenient to choose an orthogonal basis $e_1, \ldots, e_r$ for $K$ as a $\mathbf{Q}$-vector space. Then $a_j := \langle e_j, e_j\rangle \in \mathbf{Q}$ is positive, and the dual basis for $K$ is given by $f_j = e_j/a_j$. So

$$\langle x, y\rangle_2 = \sum_{j=1}^{r} \frac{1}{a_j}\langle e_j x, e_j y\rangle$$

for $x, y$ in $V$. That implies the same identity for the associated complex bilinear form $\langle,\rangle_2$ on $V \otimes_{\mathbf{Q}} \mathbf{C}$. It follows that

$$i^{a-b}(-1)^{m(m-1)/2}\langle x, \overline{x}\rangle_2 > 0$$

for all nonzero elements $x$ of $V^{a,b}$, from the corresponding inequality for $\langle , \rangle$. (Note that $\overline{e_j x} = e_j \overline{x}$ for $x$ in $V \otimes_{\mathbf{Q}} \mathbf{C}$, because $e_j \in K$ is a $\mathbf{Q}$-linear endomorphism of $V$.) $\square$

# 3 Endomorphism algebras

In the following theorem and proof, we follow Shimura's notation where possible [12, Theorem 5]. In particular, for a division algebra $L$ and a subfield $K$ of the center of $L$, write $[L : K]$ for the dimension of $L$ as a $K$-vector space.

Since the abelian category of polarizable $\mathbf{Q}$-Hodge structures is semisimple, the endomorphism algebras of all polarizable $\mathbf{Q}$-Hodge structures with Hodge numbers $(n, 0, \ldots, 0, n)$ are determined if we can find the endomorphism algebras of all simple $\mathbf{Q}$-Hodge structures with Hodge numbers of that form (including smaller values of $n$). That is solved by Theorem 3.1.

Let $V$ be a simple polarizable $\mathbf{Q}$-Hodge structure of weight $w \geq 1$ with Hodge numbers $(n, 0, \ldots, 0, n)$. The endomorphism algebra $L$ of $V$ is a division algebra with positive involution. We use Albert's classification of $L$ into Types I through IV (section 1). Let $F_0$ be the center of $L$, which is a CM field or a totally real field, and let $F$ be the subfield of $F_0$ fixed by complex conjugation.

Write $g = [F : \mathbf{Q}]$, $2n = m[L : \mathbf{Q}]$, and $q^2 = [L : F_0]$. For $V$ of Type IV, $L \otimes_{\mathbf{Q}} \mathbf{C}$ is isomorphic to the product of $2g$ copies of $M_q(\mathbf{C})$. Write the simple $L \otimes_{\mathbf{Q}} \mathbf{C}$-modules, each of complex dimension $q$, as $\chi_1, \ldots, \chi_g, \overline{\chi_1}, \ldots, \overline{\chi_g}$. Let $r_\nu$ and $s_\nu$ be the multiplicities of $\chi_\nu$ and $\overline{\chi_\nu}$, respectively, in the representation of $F_0$ on $V^{w,0} \subset V \otimes_{\mathbf{Q}} \mathbf{C}$. Then $r_\nu + s_\nu = mq$ for $\nu = 1, \ldots, g$.

As in Shimura, the proof does something more precise than determining the possible endomorphism algebras. Rather, for each division algebra $L$ with positive involution, we describe the Mumford-Tate domain $D$ of $\mathbf{Q}$-Hodge structures with a given bilinear form and a given action of $L$. For each connected component of $D$, we determine whether a very general $\mathbf{Q}$-Hodge structure in that component has endomorphism algebra equal to $L$ or bigger than $L$. In Type IV, the components of $D$ are indexed by the numbers $r_\nu$ and $s_\nu$ defined above.

**Theorem 3.1.** *Let $V$ be a simple polarizable $\mathbf{Q}$-Hodge structure of weight $w \geq 1$ with Hodge numbers $(n, 0, \ldots, 0, n)$. Let $L$ be the endomorphism algebra of $V$, and define $F_0$ and $F$ as above. Then $[L : \mathbf{Q}]$ divides $2n$ and $[F : \mathbf{Q}]$ divides $n$.*

*Conversely, every division algebra with positive involution satisfying these two bounds is the endomorphism algebra of some simple polarizable $\mathbf{Q}$-Hodge structure of weight $w$ and Hodge numbers $(n, 0, \ldots, 0, n)$, except for five cases when $w$ is odd and seven cases when $w$ is even, as follows.*

*Odd weight exceptional cases:*

*(a) Type III and $m = 1$. Then the $\mathbf{Q}$-Hodge structure $V$ is isomorphic to a direct sum $W^{\oplus 2}$, where $W$ has endomorphisms by a CM quadratic extension $F_0$ of $F$ and $\dim_{F_0}(W) = 1$. In particular, $V$ is of CM type.*

*(b) Type III, $m = 2$, $\mathrm{disc}(B, {}^-) = 1$ in $F^*/(F^*)^2$, where $B$ is the centralizer of $L$ in $\mathrm{End}_F(V)$ and ${}^-$ is its involution, coming from the $L$-invariant symmetric bilinear form $\langle , \rangle$ on $V$. In all but 2 of the $2^g$ connected components of the Mumford-Tate domain of $L$-invariant Hodge structures on $(V, \langle , \rangle)$, a generic $\mathbf{Q}$-Hodge structure $V$ has the "expected" endomorphism algebra $L$. In the other 2 components, a generic*

*$\mathbf{Q}$-Hodge structure $V$ is a direct sum $W^{\oplus 2}$, where $W$ is simple and has endomorphism algebra a Type II quaternion algebra over $F$.*

*(c) Type IV and $\sum_{\nu=1}^{g} r_\nu s_\nu = 0$. Then the $\mathbf{Q}$-Hodge structure $V$ is isomorphic to a direct sum $W^{\oplus mq^2}$ such that $W$ has endomorphisms by the CM field $F_0$ and $dim_{F_0}(W) = 1$. In particular, $V$ is of CM type.*

*(d) Type IV, $m = 2$, $q = 1$, $r_\nu = s_\nu = 1$ for $\nu = 1, \ldots, g$. Then $V$ is generically simple, with endomorphism algebra a Type II quaternion algebra over $F$.*

*(e) Type IV, $m = 1$, $q = 2$, $r_\nu = s_\nu = 1$ for $\nu = 1, \ldots, g$. Then $V$ is isomorphic to the direct sum $W^{\oplus 2}$, where $W$ is generically simple, with endomorphism algebra a Type II quaternion algebra over $F$.*

*Even weight exceptional cases:*

*(a') Type II and $m = 1$. Then the $\mathbf{Q}$-Hodge structure $V$ is isomorphic to a direct sum $V = W^{\oplus 2}$, where $W$ has endomorphisms by a CM quadratic extension $F_0$ of $F$, and $dim_{F_0} W = 1$. In particular, $V$ is of CM type.*

*(b') Type II, $m = 2$, $disc(B, ^-) = 1$ in $F^*/(F^*)^2$, where $B$ is the centralizer of $L$ in $End_F(V)$ and $^-$ is its involution, coming from the $L$-invariant symmetric bilinear form $\langle , \rangle$ on $V$. In all but 2 of the $2^g$ connected components of the Mumford-Tate domain of $L$-invariant Hodge structures on $(V, \langle , \rangle)$, a generic $\mathbf{Q}$-Hodge structure $V$ has the "expected" endomorphism algebra $L$. In the other 2 components, a generic $\mathbf{Q}$-Hodge structure $V$ is a direct sum $W^{\oplus 2}$, where $W$ is simple and has endomorphism algebra a Type III quaternion algebra over $F$.*

*(c') Type IV and $\sum_{\nu=1}^{g} r_\nu s_\nu = 0$. Then $V$ is a direct sum $V = W^{\oplus mq^2}$ for a $\mathbf{Q}$-Hodge structure $W$ with endomorphisms by $F_0$ such that $dim_{F_0}(W) = 1$. So $V$ is of CM type.*

*(d') Type IV, $m = 2$, $q = 1$, $r_\nu = s_\nu = 1$ for $\nu = 1, \ldots, g$. Then $V$ generically has endomorphism algebra a Type III quaternion algebra over $F$.*

*(e') Type IV, $m = 1$, $q = 2$, $r_\nu = s_\nu = 1$ for $\nu = 1, \ldots, g$. Then the $\mathbf{Q}$-Hodge structure $V$ is a direct sum $V = W^{\oplus 2}$, and $W$ generically has endomorphism algebra a Type III quaternion algebra over $F$.*

*(f') Type I and $m = 2$. Then the $\mathbf{Q}$-Hodge structure $V$ has endomorphisms by a CM quadratic extension $F_0$ of $F$. Since $dim_{F_0}(V) = 1$, $V$ is of CM type.*

*(g') Type I, $m = 4$, and $(V, ( , ))$ has discriminant 1 in $F^*/(F^*)^2$. In all but 2 of the $2^g$ connected components of the Mumford-Tate domain of $F$-invariant Hodge structures on $(V, \langle , \rangle)$, a generic $\mathbf{Q}$-Hodge structure $V$ has the "expected" endomorphism algebra $F$. In the other 2 components, a generic $\mathbf{Q}$-Hodge structure $V$ is simple and has endomorphism algebra a Type III quaternion algebra over $F$.*

*Proof.* At first, we consider a more general situation. Let $V$ be any simple polarizable $\mathbf{Q}$-Hodge structure. Fix a polarization $\langle , \rangle \colon V \times V \to \mathbf{Q}$. Then $L := \mathrm{End}_{\mathbf{Q}\text{-HS}}(V)$ is an algebra with positive involution, by section 1. Since $V$ is a vector space over the division algebra $L$, the dimension $[L : \mathbf{Q}]$ divides $\dim_{\mathbf{Q}} V$. When $V$ has Hodge numbers $(n, 0, \ldots, 0, n)$, this proves that $[L : \mathbf{Q}]$ divides $2n$.

Let $F_0$ be the center of $L$, which is a totally real field or a CM field, and let $F$ be the subfield of $F_0$ fixed by complex conjugation (which is also the restriction of the involution on $L$). Let $g = [F : \mathbf{Q}]$. Then $F$ is totally real, and so $F \otimes_{\mathbf{Q}} \mathbf{C}$ is the product of copies of $\mathbf{C}$ indexed by the embeddings $\sigma_1, \ldots, \sigma_g \colon F \hookrightarrow \mathbf{R}$. Each summand $V^{b,c}$ of $V \otimes_{\mathbf{Q}} \mathbf{C}$ is a module over $F \otimes_{\mathbf{Q}} \mathbf{C}$. So $V^{b,c}$ splits as a direct sum of complex linear subspaces on which $F$ acts by $\sigma_1, \ldots, \sigma_g$, respectively.

For any integers $b$ and $c$, $V^{b,c}$ is the complex conjugate of $V^{c,b}$ in $V \otimes_{\mathbf{Q}} \mathbf{C}$. Let $x$ be an an element of $V^{b,c}$ on which $L$ acts by an embedding $\sigma_j$. Since each element $a$ in $F$ acts $\mathbf{Q}$-linearly on $V$, we have $a(\overline{x}) = \overline{ax} = \overline{\sigma_j(a)x} = \sigma_j(a)\overline{x}$ for all $a$ in $F$, where the last equality uses that $\sigma_j(a)$ is real. So each embedding $L \hookrightarrow \mathbf{R}$ occurs with the same multiplicity in $V^{b,c}$ as in $V^{c,b}$. Also, $V$ is a free $F$-module, and so $V \otimes_{\mathbf{Q}} \mathbf{C}$ is a free $F \otimes_{\mathbf{Q}} \mathbf{C}$-module. That is, each embedding $F \hookrightarrow \mathbf{R}$ occurs the same number of times in $V \otimes_{\mathbf{Q}} \mathbf{C}$.

We now make our assumption that $V$ has weight $w$ and Hodge numbers $(n, 0, \ldots, 0, n)$. (To prove the following bound, it would suffice to assume that $V^{w/2,w/2} = 0$.) Then the previous paragraph implies that each embedding $F \hookrightarrow \mathbf{R}$ occurs the same number of times in $V \otimes_{\mathbf{Q}} \mathbf{C} \cong \mathbf{C}^{2n}$, and this number is even. Therefore, $[F : \mathbf{Q}]$ divides $n$, which proves the first part of the theorem.

For any positive integer $w$, the category of $\mathbf{Q}$-Hodge structures of weight 1 and Hodge numbers $(n, n)$ can be identified with the category of $\mathbf{Q}$-Hodge structures of weight $w$ and Hodge numbers $(n, 0, \ldots, 0, n)$, just by renaming $V^{1,0} \subset V \otimes_{\mathbf{Q}} \mathbf{C}$ as $V^{w,0}$. For even weights $w$, this equivalence does not preserve polarizability and hence is of little interest. But for odd weights $w$, this equivalence does preserve polarizability. Therefore, the endomorphism algebras of the simple polarizable $\mathbf{Q}$-Hodge structures of odd weight $w$ and Hodge numbers $(n, 0, \ldots, 0, n)$ are the same as the endomorphism algebras of the simple abelian varieties of dimension $n$. These were determined by Shimura [12, Theorem 5], giving the answer in the theorem.

Our proof in even weight is analogous to Shimura's argument, but we use the language of Mumford-Tate groups so that fewer explicit calculations are required. (The reader could apply the same method to reprove Shimura's classification.)

There is an equivalence of categories between $\mathbf{Q}$-Hodge structures with Hodge numbers $(n, 0, n)$ and $\mathbf{Q}$-Hodge structures of any even weight $w = 2m$ and Hodge numbers $(n, 0, \ldots, 0, n)$, just by renaming $V^{2,0} \subset V \otimes_{\mathbf{Q}} \mathbf{C}$ as $V^{2m,0}$. This equivalence preserves polarizability; we just need to replace a polarization $\langle , \rangle$ on $V$ of weight 2 by $(-1)^m \langle , \rangle$. Therefore, the same endomorphism algebras occur in any even weight.

It would be easy to argue directly with $V$ of any even weight, but we choose to work with $V$ of weight 2 and Hodge numbers $(n, 0, n)$. In this case, the polarization $\langle , \rangle$ of $V$ is a positive definite symmetric bilinear form on the $\mathbf{Q}$-vector space $V$, by section 1. For each positive definite symmetric bilinear form $\langle , \rangle$ on $V$, the space of Hodge structures on $(V, \langle , \rangle)$ with Hodge numbers $(n, 0, n)$ is the space $\mathrm{Gr}_{\mathrm{isot}}(n, 2n)$ of all isotropic $n$-dimensional $\mathbf{C}$-linear subspaces in $V \otimes_{\mathbf{Q}} \mathbf{C} \cong \mathbf{C}^{2n}$.

Let $L$ be a division algebra with positive involution. Let $F_0$ be the center of $L$, and let $F$ be the subfield of $F_0$ fixed by complex conjugation, or equivalently by the involution on $L$. Assume that $[L : \mathbf{Q}]$ divides $2n$ and $[F : \mathbf{Q}]$ divides $n$. Let $V$ be a $\mathbf{Q}$-vector space of dimension $2n$. By the first assumption, we can give $V$ the structure of a left $L$-vector space; choose such an action of $L$ on $V$. Then there is a positive definite symmetric bilinear form $\langle , \rangle \colon V \times V \to \mathbf{Q}$ which is $L$-invariant, meaning that $\langle ux, y \rangle = \langle x, \overline{u}y \rangle$ for all $u$ in $L$ and $x, y$ in $V$. Indeed, $L$ itself has an $L$-invariant positive definite symmetric bilinear form given by $\langle x, y \rangle = \mathrm{tr}^L_{\mathbf{Q}}(x\overline{y})$, and we can view $V$ as the direct sum of copies of $L$.

Fix any positive definite symmetric bilinear form $\langle , \rangle$ on the $\mathbf{Q}$-vector space $V$ which is $L$-invariant. We will show that there is an $L$-invariant $\mathbf{Q}$-Hodge structure on $(V, \langle , \rangle)$ with endomorphism algebra equal to $L$, apart from the exceptions listed in the theorem. This is slightly stronger than the theorem as stated, since we are fixing the action of $L$ and the symmetric bilinear form on $V$.

Write $g = [F : \mathbf{Q}]$, $2n = m[L : \mathbf{Q}]$ where $m$ is a positive integer (which is even for $V$ of Type I by our assumption that $[F : \mathbf{Q}]$ divides $n$), and $q^2 = [L : F_0]$. For $V$ of Type IV, recall the definition of $r_\nu$ and $s_\nu$ for $\nu = 1, \ldots, g$ from before the theorem.

We can describe the "Mumford-Tate domain" of all $L$-invariant Hodge structures on $(V, \langle , \rangle)$. It helps to observe that the ring $L \otimes_{\mathbf{Q}} \mathbf{R}$ is isomorphic to:

$$\begin{aligned} \text{Type I: } & \mathbf{R} \times \cdots \times \mathbf{R} \\ \text{Type II: } & M_2(\mathbf{R}) \times \cdots \times M_2(\mathbf{R}) \\ \text{Type III: } & \mathbf{H} \times \cdots \times \mathbf{H} \\ \text{Type IV: } & M_q(\mathbf{C}) \times \cdots \times M_q(\mathbf{C}), \end{aligned}$$

where there are $g$ factors in each case, and the involution on the right is the identity in Type I, $X \mapsto X^t$ on each copy of $M_2(\mathbf{R})$ in Type II, $X \mapsto \mathrm{tr}^{\mathbf{H}}_{\mathbf{R}}(x) - x$ on each copy of the quaternions $\mathbf{H}$ in Type III, and $X \mapsto \overline{X^t}$ on each copy of $M_q(\mathbf{C})$ in Type IV [8, pp. 201-202]. Recall from section 1 that the period domain of all Hodge structures on $(V, \langle , \rangle)$ is the isotropic Grassmannian $\mathrm{Gr}_{\mathrm{isot}}(n, 2n)$. We deduce that the Mumford-Tate domain $D$ of $L$-invariant Hodge structures on $(V, \langle , \rangle)$ is:

$$\begin{aligned} \text{Type I: } & \mathrm{Gr}_{\mathrm{isot}}(m/2, m)^g \\ \text{Type II: } & \mathrm{Gr}_{\mathrm{isot}}(m, 2m)^g \\ \text{Type III: } & \mathrm{Gr}_{\mathrm{Lag}}(m, 2m)^g \\ \text{Type IV: } & \left[ \coprod_{j=0}^{mq} \mathrm{Gr}(j, mq) \right]^g . \end{aligned}$$

Here $\mathrm{Gr}_{\mathrm{isot}}(a, b)$ is the space of isotropic linear subspaces of dimension $a$ in a complex vector space of dimension $b$ with a nondegenerate symmetric bilinear form, and $\mathrm{Gr}_{\mathrm{Lag}}(m, 2m)$ is the space of isotropic subspaces of dimension $m$ in a complex vector space of dimension $2m$ with a nondegenerate alternating bilinear form. We see that the number of connected components of the Mumford-Tate domain $D$ is $2^g$ in Type I, $2^g$ in Type II, 1 in Type III, and $(mq + 1)^g$ in Type IV.

Let $d$ be the complex dimension of the Mumford-Tate domain. Then:

$$\begin{aligned} \text{Type I: } & n = \frac{m}{2} g,\ d = \frac{1}{2} \frac{m}{2} \left( \frac{m}{2} - 1 \right) g \\ \text{Type II: } & n = 2mg,\ d = \frac{1}{2} m(m-1) g \\ \text{Type III: } & n = 2mg,\ d = \frac{1}{2} m(m+1) g \\ \text{Type IV: } & n = q^2 mg,\ d = \sum_{\nu=1}^{g} r_\nu s_\nu . \end{aligned}$$

Shimura's formula for the dimensions of the analogous Mumford-Tate domains in the period domain of abelian varieties is similar, but with the expressions $x(x+1)/2$ switched with $x(x-1)/2$ [12, section 4.1]. This is related to the switch between symplectic and orthogonal groups, in comparing polarizable Hodge structures of odd weight with those of even weight.

Let $D^0$ be a connected component of the Mumford-Tate domain $D$ of $\mathbf{Q}$-Hodge structures with Hodge numbers $(n, 0, n)$ on $(V, \langle,\rangle)$ with endomorphisms by the given homomorphism $L \to \mathrm{End}_{\mathbf{Q}}(V)$ of algebras with involution. For each larger subalgebra $L'$ of $\mathrm{End}_{\mathbf{Q}}(V)$, the subspace of $D^0$ of Hodge structures with endomorphisms by $L'$ is a closed analytic subspace of $D^0$. Therefore, there is a well-defined algebra with involution $A \subset \mathrm{End}_{\mathbf{Q}}(V)$, the *generic endomorphism algebra* for $D^0$, which is the endomorphism algebra of a very general $\mathbf{Q}$-Hodge structure $V$ in $D^0$. (That is, $A$ is the endomorphism algebra of every Hodge structure in $D^0$ outside countably many closed analytic subspaces not equal to $D^0$.) Clearly $A$ contains $L$. The main part of the theorem is to show that $A$ is equal to $L$ in most cases.

It is also convenient to consider the *generic Hodge group* $M$ of $D^0$, defined as the Hodge group (section 1) of a very general $\mathbf{Q}$-Hodge structure in $D^0$. We know that $A$ is the commutant of $M$ in $\mathrm{End}_{\mathbf{Q}}(V)$. Since $L$ is contained in $A$ and $M$ is a connected $\mathbf{Q}$-group, $M$ is contained in the connected component $H$ of the centralizer of $L$ in the $\mathbf{Q}$-group $O(V)$. We call $H$ the *expected Hodge group*. (The $\mathbf{Q}$-group $H$ depends on $(V, \langle,\rangle, L)$, but not on the particular component $D^0$.)

A crucial observation is that the generic endomorphism algebra $A \subset \mathrm{End}_{\mathbf{Q}}(V)$ and the generic Hodge group $M \subset O(V)$ are determined by $(V, \langle,\rangle, L \to \mathrm{End}_{\mathbf{Q}}(V), D^0)$. Since $H(\mathbf{Q})$ preserves these data, $H(\mathbf{Q})$ normalizes both $A$ and $M$; for $D^0$, this uses that $H$ is connected. Since $H$ is a connected group over the perfect field $\mathbf{Q}$, $H(\mathbf{Q})$ is Zariski dense in $H$ [3, Corollary 18.3], and so $A$ and $M$ are in fact normalized by the algebraic group $H$. Since $M \subset H$, we can say that $M$ is a connected normal $\mathbf{Q}$-subgroup of $H$. Thus, if $H$ is $\mathbf{Q}$-simple, then $M$ must be either 1 or $H$. But $M$ can never be 1; that would mean that the generic Hodge structure $V$ in $D^0$ has $V^{a,b} = 0$ for $a \neq b$, whereas in fact $V^{2,0}$ is not zero. So if $H$ is $\mathbf{Q}$-simple, then the generic Hodge group $M$ is equal to $H$. (This argument is inspired by Green-Griffiths-Kerr's approach to computing generic Mumford-Tate groups, although they exclude the non-connected period domains which we encounter here [4, Proposition VI.A.5].) As a result, when $H$ is $\mathbf{Q}$-simple, we know the generic endomorphism algebra $A$: it is the centralizer of the "known" $\mathbf{Q}$-group $H$ in $\mathrm{End}_{\mathbf{Q}}(V)$. In most cases, that will imply that $A$ is equal to $L$, as we want.

Suppose that $L$ is of Type I, so $L = F$. Then the expected Hodge group $H$ is $R_{F/\mathbf{Q}}SO({}_FV)$, where ${}_FV$ denotes $V$ as an $F$-vector space. Suppose that $m = \dim_F(V)$ (which is even in this case) is at least 6, or that $m = 4$ and $V$ has discriminant not equal to 1 in $F^*/(F^*)^2$. Then $SO({}_FV)$ is $F$-simple (for $m = 4$, use [5, Theorem 15.7 and section 26.B]), and so $H$ is $\mathbf{Q}$-simple. By the argument above, the generic Hodge group $M$ of $\mathbf{Q}$-Hodge structures in each component $D^0$ must be equal to $H$. So the generic endomorphism algebra $A$ is equal to the centralizer of $R_{F/\mathbf{Q}}SO({}_FV)$ in $\mathrm{End}_{\mathbf{Q}}(V)$. Clearly $A$ contains $F$. To show that $A$ is equal to $F$, note that $A \subset \mathrm{End}_{\mathbf{Q}}(V)$ must commute with the Lie algebra of $R_{F/\mathbf{Q}}SO({}_FV)$, which is $\mathfrak{so}({}_FV)$, and so it commutes with the $\mathbf{Q}$-algebra generated by this Lie algebra. The $F$-algebra generated by $\mathfrak{so}({}_FV)$ is equal to $\mathrm{End}_F(V)$, just using that $\dim_F(V)$ is at least 3 (so that ${}_FV$ is an absolutely irreducible representation of $\mathfrak{so}({}_FV)$). So $A \subset \mathrm{End}_{\mathbf{Q}}(V)$ must be contained in the commutant of $\mathrm{End}_F(V)$ in $\mathrm{End}_{\mathbf{Q}}(V)$, which is equal to $F$. Thus we have shown that for $L$ of Type I with $m \geq 6$, or with $m = 4$ and $V$ of discriminant not equal to $1 \in F^*/(F^*)^2$, the generic endomorphism algebra is equal to $L$ $(= F)$. At the same time, we found that the generic Hodge group is equal to $R_{F/\mathbf{Q}}SO({}_FV)$.

For a semisimple algebra $A$ with involution $^-$, define the group of *isometries* to be

$$\mathrm{Iso}(A, ^-) = \{g \in A^* : \overline{g} = g^{-1}\}.$$

Following the Book of Involutions [5], we write

$$\mathrm{Iso}(A, ^-) = \begin{cases} O(A, ^-) & \text{if } ^- \text{ is of orthogonal type,} \\ Sp(A, ^-) & \text{if } ^- \text{ is of symplectic type,} \\ U(A, ^-) & \text{if } ^- \text{ is of unitary type.} \end{cases}$$

For an algebra $A$ with orthogonal involution, with center $F$ and $[A : F] = q^2$, the subgroup $O^+(A, ^-) = \ker(\mathrm{Nrd}\colon O(A, ^-) \to \{\pm 1\})$ (as an algebraic group) is a form of $SO(q)$ over $F$ (meaning that the two groups become isomorphic over an algebraic closure of $F$). For $A$ with symplectic involution, $q$ must be even, and $Sp(A, ^-)$ is a form of the symplectic group $Sp(q)$ over $F$. Finally, if $A$ has a unitary involution over $F_0$ and $[A : F_0] = q^2$, with $F \subset F_0$ the subfield fixed by the involution, then the unitary group $U(A, ^-)$ is a form of $GL(q)$ over $F$.

Next, let $(V, \langle , \rangle, L, D^0)$ be of Type II. Thus $L$ is a totally indefinite quaternion algebra over a totally real field $F$, and $2n = m[L : \mathbf{Q}]$. By definition, the "expected Hodge group" $H$ is the connected component of the centralizer of $L$ in $SO(V_{\mathbf{Q}})$. Thus the Lie algebra of $H$ is the antisymmetric part of the centralizer $B$ of $L$ in $\mathrm{End}_{\mathbf{Q}}(V)$, or equivalently in $\mathrm{End}_F(V)$. Here $B$ is isomorphic to $M_m(L^{\mathrm{op}})$, with an orthogonal involution. So the expected Hodge group $H$ is $R_{F/\mathbf{Q}}O^+(B, ^-)$. Here $O^+(B, ^-)$ is an $F$-form of $SO(2m)$.

Recall from section 1 the discriminant of a central simple algebra $B$ of even degree $n = 2m$ with orthogonal involution.

Suppose that $m \geq 3$, or that $m = 2$ and $\mathrm{disc}(B, ^-) \neq 1 \in F^*/(F^*)^2$. Then $O^+(B, ^-)$ is $F$-simple (for $m = 2$, use [5, Theorem 15.7 and section 26.B]), and so $H$ is $\mathbf{Q}$-simple. By the argument above, the generic Hodge group $M$ is equal to $H$.

So the generic endomorphism algebra $A$ is the centralizer of $R_{F/\mathbf{Q}}O^+(B, ^-)$ in $\mathrm{End}_{\mathbf{Q}}(V)$. Clearly $L$ is contained in $A$. To see that equality holds, note that $A$ commutes with the Lie algebra $\mathfrak{so}(B, ^-) = \mathrm{Alt}(B, ^-) \subset \mathrm{End}_{\mathbf{Q}}(V)$, hence with the algebra generated by $\mathrm{Alt}(B, ^-)$. This algebra is all of $B$, as one can check over an algebraic closure $\overline{F}$ of $F$, using that $\overline{F}^{2m}$ is an irreducible representation of $SO(2m)$ for $m \geq 2$. So $A$ is contained in the commutant of $B \cong M_m(L^{\mathrm{op}})$ in $\mathrm{End}_{\mathbf{Q}}(V)$ or equivalently in $\mathrm{End}_F(V)$, which is equal to $L$. We have shown that in Type II with $m \geq 3$, the generic endomorphism algebra is equal to $L$, as we want.

Next, let $(V, \langle , \rangle, L, D^0)$ be of type III. Thus $L$ is a totally definite quaternion algebra over a totally real field $F$, and $2n = m[L : \mathbf{Q}]$. Let $B$ be the centralizer of $L$ in $\mathrm{End}_{\mathbf{Q}}(V)$ or equivalently in $\mathrm{End}_F(V)$; then $B \cong M_m(L^{\mathrm{op}})$ with a symplectic involution. The "expected Hodge group" $H$ is defined to be the connected component of the centralizer of $L$ in $SO(V_{\mathbf{Q}})$, that is, the connected component of $B \cap SO(V_{\mathbf{Q}})$. So $H = R_{F/\mathbf{Q}}Sp(B, ^-)$. Since $Sp(B, ^-)$ is an $F$-form of the symplectic group $Sp(2m)$, it is absolutely simple for all $m \geq 1$. So $H$ is $\mathbf{Q}$-simple. After that, the argument is the same as in Type II. We conclude that the generic Hodge group is $R_{F/\mathbf{Q}}Sp(B, ^-)$ and the generic endomorphism algebra is $L$, for $L$ of Type III with any $m \geq 1$.

Finally, let $(V, \langle,\rangle, L, D^0)$ be of Type IV. Thus $L$ is a central division algebra over a CM field $F_0$, $L$ has a unitary involution, and $2n = m[L : \mathbf{Q}]$. We write $[L : F_0] = q^2$. Let $B \cong M_m(L^{\text{op}})$ be the centralizer of $L$ in $\text{End}_{\mathbf{Q}}(V)$, or equivalently in $\text{End}_{F_0}(V)$. The "expected Hodge group" $H$ is defined as the connected component of the centralizer of $L$ in $SO(V_{\mathbf{Q}})$, that is, connected component of the identity in $B \cap SO(V_{\mathbf{Q}})$. So $H = R_{F/\mathbf{Q}} U(B, {}^-)$. Since $U(B, {}^-)$ is an $F$-form of $GL(mq)$, Type IV is more subtle, in that $H$ is never $\mathbf{Q}$-simple. It is the product of the $\mathbf{Q}$–simple group $R_{F/\mathbf{Q}} SU(B, {}^-)$ with a torus.

The dimension of the Mumford-Tate domain $D^0$ is $\sum_{\nu=1}^{g} r_\nu s_\nu$. Suppose that this is positive. Since there are only countably many Hodge structures of CM type, a very general Hodge structure $V$ in $D^0$ is not of CM type. So the generic Hodge group is not commutative. Since the generic Hodge group is normal in $R_{F/\mathbf{Q}} U(B, {}^-)$, it must contain $R_{F/\mathbf{Q}} SU(B, {}^-)$, since this is $\mathbf{Q}$-simple for all $m \geq 1$. The generic endomorphism algebra $A$ contains $L$, and is contained in the centralizer of $R_{F/\mathbf{Q}} SU(B, {}^-)$ in $\text{End}_{\mathbf{Q}}(V)$. So $A$ must commute with the Lie algebra $\mathfrak{su}(B, {}^-) = \ker(\text{tr}^B_{F_0} \colon \text{Alt}(B, {}^-) \to F_0) \subset \text{End}_{\mathbf{Q}}(V)$, hence with the algebra generated by $\mathfrak{su}(B, {}^-)$.

Suppose that $mq \geq 3$. Then $\mathfrak{su}(B, {}^-)$ generates $B$ as an algebra [5, Lemma 2.26]. So the generic endomorphism algebra $A$ is contained in the commutant of $B$ in $\text{End}_{\mathbf{Q}}(V)$, or equivalently in $\text{End}_{F_0}(V)$, which is $L$. Since $L$ is contained in $A$, we have shown that the generic endomorphism algebra $A$ is equal to $L$ in Type IV when $\sum_{\nu=1}^{g} r_\nu s_\nu > 0$ and $mq \geq 3$.

We now turn to the remaining cases, which involve Hodge structures of low dimension over the totally real field $F$. For example, case (d'): let $L$ be of Type IV with $m = 2$ and $q = 1$, while $r_\nu = s_\nu = 1$ for $\nu = 1, \ldots, g$. Thus $L$ is a CM field $F_0$, and $V$ has dimension 2 as an $L$-vector space.

The component $D^0$ of the Mumford-Tate domain of $\mathbf{Q}$-Hodge structures on $(V, \langle,\rangle, L)$ given by $r_\nu = s_\nu = 1$ is isomorphic to $\text{Gr}(1,2)^g \cong (\mathbf{CP}^1)^g$. I claim that every Hodge structure in $D^0$ has extra endomorphisms; more precisely, the generic endomorphism algebra for $\mathbf{Q}$-Hodge structures in $D^0$ is of Type III, a totally definite quaternion algebra $L_2$ over the totally real field $F$. Here $[L_2 : \mathbf{Q}] = 2[L : \mathbf{Q}]$ and so $V$ has dimension 1 as an $L_2$-vector space. We showed above that the Type III period domain for $(V, \langle,\rangle, L_2)$ is $\text{Gr}_{\text{Lag}}(1,2)^g \cong (\mathbf{CP}^1)^g$, and that the generic endomorphism algebra for $\mathbf{Q}$-Hodge structures in that domain is equal to $L_2$. So it suffices to show that there is an inclusion $L \subset L_2$ of algebras with involution, compatibly with the actions of $L$ and $L_2$ on $V$. (Then there is an obvious inclusion from the period domain for $L_2$ into the one for $L$, which must be an isomorphism since both domains are isomorphic to $(\mathbf{CP}^1)^g$.)

For convenience, we work at first with the weaker assumption that $m = 2$, $q = 1$, and $\sum_{\nu=1}^{g} r_\nu s_\nu > 0$.

One way to find $L_2$ is that $SU(B, {}^-)$ is a form of $SL(2)$ over $F$, and so it is isomorphic to $SL(1, L_2)$ for a unique quaternion algebra $L_2$ over $F$ [5, Theorem 26.9]. By our earlier discussion of Type IV, the generic Hodge group $M$ for $D^0$ is normal in the "expected Hodge group" $H = R_{F/\mathbf{Q}} U(B, {}^-)$. Since the period domain $D^0$ has dimension $g > 0$, $M$ is not commutative (as there are only countably many $\mathbf{Q}$-Hodge structures of CM type). Since $H$ is the product of the $\mathbf{Q}$-simple group $R_{F/\mathbf{Q}} SU(B, {}^-) = R_{F/\mathbf{Q}} SL(1, L_2)$ with an abelian subgroup, $M$ must contain

$R_{F/\mathbf{Q}}SL(1, L_2)$. We have $M \subset H \subset SO(V_{\mathbf{Q}})$, where $SO(V_{\mathbf{Q}})$ is $\mathbf{R}$-anisotropic (equivalently, its group of real points is compact) because $\langle , \rangle$ is positive definite on $V$. So $M$ is also $\mathbf{R}$-anisotropic, which means that the quaternion algebra $L_2$ over $F$ is totally definite, that is, of Type III.

Therefore, the generic endomorphism algebra $A$ for $D^0$ commutes with $R_{F/\mathbf{Q}}SL(1, L_2)$. So $A$ commutes with the $\mathbf{Q}$-algebra generated by the Lie algebra of $R_{F/\mathbf{Q}}SL(1, L_2)$, which is $\ker(\mathrm{tr}\colon L_2 \to F)$. That algebra is equal to $L_2$.

The homomorphism $SL(1, L_2) \to SU(B, {}^-)$ gives a homomorphism $L_2 \to B \to \mathrm{End}_{\mathbf{Q}}(V)$ of algebras with involution. Since $[L_2 : \mathbf{Q}] = 2[L : \mathbf{Q}]$, $V$ has dimension 1 as an $L_2$-vector space. So the commutant of $L_2$ in $\mathrm{End}_{\mathbf{Q}}(V)$ is isomorphic to $L_2^{\mathrm{op}}$. So $L \subset A \subset L_2^{\mathrm{op}}$. Since $[L_2^{\mathrm{op}} : \mathbf{Q}] = 2[L : \mathbf{Q}]$, the generic endomorphism algebra $A$ for $D^0$ must be either $L$ or $L_2^{\mathrm{op}}$.

We have an inclusion from the Mumford-Tate domain for $(V, \langle , \rangle, L_2^{\mathrm{op}})$ into the one for $(V, \langle , \rangle, L)$. The first is isomorphic to $(\mathbf{CP}^1)^g$ and the second is isomorphic to $D = (\coprod_{r_\nu=0}^{2} \mathrm{Gr}(r_\nu, 2))^g$, as shown earlier. So the Mumford-Tate domain for $L_2^{\mathrm{op}}$ must be equal to the unique component $D^0$ of $D$ of dimension $g$, the one with $r_\nu = s_\nu = 1$ for $\nu = 1, \ldots, g$. We have thus shown that every $\mathbf{Q}$-Hodge structure in that component $D^0$ has extra endomorphisms, with the generic endomorphism algebra being the Type III quaternion algebra $L_2^{\mathrm{op}}$ over $F$. At the same time, the argument shows that when $\sum_\nu r_\nu s_\nu$ is greater than 0 but less than $g$, the generic endomorphism algebra for that component of $D$ is equal to the "expected" algebra $L$.

Next, consider case (e'): let $L$ be of Type IV with $m = 1$ and $q = 2$, while $r_\nu = s_\nu = 1$ for $\nu = 1, \ldots, g$. Thus $L$ is a quaternion algebra over a CM field $F_0$, and $V$ has dimension 1 as an $L$-vector space.

The component $D^0$ of the Mumford-Tate domain of $\mathbf{Q}$-Hodge structures on $(V, \langle , \rangle, L)$ corresponding to $r_\nu = s_\nu = 1$ is isomorphic to $\mathrm{Gr}(1, 2)^g \cong (\mathbf{CP}^1)^g$. I claim that every Hodge structure in this domain has extra endomorphisms. More precisely, every Hodge structure in $D^0$ is non-simple, with $V = W^{\oplus 2}$. The generic endomorphism algebra for $W$ is of Type III, a totally definite quaternion algebra which we call $L_2^{\mathrm{op}}$ over the totally real field $F$. Here $[L_2^{\mathrm{op}} : \mathbf{Q}] = [L : \mathbf{Q}]/2$ and so $W$ has dimension 1 as an $L_2^{\mathrm{op}}$-vector space. We showed above that the Type III period domain for $(W, \langle , \rangle, L_2^{\mathrm{op}})$ is $\mathrm{Gr}_{\mathrm{Lag}}(1, 2)^g \cong (\mathbf{CP}^1)^g$, and that the generic endomorphism algebra for $\mathbf{Q}$-Hodge structures in that domain is equal to $L_2^{\mathrm{op}}$. So it suffices to show that there is an inclusion $L \subset M_2(L_2^{\mathrm{op}})$ of algebras with involution, compatibly with the actions of $L$ and $M_2(L_2^{\mathrm{op}})$ on $V$. (Then there is an obvious inclusion from the period domain for $L_2^{\mathrm{op}}$ into the one for $L$, which must be an isomorphism since both domains are isomorphic to $(\mathbf{CP}^1)^g$.)

For convenience, we work at first with the weaker assumption that $m = 1$, $q = 2$, and $\sum_{\nu=1}^{g} r_\nu s_\nu > 0$.

As in case (d'), the assumptions imply that the generic Hodge group $M$ for the given component $D^0$ contains $R_{F/\mathbf{Q}}SU(B, {}^-)$, and that $SU(B, {}^-)$ is isomorphic to $SL(1, L_2)$ for a unique totally definite quaternion algebra $L_2$ over the totally real field $F$. As in case (d'), the generic endomorphism algebra $A$ for $D^0$ commutes with $R_{F/\mathbf{Q}}SL(1, L_2)$ and hence with the algebra $L_2 \subset \mathrm{End}_{\mathbf{Q}}(V)$.

The homomorphism $SL(1, L_2) \to SU(B, {}^-)$ gives a homomorphism $L_2 \to B \to \mathrm{End}_{\mathbf{Q}}(V)$ of algebras with involution. Since $[L_2 : \mathbf{Q}] = [L : \mathbf{Q}]/2$, $V$ has dimension 2 as an $L_2$-vector space. So the commutant of $L_2$ in $\mathrm{End}_{\mathbf{Q}}(V)$ is isomorphic to

$M_2(L_2^{\rm op})$. So $L \subset A \subset M_2(L_2^{\rm op})$. Since $[M_2(L_2^{\rm op}) : \mathbf{Q}] = 2[L : \mathbf{Q}]$, the generic endomorphism algebra $A$ for $D^0$ must be either $L$ or $M_2(L_2^{\rm op})$.

We have an inclusion from the Mumford-Tate domain for $(W, \langle,\rangle, L_2^{\rm op})$ into the one for $(V, \langle,\rangle, L)$, by sending $W$ to $V := W^{\oplus 2}$. The first is isomorphic to $(\mathbf{CP}^1)^g$ and the second is isomorphic to $D = (\coprod_{r_\nu=0}^2 \mathrm{Gr}(r_\nu, 2))^g$, as shown earlier. So the Mumford-Tate domain for $L_2$ must be equal to the unique component $D^0$ of $D$ of dimension $g$, the one with $r_\nu = s_\nu = 1$ for $\nu = 1, \ldots, g$. We have thus shown that in case (e'), every $\mathbf{Q}$-Hodge structure in that component $D^0$ has extra endomorphisms, with $V$ being a direct sum $V = W^{\oplus 2}$ of $\mathbf{Q}$-Hodge structures and $W$ generically having endomorphism algebra equal to the Type III quaternion algebra $L_2^{\rm op}$ over $F$. At the same time, the argument shows that when $\sum_\nu r_\nu s_\nu$ is greater than 0 but less than $g$, the generic endomorphism algebra for that component of $D$ is equal to the "expected" algebra $L$.

Next, consider case (g'), with $L$ of Type I, $m = 4$, and $\mathrm{disc}(V) = 1 \in F^*/(F^*)^2$. So $L$ is equal to $F$, a totally real field. The Mumford-Tate domain $D$ for $(V, \langle,\rangle, L)$ is $\mathrm{Gr}_{\rm isot}(2, 4)^g \cong (\mathbf{CP}^1 \coprod \mathbf{CP}^1)^g$.

For each component $D^0$ of $D$, the generic Hodge group $M$ is normal in the "expected" Hodge group $H = R_{F/\mathbf{Q}} SO({}_F V)$, as shown earlier. Since ${}_F V$ has discriminant 1 in $F^*/(F^*)^2$, $SO({}_F V)$ is the product of two subgroups, $SL(1, L_2) SL(1, L_2^{\rm op})$, where $L_2$ is a quaternion algebra over $F$ [5, Corollary 15.12]. We know that $M$ is nontrivial, since a Hodge structure $V$ in $D^0$ has $V^{2,0} \neq 0$; so the generic Hodge group $M$ for $D^0$ is either $R_{F/\mathbf{Q}} SL(1, L_2)$, $R_{F/\mathbf{Q}} SL(1, L_2^{\rm op})$, or all of $H$. The generic endomorphism algebra for $D^0$ is the centralizer of the generic Hodge group in $\mathrm{End}_{\mathbf{Q}}(V)$. So the generic endomorphism algebra is $L_2^{\rm op}$, $L_2$, or $F$, respectively. (Here we are thinking of $\mathrm{End}_F(V)$ as $L_2 \otimes_F L_2^{\rm op}$; of course, $L_2^{\rm op}$ is isomorphic to $L_2$, because $L_2$ is a quaternion algebra.)

Since $V$ has dimension 1 as an $L_2$-vector space, the Mumford-Tate domain for $(V, \langle,\rangle, L_2)$ is isomorphic to $(\mathbf{CP}^1)^g$. The generic endomorphism algebra for that domain of Type III is $L_2$. Likewise, the Mumford-Tate domain for $(V, \langle,\rangle, L_2^{\rm op})$ is a different copy of $(\mathbf{CP}^1)^g$ inside $D$. We conclude that the generic endomorphism algebra for these two connected components of $D$ is isomorphic to $L_2$ ($\cong L_2^{\rm op}$), while the generic endomorphism algebra for each of the other $2^g - 2$ components of $D$ is the "expected" algebra $L = F$.

Next, consider case (b'), with $L$ of Type II, $m = 2$, and $\mathrm{disc}(B, {}^-) = 1 \in F^*/(F^*)^2$. So $L$ is a totally indefinite quaternion algebra over the totally real field $F$. Let $S_1, \ldots, S_g$ be the simple modules (of real dimension 2) for the ring $L \otimes_{\mathbf{Q}} \mathbf{R} \cong (M_2(\mathbf{R}))^g$. The Mumford-Tate domain $D$ for $(V, \langle,\rangle, L)$ is $\mathrm{Gr}_{\rm isot}(2, 4)^g \cong (\mathbf{CP}^1 \coprod \mathbf{CP}^1)^g$.

For each component $D^0$ of $D$, the generic Hodge group $M$ is normal in the "expected" Hodge group $H = R_{F/\mathbf{Q}} O^+(B, {}^-)$, as shown earlier. Since $(B, {}^-)$ has discriminant 1 in $F^*/(F^*)^2$, $O^+(B, {}^-)$ is the product of two subgroups, $SL(1, L_2) SL(1, L_3)$, where $L_2$ and $L_3$ are quaternion algebras over $F$ [5, Corollary 15.12]. We know that $M$ is nontrivial, since a Hodge structure $V$ in $D^0$ has $V^{2,0} \neq 0$; so the generic Hodge group $M$ for $D^0$ is either $R_{F/\mathbf{Q}} SL(1, L_2)$, $R_{F/\mathbf{Q}} SL(1, L_3)$, or all of $H$. The generic endomorphism algebra for $D^0$ is the centralizer of the generic Hodge group in $\mathrm{End}_{\mathbf{Q}}(V)$. So the generic endomorphism algebra is $M_2(L_2^{\rm op})$, $M_2(L_3^{\rm op})$, or $L$, respectively. (Here we are thinking of $B$ as $L_2 \otimes_F L_3$, where $L_2$ and $L_3$ both have the canonical symplectic involution. The whole algebra $\mathrm{End}_F(V)$ is the tensor product

$L \otimes_F L_2 \otimes_F L_3$, in this situation.)

Using the algebra $M_2(L_2^{\mathrm{op}}) \subset \mathrm{End}_{\mathbf{Q}}(V)$, we can view the $\mathbf{Q}$-vector space $V$ as a direct sum $V = W_2^{\oplus 2}$. Since $W_2$ has dimension 1 as an $L_2^{\mathrm{op}}$-vector space, the Mumford-Tate domain for $(W_2, \langle,\rangle, L_2^{\mathrm{op}})$ is isomorphic to $(\mathbf{CP}^1)^g$. The generic endomorphism algebra for $W_2$ in that domain of Type III is $L_2^{\mathrm{op}}$, as have shown. Likewise, the inclusion $M_2(L_3^{\mathrm{op}}) \subset \mathrm{End}_{\mathbf{Q}}(V)$ gives a different decomposition $V = W_3^{\oplus 2}$. We can view the Mumford-Tate domain for $(W_3, \langle,\rangle, L_3^{\mathrm{op}})$ as a different copy of $(\mathbf{CP}^1)^g$ inside $D$. We conclude that in case (b'), the generic $\mathbf{Q}$-Hodge structure in each of these two connected components of $D$ is non-simple, of the form $W^{\oplus 2}$ where $W$ has endomorphism algebra $L_2^{\mathrm{op}}$ or $L_3^{\mathrm{op}}$ of Type III, respectively. We also see that the generic endomorphism algebra for each of the other $2^g - 2$ components of $D$ is the "expected" algebra $L$.

It remains to consider the cases where the generic $\mathbf{Q}$-Hodge structure with endomorphisms by $L$ is in fact of CM type. Every component $D^0$ of the Mumford-Tate domain contains a CM point [4, Lemma VI.C.1], and there are only countably many CM points in any Mumford-Tate domain. So the generic Hodge structure in $D^0$ is of CM type if and only if $D^0$ is a point. By the formula for the dimension of $D^0$, these cases are:

(f') Type I, $m = 2$.

(a') Type II, $m = 1$.

(c') Type IV, $\sum_{\nu=1}^g r_\nu s_\nu = 0$.

Let $(V, \langle,\rangle, L)$ be in case (f'). So $L$ is a totally real field $F$ and $\dim_F(V) = 2$. The "expected" Hodge group $H$ as defined earlier is $R_{F/\mathbf{Q}} SO({}_F V)$, which is commutative. Since the Hodge group $M$ of $V$ is a normal subgroup of $H$, we see directly that $M$ is commutative; that is, $V$ is of CM type.

As an $F$-vector space of dimension 2, $V$ has a canonical symmetric bilinear form $(,)$ (Lemma 2.1). This form is positive definite, and so its discriminant (the negative of the determinant, in this case) is totally negative in $F^*/(F^*)^2$. So $F_0 := F(\sqrt{\mathrm{disc}({}_F V)})$ is a totally imaginary quadratic extension of $F$. The "expected" endomorphism algebra of $V$ is the centralizer of $R_{F/\mathbf{Q}} SO({}_F V)$ in $\mathrm{End}_{\mathbf{Q}}(V)$, which is the CM field $F_0$. The actual endomorphism algebra contains $F_0$. Since $\dim_{F_0}(V) = 1$, we have constructed enough endomorphisms to show again that $V$ is of CM type.

Next, let $(V, \langle,\rangle, L)$ be in case (a'): Type II and $m = 1$. So $L$ is a totally indefinite quaternion algebra over a totally real field $F$, and $\dim_L(V) = 1$. Let $B$ be the centralizer of $L$ in $\mathrm{End}_{\mathbf{Q}}(V)$, or equivalently in $\mathrm{End}_F(V)$. The "expected" Hodge group $H$ as defined earlier is $R_{F/\mathbf{Q}} O^+(B, {}^-)$, which is commutative since $O^+(B, {}^-)$ is an $F$-form of $SO(2m) = SO(2)$. So we see directly that $V$ is of CM type.

The discriminant $\mathrm{disc}(B, {}^-)$ in $F^*/(F^*)^2$ is totally negative, using the positivity of the symmetric bilinear form $(,)$ on the 4-dimensional $F$-vector space $V$. So $F_0 := F(\sqrt{\mathrm{disc}(B, {}^-)})$ is a totally imaginary quadratic extension field of $F$. The "expected" endomorphism algebra as defined earlier is the centralizer of $R_{F/\mathbf{Q}} O^+(B, {}^-)$, which is the matrix algebra $M_2(F_0)$. The actual endomorphism algebra of the $\mathbf{Q}$-Hodge structure $V$ contains $M_2(F_0)$. We conclude that the $\mathbf{Q}$-Hodge structure $V$ is isomorphic to a direct sum $V = W^{\oplus 2}$, where $W$ has endomorphisms by the CM field $F_0$, and $\dim_{F_0} W = 1$. We have constructed enough endomorphisms to see again that $V$ is of CM type.

Finally, let $(V, \langle,\rangle, L)$ be in case (c'): Type IV with $\sum_{\nu=1}^g r_\nu s_\nu = 0$. So $L$ is a central simple algebra with unitary involution over a CM field $F_0$. We write $2n = m[L : \mathbf{Q}]$ and $[L : F_0] = q^2$. Let $B \cong M_m(L^{\mathrm{op}})$ be the centralizer of $L$ in $\mathrm{End}_{\mathbf{Q}}(V)$, or equivalently in $\mathrm{End}_{F_0}(V)$. The "expected Hodge group" $H$ as defined earlier is $H = R_{F/\mathbf{Q}}U(B, ^-)$; here $U(B, ^-)$ is an $F$-form of $GL(mq)$. The Hodge group $M$ of $V$ is a normal connected $\mathbf{Q}$-subgroup of $H$, and we also know that $M$ is commutative because the given component $D^0$ of the Mumford-Tate domain has dimension zero. So $M$ is a subgroup of the center of $H$, which is the torus $R_{F/\mathbf{Q}}T$, where $T$ is the 1-dimensional torus $\ker(N\colon R_{F_0/F}G_m \to G_m)$ over $F$.

So the endomorphism algebra of the $\mathbf{Q}$-Hodge structure $V$ contains the centralizer of $R_{F/\mathbf{Q}}T$ in $\mathrm{End}_{\mathbf{Q}}(V)$, which is the matrix algebra $\mathrm{End}_{F_0}(V) \cong M_{mq^2}(F_0)$. Therefore, the $\mathbf{Q}$-Hodge structure $V$ is a direct sum $V = W^{\oplus mq^2}$ for a $\mathbf{Q}$-Hodge structure $W$ with endomorphisms by $F_0$ such that $\dim_{F_0}(W) = 1$. Thus we have constructed enough endomorphisms to see again that $V$ is of CM type. □

*Remark* 3.2. As an addendum to Theorem 3.1, we can say when a CM Hodge structure has more than the expected endomorphism algebra. For CM abelian varieties, this was worked out by Shimura [13, Proposition 26]. Namely, let $X$ be a complex abelian variety of dimension $g$ with a homomorphism $F_0 \to \mathrm{End}(X)_{\mathbf{Q}}$ for a CM field $F_0$ of degree $2g$ over $\mathbf{Q}$. The isogeny type of $X$ is described by a *CM type* on $F_0$, meaning a set $\Phi$ of $g$ complex embeddings of $F_0$ such that every complex embedding is in $\Phi \cup \overline{\Phi}$. Then $\mathrm{End}(X)_{\mathbf{Q}}$ is strictly larger than $F_0$ if and only if there is a strictly smaller CM subfield $K_0$ of $F_0$, with subfield $K$ fixed by complex conjugation, such that any two elements of $\Sigma$ which agree on $K$ also agree on $K_0$. When this happens, $X$ is isogenous to a power of the CM abelian variety with endomorphisms by $K_0$ and CM type $\Sigma|_{K_0}$.

Essentially the same statement holds for CM Hodge structures of any weight $w \geq 1$ with Hodge numbers $(g, 0, \ldots, 0, g)$. Namely, the CM Hodge structures with Hodge numbers $(g, 0, \ldots, 0, g)$ and with a homomorphism $F_0 \to \mathrm{End}_{\mathbf{Q}\text{-HS}}(V)$ for a CM field $F_0$ of degree $2g$ are classified by CM types on $F_0$, just as in weight 1. (In particular, these Hodge structures are all polarizable.) It follows that the equivalence of categories from Hodge structures with Hodge numbers $(g, g)$ to Hodge structures with Hodge numbers $(g, 0, \ldots, 0, g)$ given by renaming $V^{1,0} \subset V \otimes_{\mathbf{Q}} \mathbf{C}$ as $V^{w,0}$ restricts to an equivalence from the CM Hodge structures of weight 1 to those of weight $w$. Therefore, we have the same criterion as in the previous paragraph for when a CM Hodge structure with Hodge numbers $(g, 0, \ldots, 0, g)$ has more than the expected endomorphism algebra.

# 4 Hodge structures not generated by curves

In this section, we show that the $\mathbf{Q}$-Hodge structures considered in this paper, those of weight at least 2 with Hodge numbers $(n, 0, \ldots, 0, n)$, are not in the tensor category generated by curves (or equivalently by abelian varieties), except when they are of CM type.

Define the tensor category of $\mathbf{Q}$-Hodge structures *generated by curves* to be the subcategory of Hodge structures generated by $H^1$ of smooth complex projective curves together with the Hodge structure $\mathbf{Q}(j)$ for integers $j$ by taking direct sums, tensor products, and direct summands. This can also be described as the tensor category generated by abelian varieties. Every Hodge structure of CM type belongs to the tensor category generated by curves. The Kuga-Satake construction shows that every polarizable $\mathbf{Q}$-Hodge structure with Hodge numbers $(1, b, 1)$ is in the tensor category generated by curves [16].

We use the following result of Deligne's [10, Lemma 5]. We say that a Hodge structure $V$ has *type* $\{(a_1, b_1), \ldots, (a_m, b_m)\}$ if $V_{\mathbf{C}} = \oplus_j V^{a_j, b_j}$. Thus we do not specify the actual Hodge numbers.

**Theorem 4.1.** *Let $V$ be a $\mathbf{Q}$-Hodge structure which belongs to the tensor category generated by curves. Then the Hodge structure on the Lie algebra of the Mumford-Tate group of $V$ is of type $\{(-1, 1), (0, 0), (1, -1)\}$.*

**Corollary 4.2.** *Let $V$ be a $\mathbf{Q}$-Hodge structure of weight $w \geq 2$ with Hodge numbers $(n, 0, \ldots, 0, n)$. If $V$ is not of CM type, then it is not in the tensor category generated by curves.*

*Proof.* The Hodge structure on $\mathrm{End}_{\mathbf{Q}}(V)$ is of type $\{(-w, w), (0, 0), (w, -w)\}$. The Lie algebra $\mathfrak{mt}$ of the Mumford-Tate group $\mathrm{MT}(V)$ is a sub-Hodge structure of $\mathrm{End}_{\mathbf{Q}}(V)$. If the Hodge structure $V$ is in the tensor category generated by curves, then in particular it is polarizable. Also, by Theorem 4.1, $\mathfrak{mt}$ is of type $\{(-1, 1),$ $(0, 0), (1, -1)\}$, and so $\mathfrak{mt}$ must be of type $\{(0, 0)\}$. Equivalently, the homomorphism $R_{\mathbf{C}/\mathbf{R}} G_m \to \mathrm{MT}(V)_{\mathbf{R}} \to GL(\mathfrak{mt})$ that describes the Hodge structure on $\mathfrak{mt}$ is trivial. Since $R_{\mathbf{C}/\mathbf{R}} G_m$ is Zariski dense in the Mumford-Tate group as a $\mathbf{Q}$-group, it follows that the conjugation homomorphism $\mathrm{MT}(V) \to GL(\mathfrak{mt})$ is trivial. That is, the connected $\mathbf{Q}$-group $\mathrm{MT}(V)$ is commutative. So $V$ is of CM type. $\square$

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