Topology of Singular Algebraic Varieties

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Abstract

I will discuss recent progress by many people in the program of extending natural topological invariants from manifolds to singular spaces. Intersection homology theory and mixed Hodge theory are model examples of such invariants. The past 20 years have seen a series of new invariants, partly inspired by string theory, such as motivic integration and the elliptic genus of a singular variety. These theories are not defined in a topological way, but there are intriguing hints of their topological significance.

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1. Introduction

The most useful fact about singular complex algebraic varieties is Hironaka's theorem that there is always a resolution of singularities [20]. It has long been clear that the non-uniqueness of resolutions poses a difficulty in many applications. Many different methods have been used to get around this difficulty so as to define invariants of singular varieties. One approach is to try to describe the relation between any two resolutions, leading to ideas such as cubical hyperresolutions [18] and the weak factorization theorem ([1], [31]). Another idea, coming from minimal model theory, is to insist on the special importance of crepant resolutions, and more generally to emphasize the role of the canonical bundle. Recently the interplay between these two approaches has been very successful, as I will describe.

The recent methods tend to be more roundabout than the direct topological definition of intersection homology groups. It is tempting to try to define suitable generalizations of intersection homology groups in order to "explain" various results below (3.2, 3.4, 4.1, 5.2).

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2.   The weight filtration

Deligne discovered a remarkable structure on the rational cohomology of any complex algebraic variety, not necessarily smooth or compact: the weight filtration [9]. This filtration expresses the way in which the cohomology of any variety is related to the cohomology of smooth compact varieties. It is a deep fact that the resulting filtration is well-defined. For example, an immediate consequence of the well-definedness of the weight filtration on cohomology with compact support is the following fact, originally conjectured by Serre ([11], [6], [12], p. 92).

Theorem 2.1. *For any complex algebraic variety X, not necessarily smooth or compact, one can define "virtual Betti numbers" $a_i X \in \mathbf{Z}$ for $i \geq 0$ such that*

(1) if X is smooth and compact, then the numbers $a_i X$ are the Betti numbers $b_i X = \dim_{\mathbf{Q}} H^i(X, \mathbf{Q})$;

(2) for any Zariski-closed subset $Y \subset X$, $a_i X = a_i Y + a_i(X - Y)$.

Using resolution of singularities, it is clear that the numbers $a_i X$ are uniquely characterized by these properties. What is less clear is the existence of such numbers. It follows, for example, that if two smooth compact varieties X and Y can be written as finite disjoint unions of locally closed subsets, $X = \coprod X_i$ and $Y = \coprod Y_i$, with isomorphisms $X_i \cong Y_i$ for all i, then X and Y have the same Betti numbers. This is a topological property of algebraic varieties which has no obvious analogue in a purely topological context.

The existence of the weight filtration, and consequently of the virtual Betti numbers $a_i X$, was originally suggested by Grothendieck's approach to the Weil conjectures on counting rational points on varieties over finite fields. Indeed, the number of $\mathbf{F}_q$-points of a variety clearly has an additive property analogous to property (2) above. One proof of the existence of the weight filtration for complex varieties reduces the problem to the full Weil conjecture for varieties over finite fields, proved by Deligne [8]. Around the same time, Deligne gave a more direct proof of the existence of the weight filtration for complex varieties, using Hodge theory [7]. This is a classic example of the philosophy that the deepest properties of algebraic varieties can often be proved using either number theory or analysis, while they have no "purely geometric" proof.

In 1995, however, Gillet and Soulé gave a new proof of the existence of the weight filtration [13]. They used "only" resolution of singularities and algebraic K-theory, specifically the Gersten resolution. As a result of their more geometric proof, they were able to define the weight filtration on the integral cohomology or $\mathbf{F}_l$-cohomology of a complex algebraic variety, not only on rational cohomology.

To understand what this means, let me describe the weight filtration for a smooth complex variety U, not necessarily compact. Using resolution of singularities, we can write U as the complement of a divisor with normal crossings D in some smooth compact variety X. For $i \geq 0$, let $X^{(i)}$ be the disjoint union of the i-fold intersections of divisors. Then there is a spectral sequence

$$E_1 = H^j(X^{(i)}, k) \Rightarrow H_c^{i+j}(U, k)$$

for any coefficient ring k. The weight filtration on the compactly supported cohomology of U is defined as the filtration associated to this spectral sequence. Gillet

and Soulé show that for any coefficient ring k, this filtration is an invariant of U, independent of the choice of compactification U. This is not at all clear from the known invariance of this filtration for $k = \mathbf{Q}$.

In fact, Gillet and Soulé proved more: for any coefficient ring k, the spectral sequence is an invariant of U from the E_2 term on. For $k = \mathbf{Q}$, the spectral sequence degenerates at E_2, but this is not true with coefficients in $\mathbf{Z}$ or $\mathbf{F}_l$. As a result, for general coefficients k, the groups in the E_2 term are interesting new invariants of U which are not simply the associated graded groups to the weight filtration. They satisfy Mayer-Vietoris sequences, and so can be considered as a cohomology theory on algebraic varieties.

I can now explain a new application of the geometric proof that the weight filtration is well-defined. Namely, one can try to define the weight filtration not only for algebraic varieties. The point is that resolution of singularities holds more generally, for complex analytic spaces, and even for real analytic spaces. Gillet and Soulé's construction of the weight filtration uses algebraic K-theory as well as resolution of singularities, and it is not clear how to adapt the argument to an analytic setting. But Guillen and Navarro Aznar improved Gillet and Soulé's argument so as to construct the weight filtration using only resolution of singularities [17]. The details of their argument use their idea of "cubical hyperresolutions" [18].

Using the method of Guillen and Navarro Aznar, I have been able to define the weight filtration for complex and real analytic spaces. In more detail, let us define a compactification of a complex analytic space X to be a compact complex analytic space $\overline{X}$ containing X as the complement of a closed analytic subset. Of course, not every complex analytic space has a compactification in this sense. We say that two compactifications of X are equivalent if there is a third which lies over both of them.

Theorem 2.2. *Let k be any commutative ring. Then the compactly supported cohomology $H_c^*(X, k)$ has a well-defined weight filtration for every complex analytic space X with an equivalence class of compactifications.*

Any algebraic variety comes with a natural equivalence class of compactifications, but in the analytic setting this has to be considered as an extra piece of structure. On the other hand, the theorem says that the weight filtration is well-defined on all compact complex analytic spaces, with no extra structure needed.

For real analytic spaces, one has the difficulty that there is no natural orientation, unlike the complex analytic situation. This is not a problem if one uses $\mathbf{F}_2$-coefficients, and therefore one can prove:

Theorem 2.3. *For every real analytic space X with an equivalence class of compactifications, the compactly supported cohomology of the space $X(\mathbf{R})$ of real points with $\mathbf{F}_2$ coefficients has a well-defined weight filtration.*

In particular, one can define virtual Betti numbers $a_i X$ for a real analytic space X with an equivalence class of compactifications, the integers $a_i X$ being the usual $\mathbf{F}_2$-Betti numbers in the case of a closed real analytic manifold.

Example. Let X be the compact real analytic space obtained by identifying two copies at the circle at a point, and let Y be the compact real analytic space obtained by identifying two points on a single circle (the figure eight). It is imme-

diate to compute that $a_0 X = 1$ and $a_1 X = 2$, whereas $a_0 Y = 0$ and $a_1 = Y$. The interesting point here is that the spaces $X(\mathbf{R})$ and $Y(\mathbf{R})$ of real points are homeomorphic. Thus the numbers a_i for a compact real analytic space are not topological invariants of the space of real points. In a similar vein, Steenbrink showed that the weight filtration on the rational cohomology of complex algebraic varieties is not a topological invariant, using 3-folds [27].

Nonetheless, it seems fair to say that extending the weight filtration and the virtual Betti numbers to complex and real analytic spaces helps to bring out more of the topological meaning of these invariants of algebraic varieties. A real analytic space has in some ways a weak structure; for example, the classification of closed real analytic manifolds up to isomorphism is the same as the classification of closed differentiable manifolds up to diffeomorphism. From this point of view, it is surprising that compactified real analytic spaces have the extra structure of the weight filtration on their $\mathbf{F}_2$-cohomology. It seems natural to ask for an $\mathbf{F}_2$-linear abelian category of "mixed motives" associated to compactified real analytic spaces X, such that the $\mathbf{F}_2$-cohomology groups of X with their weight filtration are determined by the mixed motive of X. On Beilinson's conjectured abelian category of mixed motives in algebraic geometry, see for example Jannsen [21], 11.3, and [22]; on various approximations to this category, see the triangulated categories defined by Hanamura [19], Levine [26], and Voevodsky [29], and the abelian category defined by Nori.

It should be much easier to define mixed motives for real analytic spaces than to do so for algebraic varieties. In particular, one might speculate that the mixed motive of a real analytic space should not involve much more information than the weight spectral sequence converging to its $\mathbf{F}_2$-cohomology (starting at E_2), perhaps considered together with an action of the Steenrod algebra. In low dimensions, one could hope for precise classifications of mixed motives along these lines.

3.   Stringy Betti numbers

The following result of Batyrev's [4] is related to his famous result that two birational Calabi-Yau manifolds have the same Betti numbers. The proof uses Kontsevich's idea of motivic integration [24], as developed by Denef and Loeser [10]. To be precise, Batyrev's statement involves Hodge numbers, but I will only state what it gives about Betti numbers.

Theorem 3.1. *Let Y be a complex projective variety with log-terminal singularities. Then one can define the "stringy Poincaré function" $p_{str}(Y)$, which is a rational function, such that for any crepant resolution of singularities $\pi : X \to Y$, the stringy Poincaré function of Y is the usual Poincaré polynomial of X.*

We recall Reid's important definitions which are used here. First, let Y be any normal complex variety such that the canonical divisor K_Y is $\mathbf{Q}$-Cartier. By Hironaka, Y has a resolution of singularities $\pi : X \to Y$ such that the exceptional divisors E_i, $i \in I$, are smooth with normal crossings. The discrepancies a_i of E_i are defined by

$$K_X = \pi^* K_Y + \sum a_i E_i.$$

The variety Y is defined to have log-terminal singularities if and only if $a_i > -1$ for all i. A resolution $X \to Y$ is said to be crepant if $K_X = \pi^* K_Y$.

Batyrev defines the stringy Poincaré function of Y by the formula:

$$p_{\mathrm{str}}(Y) = \sum_{J \subset I} p(E_J^0) \prod_{j \in J} \frac{q-1}{q^{a_j+1}-1}.$$

Here E_J^0 is the open stratum of $E_J := \cap_{j \in J} E_j$, and $p(E_J^0)$ denotes the virtual Poincaré polynomial of E_J^0, written as a polynomial in $q^{1/2}$. Thus $p_{\mathrm{str}}(Y)$ is a rational function in $q^{1/2}$ for Y Gorenstein, and in $q^{1/n}$ for some n in general.

Batyrev's proof that the stringy Poincaré function of Y is independent of the choice of resolution, using motivic integration, rests on the additivity properties of the virtual Poincaré polynomial. Using our extension of virtual Betti numbers to complex analytic spaces, we find:

Theorem 3.2. *The stringy Poincaré function can be defined as a rational function for any compactified complex analytic space with log-terminal singularities. For any crepant resolution $X \to Y$ with Y compact, the stringy Poincaré function of Y is the usual Poincaré polynomial of X.*

Likewise for real analytic spaces:

Theorem 3.3. *An $\mathbf{F}_2$-analogue of the stringy Poincaré function can be defined as a rational function for compactified real analytic spaces with log-terminal singularities. For any crepant resolution $X \to Y$ with Y compact, the stringy Poincaré function of Y is the usual Poincaré polynomial of the $\mathbf{F}_2$-cohomology of X.*

In particular, this answers part of Goresky and MacPherson's Problem 7 in [15]:

Corollary 3.4. *Given a compact real algebraic variety Y, the $\mathbf{F}_2$-Betti numbers of any two projective IH-small resolutions of Y are the same.*

This uses the relation between IH-small resolutions and crepant resolutions, which I worked out in [28] using results of Kawamata [23] and Wisniewski [30]. In the complex situation, the corollary (for Betti numbers with any coefficients) has a more direct proof, since the Betti numbers of any small resolution of Y are equal to the dimensions of the intersection homology groups of Y. It is not yet known whether one can define a new version of intersection homology groups with $\mathbf{F}_2$-coefficients which would be self-dual for all compact real analytic spaces. A possible framework for defining such a theory has been set up by Banagl [2].

4. The elliptic genus of a singular variety

I found that any characteristic number which can be extended from smooth compact complex varieties to singular varieties, compatibly with small resolutions, must be a specialization of the elliptic genus [28]. It was then an important problem to define the elliptic genus for singular varieties. This was solved in a completely satisfying way by Borisov and Libgober [5]:

Theorem 4.1. *Let Y be a projective variety with log-terminal singularities. Then one can define the elliptic genus of Y, $\varphi(Y)$, such that for any crepant resolution $X \to Y$, we have $\varphi(Y) = \varphi(X)$.*

Here is Borisov and Libgober's definition of $\varphi(Y)$. Let $\pi : X \to Y$ be a resolution whose exceptional divisors E_k have simple normal crossings, and let a_k be the discrepancies as in section 3. Formally, let y_l denote the Chern roots of X so that $c(TX) = \prod_l (1 + y_l)$, and let e_k be the cohomology classes on X of the divisors E_k. Then $\varphi(Y)$ is the analytic function of variables z and τ defined by

$$\varphi(Y) = \int_Y (\prod_l \frac{(\frac{y_l}{2\pi i})\theta(\frac{y_l}{2\pi i} - z)\theta'(0)}{\theta(-z)\theta(\frac{y_l}{2\pi i})}) \times (\prod_k \frac{\theta(\frac{e_k}{2\pi i} - (\alpha_k + 1)z)\theta(-z)}{\theta(\frac{e_k}{2\pi i} - z)\theta(-(\alpha_k + 1)z)})$$

where $\theta(z, \tau)$ is the Jacobi theta function. The proof that $\varphi(Y)$ is independent of the choice of resolution for log-terminal Y uses the weak factorization theorem of Abramovich, Karu, Matsuki, and Wlodarczyk ([1], [31]).

In the spirit of earlier sections, the singular elliptic genus extends to compact complex analytic spaces with log-terminal singularities. But it remains a mystery how to define the elliptic genus for some topologically defined class of singular spaces that would include singular analytic spaces with log-terminal singularities.

5. Possible characteristic numbers for real analytic spaces

In my paper [28], in trying to define characteristic numbers for singular complex varieties, it was very helpful to require that these numbers are compatible with IH-small resolutions, as Goresky and MacPherson had suggested ([15], Problem 10). The problem thereby becomes more precise: it may be possible to show that some characteristic numbers extend to singular varieties and some do not. This can help to suggest valuable invariants for singular varieties, such as Borisov and Libgober's elliptic genus for singular varieties, even if one is not a priori interested in IH-small resolutions. (The same comments apply to crepant resolutions.)

With this in mind, we here begin to analyze which characteristic numbers can be defined for real analytic spaces, or for topological spaces with similar singularities, compatibly with IH-small resolutions. In the complex situation, the fundamental example of a singularity with two different IH-small resolutions is the 3-fold node; one says that the two IH-small resolutions are related by the simplest type of "flop." Likewise, in the real situation, the real 3-fold node has two different IH-small resolutions. For convenience, let us say that two closed n-manifolds are related by a "real flop" if they are the two different IH-small resolutions X_1 and X_2 of a singular space with singular set of real codimension 3 that is locally isomorphic to the product of the 3-fold node with an $(n-3)$-manifold.

Let us first consider characteristic numbers for unoriented spaces. By Thom, the bordism ring MO_* for unoriented manifolds is detected by Stiefel-Whitney numbers. Therefore we can ask which Stiefel-Whitney numbers (meaning $\mathbf{F}_2$-linear combinations of Stiefel-Whitney monomials) are unchanged under real flops. Or, more or less equivalently: what is the quotient of the bordism ring MO_* by the ideal of real flops $X_1 - X_2$, for X_1 and X_2 as above? There is a good answer:

Theorem 5.1. *The $\mathbf{F}_2$-vector space of Stiefel-Whitney numbers which are invariant under real flops of n-manifolds is spanned by the numbers $w_1^i w_{n-i}$ for $0 \leq i \leq n$, or equivalently by the numbers $w_1^{n-2i} v_i^2$ for $0 \leq i \leq n/2$, modulo those Stiefel-Whitney numbers which vanish for all n-manifolds. Here $v_i = v_i(w_1, w_2, \dots)$ denotes the Wu class. The dimension of this space of invariant Stiefel-Whitney numbers, modulo those which vanish for all n-manifolds, is 0 for n odd and $\lfloor n/2 \rfloor + 1$ for n even. The quotient ring of MO_* by the ideal of real flops is isomorphic to:*

$$\mathbf{F}_2[\mathbf{RP}^2, \mathbf{RP}^4, \mathbf{RP}^8, \dots]/((\mathbf{RP}^{2^a})^2 = (\mathbf{RP}^2)^{2^a} \ for \ all \ a \geq 2).$$

This class of Stiefel-Whitney numbers has occurred before, in Goresky and Pardon's calculation of the bordism ring of locally orientable $\mathbf{F}_2$-Witt spaces [16]. To be precise, the latter ring coincides with the above ring in even dimensions but is also nonzero in odd dimensions. Goresky defined a Wu class v_i in intersection homology for $\mathbf{F}_2$-Witt spaces [14], so that the square v_i^2 lives in ordinary homology, and the characteristic numbers for locally orientable $\mathbf{F}_2$-Witt spaces Y are obtained by multiplying these homology classes by powers of the cohomology class w_1.

This does not explain the invariance of these Stiefel-Whitney numbers for real flops, however. The problem is that the 3-fold node is not an $\mathbf{F}_2$-Witt space. (Topologically, it is the cone over $S^1 \times S^1$, whereas the cone over an even-dimensional manifold is a Witt space if and only if the homology in the middle dimension is zero.) That is, the standard definition of intersection homology is not self-dual on a space with 3-fold node singularities. This again points to the problem of defining a new version of intersection homology with $\mathbf{F}_2$ coefficients which is self-dual on real analytic spaces. That should yield an L-class in the $\mathbf{F}_2$-homology of such a space, which we can also identify with the square of the Wu class, and which therefore should allow the above characteristic numbers to be defined for a large class of real analytic spaces. There are related results by Banagl [3], for spaces which admit an extra "Lagrangian" structure.

We now ask the analogous question for oriented singular spaces: what characteristic numbers can be defined, compatibly with IH-small resolutions? We could begin by asking for the quotient ring of the oriented bordism ring MSO_* by oriented real flops $X_1 - X_2$, defined exactly as in the unoriented case (X_1 and X_2 are the two small resolutions of a family of real 3-fold nodes), except that we require X_1 and X_2 to be compatibly oriented. It turns out that this is not enough: all Pontrjagin numbers are invariant under oriented real flops, whereas they can change under other changes from one IH-small resolution to another, such as complex flops (between the two small resolutions of a complex family of complex 3-fold nodes). By considering both real and complex flops, we get a reasonable answer:

Theorem 5.2. *The quotient ring of MSO_* by the ideal generated by oriented real flops and complex flops is:*

$$\mathbf{Z}[\delta, 2\gamma, 2\gamma^2, 2\gamma^4, \dots],$$

where $\mathbf{CP}^2$ maps to δ and $\mathbf{CP}^4$ maps to $2\gamma + \delta^2$. This quotient ring is exactly the image of MSO_* under the Ochanine elliptic genus ([25], p. 63).

This result suggests that it should be possible to define the Ochanine genus for a large class of compact oriented real analytic spaces, or even more general singular spaces.

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