Pseudo-abelian varieties

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The theory of algebraic groups is divided into two parts with very different flavors: affine algebraic groups (which can be viewed as matrix groups) and abelian varieties. Concentrating on these two types of groups makes sense in view of Chevalley's theorem: for a perfect field k, every smooth connected k-group G is an extension of an abelian variety A by a smooth connected affine k-group N [8, 9]:

$$1 \to N \to G \to A \to 1.$$

But Chevalley's theorem fails over every imperfect field. What can be said about the structure of a smooth connected algebraic group over an arbitrary field k? (Group schemes which are neither affine nor proper come up naturally, for example as the automorphism group scheme or the Picard scheme of a projective variety over k. Groups over an imperfect field such as the rational function field $\mathbf{F}_p(t)$ arise geometrically as the generic fiber of a family of groups in characteristic p.)

One substitute for Chevalley's theorem that works over an arbitrary field is that every connected group scheme (always assumed to be of finite type) over a field k is an extension of an abelian variety by a connected affine group scheme, not uniquely [27, Lemme IX.2.7]. But when this result is applied to a smooth k-group, the affine subgroup scheme may have to be non-smooth. And it is desirable to understand the structure of smooth k-groups as far as possible without bringing in the complexities of arbitrary k-group schemes. To see how far group schemes can be from being smooth, note that every group scheme G of finite type over a field k has a unique maximal smooth closed k-subgroup [11, Lemma C.4.1], but (for k imperfect) that subgroup can be trivial even when G has positive dimension. (A simple example is the group scheme $G = \{(x, y) \in (\mathbf{G}_a)^2 : x^p = ty^p\}$ for $t \in k$ not a pth power, where p is the characteristic of k. The dimension of G is 1, but the maximal smooth k-subgroup of G is the trivial group.)

Brion gave a useful structure theorem for smooth k-groups by putting the smooth affine group "on top". Namely, for any field k of positive characteristic, every smooth connected k-group is a central extension of a smooth connected affine k-group by a semi-abelian variety (an extension of an abelian variety by a torus) [7, Proposition 2.2]. (Another proof was given by C. Sancho de Salas and F. Sancho de Salas [30].) One can still ask what substitute for Chevalley's theorem works over arbitrary fields, with the smooth affine group "on the bottom". We can gain inspiration from Tits's theory of pseudo-reductive groups [36, 37], developed by Conrad-Gabber-Prasad [11]. By definition, a pseudo-reductive group over a field k is a smooth connected affine k-group G such that every smooth connected unipotent normal k-subgroup of G is trivial. That suggests the definition:

Definition 0.1. A *pseudo-abelian variety* over a field k is a smooth connected k-group G such that every smooth connected affine normal k-subgroup of G is trivial.

It is immediate that every smooth connected group over a field k is an extension of a pseudo-abelian variety by a smooth connected affine group over k, in a unique way. Whether this is useful depends on what can be said about the structure of pseudo-abelian varieties. Chevalley's theorem implies that a pseudo-abelian variety over a perfect field is simply an abelian variety.

Over any imperfect field, Raynaud constructed pseudo-abelian varieties which are not abelian varieties [13, Exp. XVII, App. III, Prop. 5.1]. Namely, for any finite purely inseparable extension l/k and any abelian variety B over l, the Weil restriction $R_{l/k}B$ is a pseudo-abelian variety, and it is not an abelian variety if $l \neq k$ and $B \neq 0$. (Weil restriction produces a k-scheme $R_{l/k}B$ whose set of k-rational points is equal to the set of l-rational points of B.) Indeed, over an algebraic closure $\overline{k}$ of l, $R_{l/k}B$ becomes an extension of $B_{\overline{k}}$ by a smooth unipotent group of dimension $([l\colon k]-1)\dim(B)$, and so $R_{l/k}B$ is not an abelian variety. (This example shows that the notion of a pseudo-abelian variety is not geometric, in the sense that it is not preserved by arbitrary field extensions. It is preserved by separable field extensions, however.)

One main result of this paper is that every pseudo-abelian variety over a field k is *commutative*, and every pseudo-abelian variety is an extension of a smooth connected commutative unipotent k-group by an abelian variety (Theorem 2.1). In this sense, pseudo-abelian varieties are reasonably close to abelian varieties. So it is a meaningful generalization of Chevalley's theorem to say that every smooth connected group over a field k is an extension of a pseudo-abelian variety by a smooth connected affine group over k.

One can expect many properties of abelian varieties to extend to pseudo-abelian varieties. For example, the Mordell-Weil theorem holds for pseudo-abelian varieties (Proposition 4.1). Like abelian varieties, pseudo-abelian varieties can be characterized among all smooth connected groups G over a field k without using the group structure, in fact using only the birational equivalence class of G over k: G is a pseudo-abelian variety if and only if G is not "smoothly uniruled" (Theorem 5.1).

The other main result is that, over an imperfect field of characteristic p, *every* smooth connected commutative group of exponent p occurs as the unipotent quotient of some pseudo-abelian variety (Corollaries 6.5 and 7.3). Over an imperfect field, smooth commutative unipotent groups form a rich family, studied by Serre, Tits, Oesterlé, and others over the past 50 years [18], [26], [11, Appendix B]. So there are far more pseudo-abelian varieties (over any imperfect field) than the initial examples, Weil restrictions of abelian varieties.

Lemma 8.1 gives a precise relation between the structure of certain pseudo-abelian varieties and the (largely unknown) structure of commutative pseudo-reductive groups. We prove some new results about commutative pseudo-reductive groups. First, a smooth connected unipotent group of dimension 1 over a field k occurs as the unipotent quotient of some commutative pseudo-reductive group if and only if it is not isomorphic to the additive group $\mathbf{G}_a$ over k (Corollary 9.5). But an analogous statement fails in dimension 2 (Example 9.7). The proofs include some tools for computing the invariants $\mathrm{Ext}^1(U, \mathbf{G}_m)$ and $\mathrm{Pic}(U)$ of a unipotent group U. Finally, Question 7.4 conjectures a calculation of $\mathrm{Ext}^2(\mathbf{G}_a, \mathbf{G}_m)$ over any field by generators and relations, in the spirit of the Milnor conjecture. Question 9.11 attempts to describe the commutative pseudo-reductive groups over 1-dimensional fields.

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1 Notation

A *variety* over a field k means an integral separated scheme of finite type over k. Let k be a field with algebraic closure $\overline{k}$ and separable closure k_s. A field extension F of k (not necessarily algebraic) is *separable* if the ring $F \otimes_k \overline{k}$ contains no nilpotent elements other than zero. For example, the function field of a variety X over k is separable over k if and only if the smooth locus of X over k is nonempty [4, section X.7, Theorem 1, Remark 2, Corollary 2].

We use the convention that a connected topological space is nonempty.

A group scheme over a field k is *unipotent* if it is isomorphic to a k-subgroup scheme of the group of strictly upper triangular matrices in $GL(n)$ for some n (see [13, Théorème XVII.3.5] for several equivalent conditions). Being unipotent is a geometric property, meaning that it does not change under field extensions of k. Unipotence passes to subgroup schemes, quotient groups, and group extensions.

We write $\mathbf{G}_a$ for the additive group. Over a field k of characteristic $p > 0$, we write α_p for the k-group scheme $\{x \in \mathbf{G}_a : x^p = 0\}$. A group scheme over k is unipotent if and only if it has a composition series with successive quotients isomorphic to α_p, $\mathbf{G}_a$, or k-forms of $(\mathbf{Z}/p)^r$ [13, Théorème XVII.3.5].

Tits defined a smooth connected unipotent group over a field k to be *k-wound* if it does not contain $\mathbf{G}_a$ as a subgroup over k. When k has characteristic $p > 0$, a smooth connected commutative k-group of exponent p can be described in a unique way as an extension of a k-wound group by a subgroup isomorphic to $(\mathbf{G}_a)^n$ for some $n \geq 0$ [11, Theorem B.3.4]. Over a perfect field, a k-wound group is trivial. An example of a nontrivial k-wound group is the smooth connected subgroup $\{(x, y) : y^p = x - tx^p\}$ of $(\mathbf{G}_a)^2$ for any $t \in k - k^p$, discussed in Example 9.6.

Over an imperfect field k of characteristic p, there are many smooth connected commutative groups of exponent p (although they all become isomorphic to $(\mathbf{G}_a)^n$ over the algebraic closure of k). One striking phenomenon is that some of these groups are k-rational varieties, while others contain no k-rational curves [18, Theorem 6.9.2], [26, Theorem VI.3.1]. Explicitly, define a *p-polynomial* to be a polynomial with coefficients in k such that every monomial in f is a single variable raised to some power of p. Then every smooth connected commutative k-group of exponent p and dimension n is isomorphic to the subgroup of $(\mathbf{G}_a)^{n+1}$ defined by some p-polynomial f with nonzero degree-1 part [26, Proposition V.4.1], [11, Proposition B.1.13].

A smooth connected affine group G over a field k is *pseudo-reductive* if every smooth connected unipotent normal k-subgroup of G is trivial. The stronger property that G is *reductive* means that every smooth connected unipotent normal subgroup of $G_{\overline{k}}$ is trivial.

We write $\mathbf{G}_m$ for the multiplicative group over k. For each positive integer n, the k-group scheme $\{x \in \mathbf{G}_m : x^n = 1\}$ of nth roots of unity is called μ_n. A k-group scheme M is of *multiplicative type* if it is the dual of some $\mathrm{Gal}(k_s/k)$-module L which is finitely generated as an abelian group, meaning that $M = \mathrm{Spec}(k_s[L])^{\mathrm{Gal}(k_s/k)}$ [13, Proposition X.1.4]. Dualizing the surjection $L \to L/L_{\mathrm{tors}}$ shows that every k-group scheme M of multiplicative type contains a k-torus T with M/T finite. (Explicitly, T is the identity component of M with reduced scheme structure.)

2 Structure of pseudo-abelian varieties

Theorem 2.1. *Every pseudo-abelian variety E over a field k is commutative. Moreover, E is in a unique way an extension of a smooth connected commutative unipotent k-group U by an abelian variety A:*

$$1 \to A \to E \to U \to 1$$

Finally, E can be written (not uniquely) as $(A \times H)/K$ for some commutative affine k-group scheme H and some commutative finite k-group scheme K which injects into both A and H, with $H/K \cong U$.

Proof. Since E is a smooth connected k-group, the commutator subgroup $[E, E]$ is a smooth connected normal k-subgroup of E [13, Proposition VIB.7.1]. Since abelian varieties are commutative, Chevalley's theorem applied to $E_{\overline{k}}$ gives that $[E, E]_{\overline{k}}$ is affine [8, 9]. Therefore the k-subgroup $[E, E]$ is affine. Since E is a pseudo-abelian variety over k, it follows that $[E, E]$ is trivial. That is, E is commutative.

If the field k is perfect, then the pseudo-abelian variety E is an abelian variety by Chevalley's theorem. So we can assume that k is imperfect; in particular, k has characteristic $p > 0$. By Brion's theorem, E is an extension

$$1 \to A \to E \to U \to 1$$

with A a semi-abelian variety and U a smooth connected affine k-group [7, Proposition 2.2]. The maximal k-torus in A is trivial because E is a pseudo-abelian variety. That is, A is an abelian variety. So the morphism $E \to U$ is proper and flat, with geometrically reduced and connected fibers. It follows that the pullback map $O(U) \to O(E)$ on rings of regular functions is an isomorphism [15, Proposition 7.8.6]. Since U is affine, it follows that $U = \mathrm{Spec}\, O(E)$ and hence the exact sequence is uniquely determined by E. (The idea of considering $\mathrm{Spec}\, O(E)$ goes back to Rosenlicht [28, p. 432].)

Like any connected group scheme of finite type over k, E can also be written (not uniquely) as an extension

$$1 \to H \to E \to B \to 1$$

with H a connected affine group scheme over k and B an abelian variety [27, Lemme IX.2.7]. Let K be the intersection of H and A in E. Then K is both affine and proper over k, and so K has dimension 0. Also, H/K injects into U, and the abelian variety B maps onto the quotient group $U/(H/K)$. Since $U/(H/K)$ is both affine (being a quotient group of U) and an abelian variety, it is trivial. That is, H/K maps isomorphically to U. Since E is commutative, this means that E is isomorphic to $(A \times H)/K$.

It remains to show that U is unipotent. Since H is a commutative affine k-group scheme, it is an extension of a unipotent k-group scheme by a k-group scheme M of multiplicative type [13, Théorème XVII.7.2.1]. Because $M \subset H \subset E$ where E is a pseudo-abelian variety, every k-torus in M is trivial. By section 1, it follows that M is finite. Thus H is an extension of a unipotent k-group scheme by a finite k-group scheme. So the quotient group U of H is also an extension of a unipotent k-group scheme by a finite k-group scheme; in particular, every k-torus in U is trivial. Since U is a smooth connected affine k-group, it follows that U is unipotent [13, Proposition XVII.4.1.1]. □

Question 2.2. (Suggested by Michel Brion.) How can Raynaud's examples of pseudo-abelian varieties, purely inseparable Weil restrictions of abelian varieties, be described explicitly as extensions $1 \to A \to E \to U \to 1$ or as quotients $(A \times H)/K$, in the terminology of Theorem 2.1?

For a finite purely inseparable extension l/k and an abelian variety B over l, the maximal abelian subvariety of the Weil restriction $R_{l/k}B$ is the Chow l/k-trace of B [10]. Question 2.2 asks for a description of the unipotent quotient of $R_{l/k}B$, too.

Lemma 2.3. *Let G be a smooth connected group over a field k, and let K be a separable extension field of k. Then G is a pseudo-abelian variety over k if and only if it becomes a pseudo-abelian variety over K.*

Proof. If G becomes a pseudo-abelian variety over K, it is clearly a pseudo-abelian variety over k. For the converse, by considering the separable closure of K, it suffices to treat the cases where (1) K is the separable closure of k or (2) k is separably closed. To prove (1): there is a unique maximal smooth connected affine normal k_s-subgroup of G_{k_s}. By uniqueness, it is $\mathrm{Gal}(k_s/k)$-invariant, and therefore comes from a subgroup H over k. Clearly H is a smooth connected affine normal k-subgroup of G. To prove (2), reduce to the case where K is finitely generated over k, so that K is the fraction field of a smooth k-variety X, shrink X so that the maximal smooth connected affine normal K-subgroup of G_K comes from a subgroup scheme of G_X, and specialize to a k-point of X (which exists because k is separably closed). This is essentially the same as the proof that pseudo-reductivity remains unchanged under separable extensions [11, Proposition 1.1.9(1)]. □

3 Example

Pseudo-abelian varieties occur in nature, in the following sense.

Example 3.1. For every odd prime p, there is a regular projective curve X over a field k of characteristic p such that the Jacobian $\mathrm{Pic}^0_{X/k}$ is a pseudo-abelian variety which is not an abelian variety.

We leave it to the reader to seek a curve with these properties in characteristic 2. (The simpler the example, the better.)

Proof. Let k be the rational function field $\mathbf{F}_p(t)$. Let X be the regular compactification of the regular affine curve $y^2 = x(x-1)(x^p - t)$ over k. Rosenlicht considered this curve for a closely related purpose [29, pp. 49–50]. (To find the non-regular locus of the given affine curve, compute the zero locus of all derivatives of the equation with respect to x, y and also t: this gives that $(2x-1)(x^p-t)=0$, $2y=0$, and $x(x-1)=0$, which defines the empty set in $\mathbf{A}^2_k=\mathbf{A}^2_{\mathbf{F}_p(t)}$.) Then X is a geometrically integral projective curve of arithmetic genus $(p+1)/2$, and so $G := \mathrm{Pic}^0_{X/k} = \ker(\deg\colon \mathrm{Pic}_{X/k} \to \mathbf{Z})$ is a smooth connected commutative k-group of dimension $(p+1)/2$ [3, Theorem 8.2.3 and Proposition 8.4.2]. Over an algebraic closure $\overline{k}$, the curve $X_{\overline{k}}$ is not regular: it has a cusp (of the form $z^2=w^p$) at the point $(x,y)=(u,0)$, where we define $u=t^{1/p}$ in $\overline{k}$. The normalization C of $X_{\overline{k}}$ is the regular compactification of the regular affine curve $y^2=x(x-1)(x-u)$ over $\overline{k}$, with normalization map $C\to X_{\overline{k}}$ given by $(x,y)\mapsto (x, y(x-u)^{(p-1)/2})$. Since C has genus 1, $\mathrm{Pic}^0_{C/\overline{k}}$ is an elliptic curve over $\overline{k}$. Pulling back by $C\to X_{\overline{k}}$ gives a homomorphism from $G_{\overline{k}}$ onto $\mathrm{Pic}^0_{C/\overline{k}}$,

$$1\to N\to G_{\overline{k}}\to \mathrm{Pic}^0_{C/\overline{k}}\to 1,$$

with kernel N isomorphic to $(\mathbf{G}_a)^{(p-1)/2}$ over $\overline{k}$ [32, section V.17], [3, Proposition 9.2.9]. It follows that G is not an abelian variety over k.

To show that G is a pseudo-abelian variety over k, we have to show that every smooth connected affine k-subgroup S of G is trivial. For such a subgroup, $S_{\overline{k}}$ must map trivially into the elliptic curve $\mathrm{Pic}^0_{C/\overline{k}}$. So it suffices to show that every smooth connected k-subgroup S of G with $S_{\overline{k}}$ contained in N is trivial. It will be enough to prove the corresponding statement at the level of Lie algebras. Namely, we have an exact sequence of $\overline{k}$-vector spaces

$$0\to N\to H^1(X,O)\otimes_k \overline{k}\to H^1(C,O)\to 0,$$

and it suffices to show that the codimension-1 $\overline{k}$-linear subspace N has zero intersection with the k-vector space $H^1(X,O)$.

The dual of the surjection $H^1(X,O)\otimes_k\overline{k}\to H^1(C,O)$ is the inclusion $H^0(C,K_C)\to H^0(X,K_X)\otimes_k\overline{k}$ given by the trace map associated to the finite birational morphism $C\to X_{\overline{k}}$. Here K_X denotes the canonical line bundle of the Gorenstein curve X. It is a standard calculation for hyperelliptic curves that $H^0(C,K_C)$ has a $\overline{k}$-basis given by dx/y and $H^0(X,K_X)$ has a k-basis given by $x^i dx/y$ for $0\le i\le (p-1)/2$ [33, section 2]. By the formula for the normalization map $C\to X_{\overline{k}}$, this map sends dx/y to $(x-u)^{(p-1)/2}dx/y$.

To show that $N\subset H^1(X,O)\otimes_k\overline{k}$ has zero intersection with the k-linear space $H^1(X,O)$, it is equivalent to show that the coefficients $a_0,\ldots,a_{(p-1)/2}\in\overline{k}$ of $(x-u)^{(p-1)/2}dx/y$ in terms of our k-basis for $H^0(X,K_X)$ are k-linearly independent. These coefficients are $\binom{(p-1)/2}{i}(-u)^{(p-1)/2-i}$ for $0\le i\le (p-1)/2$. Since nonzero factors in k do not matter, it suffices to show that $1,u,u^2,\ldots,u^{(p-1)/2}\in\overline{k}$ are k-linearly independent. Since $t\in k$ is not a pth power, $u=t^{1/p}$ has degree p over k, and so even $1,u,u^2,\ldots,u^{p-1}$ are k-linearly independent. This completes the proof that $G=\mathrm{Pic}^0_{X/k}$ is a pseudo-abelian variety over k. $\square$

We remark that for any odd prime p, the genus $(p+1)/2$ in this example is the smallest possible for a geometrically integral projective curve X over a field k of characteristic p whose Jacobian G is a pseudo-abelian variety over k but not an abelian variety. Indeed, such a curve X must be regular; otherwise the kernel K of the homomorphism from G to the Jacobian of the normalization of X would be a nontrivial smooth connected affine k-subgroup of G. (It suffices to check that $K_{\overline{k}}$ is a nontrivial smooth connected affine group over $\overline{k}$. To do that, let $f : D \to X$ be the normalization; this is not an isomorphism if X is not regular. Then $f : D_{\overline{k}} \to X_{\overline{k}}$ is a birational morphism of (possibly singular) integral projective curves. The kernel $K_{\overline{k}}$ of the surjection $G_{\overline{k}} = \mathrm{Pic}^0_{X/\overline{k}} \to \mathrm{Pic}^0_{D/\overline{k}}$ has $K(\overline{k}) = H^0(X_{\overline{k}}, (R_{D/X}\mathbf{G}_{m,D})/\mathbf{G}_{m,X})$, which is nontrivial if f is not an isomorphism. More precisely, $K_{\overline{k}}$ is a quotient of the product of the groups $(O_{D,y}/\mathfrak{m}^N)^*$ viewed as $\overline{k}$-groups for a nonempty finite set of points $y \in D(\overline{k})$ and some positive integers N, and so $K_{\overline{k}}$ is smooth, connected, and affine over $\overline{k}$.)

Next, X is not smooth over k; otherwise its Jacobian G would be an abelian variety. So the geometric genus of $X_{\overline{k}}$ (the genus of the normalization of $X_{\overline{k}}$) is less than the arithmetic genus of $X_{\overline{k}}$ (or equivalently of X), by considering the exact sequence of sheaves $0 \to O_{X_{\overline{k}}} \to g_*O_C \to L \to 0$ associated to the normalization $g\colon C \to X_{\overline{k}}$. Finally, the geometric genus of $X_{\overline{k}}$ is not zero (otherwise G would be affine; the Jacobian $\mathrm{Pic}^0_{X/\overline{k}}$ is an extension of the Jacobian of the normalization by a smooth connected affine group). Tate showed that the geometric and arithmetic genera differ by a multiple of $(p-1)/2$ for a geometrically integral regular projective curve X over a field of characteristic p [34]; see Schröer [31] for a proof in the language of schemes. So X must have arithmetic genus at least $1 + (p-1)/2 = (p+1)/2$, as claimed.

4 Mordell-Weil theorem for pseudo-abelian varieties

One can expect many properties of abelian varieties to extend to pseudo-abelian varieties. We show here that the Mordell-Weil theorem holds for pseudo-abelian varieties.

Proposition 4.1. *Let E be a pseudo-abelian variety over a field k which is finitely generated over the prime field. Then the abelian group $E(k)$ is finitely generated.*

Proof. If k has characteristic zero, then E is an abelian variety and this is the usual Mordell-Weil theorem [24, Chapter 6]. So let k be a finitely generated field over $\mathbf{F}_p$. As with any connected group scheme over k, we can write E as an extension

$$1 \to H \to E \to B \to 1$$

with H a connected affine k-group scheme and B an abelian variety [27, Lemme IX.2.7]. Since E is a pseudo-abelian variety, E is commutative and the maximal smooth connected k-subgroup of H is trivial. Note that we can define the maximal smooth k-subgroup of any k-group scheme H as the Zariski closure of the group $H(k_s)$ [11, Lemma C.4.1]. So $H(k_s)$ is finite, and so $H(k)$ is finite. By the exact sequence $H(k) \to E(k) \to B(k)$, where $B(k)$ is finitely generated by Mordell-Weil, $E(k)$ is finitely generated. □

5 Birational characterization of pseudo-abelian varieties

In this section we show that pseudo-abelian varieties can be characterized among all smooth algebraic groups without using the group structure. In fact, the birational equivalence class of a smooth connected group G over a field k is enough to determine whether G is a pseudo-abelian variety. This makes pseudo-abelian varieties a very natural class of algebraic groups. Theorem 5.1 says that a smooth connected k-group is pseudo-abelian if and only if it is not "smoothly uniruled", a notion which we will define.

As usual, a variety X over a field k is *uniruled* if there is a variety Y over k and a dominant rational map $Y \times \mathbf{P}^1 \dashrightarrow X$ over k which does not factor through Y [22, Proposition IV.1.3]. We say that a variety X is *rationally connected* if a compactification of X is rationally connected in the usual sense [22, Definition IV.3.2.2]. Equivalently, X is rationally connected if and only if there is a variety Y over k and a rational map $u\colon Y \times \mathbf{P}^1 \dashrightarrow X$ over k such that the associated map $u^{(2)}\colon Y \times \mathbf{P}^1 \times \mathbf{P}^1 \dashrightarrow X \times_k X$ is dominant. Next, a variety X over a field k is *generically smooth* if the smooth locus of X over k is nonempty. Over a perfect field, every variety is generically smooth.

We now make a new definition. A generically smooth variety X over a field k is *smoothly uniruled* if there are generically smooth k-varieties B and E with dominant rational maps

$$\begin{tikzcd} E \arrow[r, dashed] \arrow[d, dashed] & X \\ B & \end{tikzcd}$$

such that the generic fiber of $E \dashrightarrow B$ is a generically smooth and rationally connected variety over $k(B)$, and $E \dashrightarrow X$ does not factor through B. Smooth uniruledness depends only on the birational equivalence class of X over k.

It is clear that a smoothly uniruled variety is uniruled. The converse holds for k perfect, but not in general, as Theorem 5.1 will show. (Being "smoothly uniruled" does not imply being "separably uniruled", which is stronger than uniruledness even over an algebraically closed field of positive characteristic [22, Definition IV.1.1].) Note that uniruledness is a geometric notion; that is, a k-variety X (which need not be generically smooth) is uniruled if and only if $X_{\overline{k}}$ has uniruled irreducible components [22, Proposition IV.1.3]. That is not true for smooth uniruledness (over an imperfect field k), as Theorem 5.1 will imply. At least smooth uniruledness does not change under separable algebraic extensions of k. Since smooth uniruledness turns out to be an interesting property of algebraic groups, it should be worthwhile to study smooth uniruledness for other classes of varieties over imperfect fields.

Theorem 5.1. *Let G be a smooth connected group over a field k. Then G is an abelian variety if and only if G is not uniruled. And G is a pseudo-abelian variety if and only if G is not smoothly uniruled. In particular, whether G is a pseudo-abelian variety depends only on the birational equivalence class of G over k.*

Proof. If G is not an abelian variety, then $G_{\overline{k}}$ has a nontrivial smooth connected affine normal subgroup N over $\overline{k}$, by Chevalley's theorem. Such a group N is rational [2, Remark 14.14] and has positive dimension. Using the product map $G_{\overline{k}} \times N \to G_{\overline{k}}$, it follows that $G_{\overline{k}}$ is uniruled. Equivalently, G is uniruled. Conversely, if G is an abelian variety, then $G_{\overline{k}}$ contains no rational curves, and so G is not uniruled.

If G is not a pseudo-abelian variety, then G has a nontrivial smooth connected affine normal k-subgroup N. Then $N_{\overline{k}}$ is rational and so N is rationally connected, as that is a geometric property [22, Ex. IV.3.2.5]. The diagram

$$\begin{tikzcd} G \times N \arrow[r, "gn"] \arrow[d, "g"'] & G \\ G & \end{tikzcd}$$

has the properties needed to show that G is smoothly uniruled: the base variety G is generically smooth, the generic fiber $N_{k(G)}$ of the vertical map is generically smooth and rationally connected, and the horizontal map $G \times N \to G$ is dominant and does not factor through the vertical map.

Conversely, let G be a pseudo-abelian variety over a field k. Suppose that G is smoothly uniruled. Let

$$\begin{tikzcd} E \arrow[r, dashed] \arrow[d, dashed] & G \\ B & \end{tikzcd}$$

be a diagram as in the definition of smooth uniruledness. Thus the generic fiber of $E \dashrightarrow B$ is a generically smooth and rationally connected variety over $k(B)$, and $E \dashrightarrow G$ does not factor through B. It follows that these properties hold over a dense open subset of B. Because B is a generically smooth k-variety, $B(k_s)$ is Zariski dense in B. So there is a point in $B(k_s)$ whose inverse image Y in E is a generically smooth, rationally connected variety over k_s with a nonconstant rational map $f: Y \dashrightarrow G_{k_s}$. In particular, Y has positive dimension. Here $Y(k_s)$ is Zariski dense in Y because Y is generically smooth.

By Theorem 2.1, we can write the pseudo-abelian variety G as $(A \times H)/K$ for some abelian variety A, commutative affine k-group scheme H, and commutative finite k-group scheme K. The image of the rationally connected k_s-variety Y in the abelian variety A/K must be a k_s-rational point. So f maps Y into the inverse image of this point in G_{k_s}, which is a principal H_{k_s}-bundle over $\mathrm{Spec}(k_s)$. Since $Y(k_s)$ is Zariski dense in Y, this principal bundle has a k_s-rational point and hence is trivial. Thus we get a nonconstant rational map from the generically smooth variety Y to H_{k_s}. It follows that $H(k_s)$ is infinite, and so the maximal smooth k-subgroup of H has positive dimension. Such a subgroup is affine and contained in G, contradicting that G is a pseudo-abelian variety. $\square$

6 Construction of pseudo-abelian varieties: supersingular case

The unipotent quotient of a pseudo-abelian variety over a field k is a smooth connected commutative unipotent group over k. In this section, we show that when k is imperfect of characteristic p, every smooth connected commutative group of exponent p over k occurs as the unipotent quotient of some pseudo-abelian variety E, even in the special case where the abelian subvariety of E is a supersingular elliptic curve (Corollary 6.5). Thus there are many more pseudo-abelian varieties over an imperfect field than Raynaud's original examples, Weil restrictions of abelian varieties. (Weil restrictions occur only in certain dimensions. For example, if a Weil restriction $R_{l/k}B$ for a purely inseparable extension l/k has its maximal abelian subvariety of dimension 1, then the abelian variety B has dimension 1, and so the unipotent quotient of $R_{l/k}B$ has dimension $p^r - 1$ for some r.)

Definition 6.1. Let U be a smooth connected commutative unipotent group over a field k. Let K be a finite commutative k-group scheme. We say that a commutative extension

$$1 \to K \to H \to U \to 1$$

is *highly nontrivial* if the maximal smooth connected k-subgroup of H (which is necessarily unipotent) is trivial.

For us, the point of the notion of highly nontrivial extensions is:

Lemma 6.2. *(1) Let $1 \to K \to H \to U \to 1$ be a highly nontrivial extension of a smooth connected commutative unipotent group U over a field k. Let A be an abelian variety over k that contains K as a subgroup scheme. Then $E := (A \times H)/K$ is a pseudo-abelian variety which is an extension*

$$1 \to A \to E \to U \to 1.$$

(2) Conversely, let E be any pseudo-abelian variety over a field k of characteristic p. Write E as an extension $1 \to A \to E \to U \to 1$ with A an abelian variety and U a smooth connected commutative unipotent group. Let p^r be the exponent of U. Then E can be written as $(A \times H)/A[p^r]$ for some highly nontrivial extension $1 \to A[p^r] \to H \to U \to 1$.

Proof. Let us prove (1). Clearly E is an extension $1 \to A \to E \to U \to 1$. It follows that E is a smooth connected k-group. Clearly E is commutative.

Let N be a smooth connected affine k-subgroup of E. Then N must map trivially into the abelian variety $E/H = A/K$. Therefore N is contained in the subgroup scheme H of E. Since H is a highly nontrivial extension, N is trivial. Thus E is a pseudo-abelian variety, proving (1).

We turn to (2). Since U has exponent p^r, the abelian group $\mathrm{Ext}^1(U, A)$ is killed by p^r. Consider the exact sequence

$$\mathrm{Ext}^1(U, A[p^r]) \to \mathrm{Ext}^1(U, A) \xrightarrow{p^r} \mathrm{Ext}^1(U, A).$$

(Such exact sequences hold for Ext in any abelian category, in this case the category of commutative k-group schemes of finite type [13, Théorème VIA.5.4.2].) The exact sequence shows that the extension E comes from a commutative extension $1 \to A[p^r] \to H \to U \to 1$, with $H \subset E$. Clearly H is affine. Since E is a pseudo-abelian variety, the maximal smooth connected k-subgroup of H is trivial, and so H is a highly nontrivial extension. $\square$

Lemma 6.3. *Let U be a smooth connected commutative group of exponent p over a field k of characteristic p. If k is imperfect, then there is a highly nontrivial extension of U by α_p.*

Proof. It suffices to show that there are highly nontrivial extensions of $(\mathbf{G}_a)^s$ by α_p over k for some arbitrarily large numbers s. Indeed, having a highly nontrivial extension is a property which passes from one smooth connected commutative unipotent k-group to any smooth connected k-subgroup. And every smooth connected commutative k-group of exponent p and dimension n is isomorphic to the subgroup of $(\mathbf{G}_a)^{n+1}$ defined by some p-polynomial over k.

Since k is imperfect, we can choose an element t in k^* which is not a pth power. We will exhibit a highly nontrivial extension $1 \to \alpha_p \to H \to (\mathbf{G}_a)^{(p-1)p^{r-1}} \to 1$ over k, for any $r \geq 1$. For clarity, first take $r = 1$. That is, we want to construct a highly nontrivial extension $1 \to \alpha_p \to H \to (\mathbf{G}_a)^{p-1} \to 1$. Let $l = k(u)$ where $u = t^{1/p}$; thus l is a field of degree p over k. We will take H to be the Weil restriction $R_{l/k}\alpha_p$. A general reference on Weil restriction is [11, Appendix A.5]. Note that Weil restriction need not multiply dimensions by $p = [l \colon k]$ for non-smooth schemes such as the 0-dimensional scheme α_p. In fact, $R_{l/k}\alpha_p$ has dimension $p-1$; explicitly, it is the k-subgroup scheme

$$\{(a_0, a_1, \ldots, a_{p-1}) \in (\mathbf{G}_a)^p : a_0^p + t a_1^p + \cdots + t^{p-1} a_{p-1}^p = 0\},$$

as we find by writing out the equation $(a_0 + a_1 u + \ldots + a_{p-1} u^{p-1})^p = 0$. We check immediately that the kernel of the natural homomorphism $R_{l/k}\alpha_p \to (R_{l/k}\mathbf{G}_a)/\mathbf{G}_a$ is α_p. The resulting injection

$$(R_{l/k}\alpha_p)/\alpha_p \to (R_{l/k}\mathbf{G}_a)/\mathbf{G}_a$$

is an isomorphism, because the two k-group schemes have the same dimension and $(R_{l/k}\mathbf{G}_a)/\mathbf{G}_a$ is smooth and connected. The quotient group $(R_{l/k}\mathbf{G}_a)/\mathbf{G}_a$ is isomorphic to $(\mathbf{G}_a)^p/\mathbf{G}_a \cong (\mathbf{G}_a)^{p-1}$. Thus H is an extension of $(\mathbf{G}_a)^{p-1}$ by α_p, as we want. By construction, H is commutative of exponent p.

It remains to show that the maximal smooth connected k-subgroup of H is trivial. We have $H(k_s) = (R_{l/k}\alpha_p)(k_s) = \alpha_p(l_s) = 0$, since l_s is a field of characteristic p. Therefore every smooth k-subgroup of H, connected or not, is trivial. So H is a highly nontrivial extension as we want, in the case $r = 1$.

We now generalize the construction to exhibit a highly nontrivial extension $1 \to \alpha_p \to H \to (\mathbf{G}_a)^{(p-1)p^{r-1}} \to 1$ over k for any $r \geq 1$. Again, let t be an element of k^* which is not a pth power.

Let $u = t^{1/p}$, $v = t^{1/p^{r-1}}$, and $w = t^{1/p^r}$. First define

$$\begin{aligned} U &:= (R_{k(w)/k}\mathbf{G}_a)/(R_{k(v)/k}\mathbf{G}_a) \\ &= R_{k(v)/k}((R_{k(w)/k(v)}\mathbf{G}_a)/\mathbf{G}_a) \\ &\cong (\mathbf{G}_a)^{(p-1)p^{r-1}}. \end{aligned}$$

We define an extension group $1 \to \alpha_p \to H \to U \to 1$ as the fiber product

$$\begin{aligned} H &:= [R_{k(v)/k}((R_{k(w)/k(v)}\mathbf{G}_a)/\mathbf{G}_a)] \times_{(R_{k(u)/k}\mathbf{G}_a)/\mathbf{G}_a} R_{k(u)/k}\alpha_p \\ &= U \times_{(\mathbf{G}_a)^{p-1}} R_{k(u)/k}\alpha_p. \end{aligned}$$

Here the homomorphism $R_{k(v)/k}((R_{k(w)/k(v)}\mathbf{G}_a)/\mathbf{G}_a) \to (R_{k(u)/k}\mathbf{G}_a)/\mathbf{G}_a$ on the left corresponds on k-rational points to taking the p^{r-1}st power, and the homomorphism on the right is $R_{k(u)/k}\alpha_p \to (R_{k(u)/k}\alpha_p)/\alpha_p = (R_{k(u)/k}\mathbf{G}_a)/\mathbf{G}_a$. Since the latter homomorphism is a surjection with kernel α_p, it is clear that H is an extension $1 \to \alpha_p \to H \to U \to 1$. The definition shows that H is commutative of exponent p.

It remains to show that H is a highly nontrivial extension. We will prove the stronger statement that the maximal smooth k-subgroup of H is trivial. That holds if $H(k_s) = 1$. By definition of H, $H(k_s)$ is the fiber product

$$H(k_s) = [k_s(w)/k_s(v)] \times_{k_s(u)/k_s} \alpha_p(k_s(u)),$$

where the left homomorphism is the p^{r-1}st power. Since $k_s(u)$ is a field of characteristic p, $\alpha_p(k_s(u)) = 0$. So $H(k_s) = \{y \in k_s(w)/k_s(v) : y^{p^{r-1}} \in k_s\}$. This group is zero by the following lemma, applied to the field $F = k_s$.

Lemma 6.4. *Let F be a field of characteristic $p > 0$ with an element $t \in F$ that is not a pth power in F. Let r be a positive integer. Let $u = t^{1/p}$, $v = t^{1/p^{r-1}}$, and $w = t^{1/p^r}$. Then*

$$F(w) \cap F^{1/p^{r-1}} = F(v).$$

Proof. The intersection $F(w) \cap F^{1/p^{r-1}}$ is a subfield of $F(w)$ that contains $F(v)$. It is equal to $F(v)$ because $[F(w)\colon F(v)] = p$ is prime and w is not in $F^{1/p^{r-1}}$ (because $w^{p^r} = t$ and t is not a pth power in F). □

Thus $H(k_s) = 0$. We have shown that the extension $1 \to K \to H \to (\mathbf{G}_a)^{p^{r-1}(p-1)} \to 1$ is highly nontrivial, proving Lemma 6.3. □

Corollary 6.5. *For any smooth connected commutative group U of exponent p over an imperfect field k and any supersingular elliptic curve A over k, there is a pseudo-abelian variety over k which is an extension of U by A.*

The assumption that k is imperfect is essential, by Chevalley's theorem: every pseudo-abelian variety over a perfect field is an abelian variety. In particular, there is a pseudo-abelian variety $1 \to A \to E \to \mathbf{G}_a \to 1$ over k with A a supersingular elliptic curve whenever k is imperfect, but not when k is perfect.

Proof. Let A be a supersingular elliptic curve over k. Then the kernel of the Frobenius homomorphism on A is isomorphic to α_p over k. Since k is imperfect, Lemma 6.3 shows that there is a highly nontrivial extension H of U by α_p. By Lemma 6.2(1), $E = (A \times H)/\alpha_p$ is a pseudo-abelian variety. It is an extension of U by A. □

7 Construction of pseudo-abelian varieties: ordinary case

This section shows again that there are many pseudo-abelian varieties over an imperfect field k. Namely, for any ordinary elliptic curve A over k which cannot be defined over the subfield k^p, every smooth connected commutative group of exponent p over k occurs as the unipotent quotient of a pseudo-abelian variety with abelian subvariety A, possibly after a finite separable extension of k (Corollary 7.3). This is somewhat harder than the analogous result for supersingular elliptic curves, Corollary 6.5. The analysis leads to a conjectural computation of $\mathrm{Ext}^2_k(\mathbf{G}_a, \mathbf{G}_m)$ by generators and relations (Question 7.4).

The situation is different for pseudo-abelian varieties E over k whose abelian subvariety is an ordinary elliptic curve which can be defined over k^p. In that case, the unipotent quotient of E is very restricted, by Lemma 8.1 and Example 9.7.

Lemma 7.1. *Let k be a field of characteristic $p > 0$. Let K be a commutative k-group scheme which is a nontrivial extension of $\mathbf{Z}/p$ by μ_p. Let U be a smooth connected commutative k-group of exponent p. Then there is a highly nontrivial extension of U by K over k.*

Proof. As in the proof of Lemma 6.3, it suffices to show that there are highly nontrivial extensions of $(\mathbf{G}_a)^s$ by K over k for some arbitrarily large numbers s.

We will exhibit a highly nontrivial extension $1 \to K \to H \to (\mathbf{G}_a)^{(p-1)p^{r-1}} \to 1$ over k, for any $r \geq 1$. We are assuming that the class of K in $\mathrm{Ext}^1(\mathbf{Z}/p, \mu_p) = \mathrm{Ext}^1(\mathbf{Z}/p, \mathbf{G}_m) = k^*/(k^*)^p$ [12, Corollaire III.6.4.4] is nontrivial. (Here Ext is taken in the abelian category of commutative k-group schemes of finite type.) Let $t \in k^*$ represent this extension; then t is not a pth power in k.

For clarity, first take $r = 1$. That is, we want to construct a highly nontrivial extension $1 \to K \to H \to (\mathbf{G}_a)^{p-1} \to 1$. Let $l = k(u)$ where $u = t^{1/p}$; thus l is a field of degree p over k. We will take H to be the Weil restriction $R_{l/k}\mu_p$. Like the Weil restriction $R_{l/k}\alpha_p$ in the proof of Lemma 6.3, $R_{l/k}\mu_p$ has dimension $p-1$; explicitly, it is the hypersurface

$$\{(a_0, a_1, \ldots, a_{p-1}) \in \mathbf{A}^p_k : a_0^p + t a_1^p + \cdots + t^{p-1} a_{p-1}^p = 1\},$$

as we find by writing out the equation $(a_0 + a_1 u + \ldots + a_{p-1} u^{p-1})^p = 1$. It is straightforward to check that the natural homomorphism

$$(R_{l/k}\mu_p)/\mu_p \to (R_{l/k}\mathbf{G}_m)/\mathbf{G}_m$$

is an isomorphism. The quotient group $(R_{l/k}\mathbf{G}_m)/\mathbf{G}_m$ is a smooth connected commutative group of exponent p, described explicitly as the subgroup

$$U := \{(x_0, \ldots, x_{p-1}) \in (\mathbf{G}_a)^p : x_0^p + t x_1^p + \cdots + t^{p-1} x_{p-1}^p = x_{p-1}\}$$

[26, Proposition VI.5.3]. Under Oesterlé's isomorphism $(R_{l/k}\mathbf{G}_m)/\mathbf{G}_m \to U$, the point $u = t^{1/p}$ in $(R_{l/k}\mathbf{G}_m)(k) = l^*$ maps to $(0, \ldots, 0, 1/t)$ in $U(k)$.

The homomorphism $f\colon U \to (\mathbf{G}_a)^{p-1}$ given by $(x_0, \ldots, x_{p-1}) \mapsto (x_0, \ldots x_{p-2})$ has kernel isomorphic to $\mathbf{Z}/p$, generated by the point $(0, \ldots, 0, 1/t)$. By counting dimensions, it follows that f is surjective and gives an isomorphism $U/(\mathbf{Z}/p) \cong (\mathbf{G}_a)^{p-1}$. Therefore $H = R_{l/k}\mu_p$ is a three-step extension $\begin{pmatrix} (\mathbf{G}_a)^{p-1} \\ \mathbf{Z}/p \\ \mu_p \end{pmatrix}$. (The notation means that H maps onto the top group $(\mathbf{G}_a)^{p-1}$, the kernel maps onto the middle group $\mathbf{Z}/p$, and so on.) Write K_1 for the subgroup $\begin{pmatrix} \mathbf{Z}/p \\ \mu_p \end{pmatrix}$ in H.

We want to show that K_1 is the nontrivial extension classified by $t \in \mathrm{Ext}^1(\mathbf{Z}/p, \mu_p) = (k^*)/(k^*)^p$. We can use that $\mathrm{Ext}^1(\mathbf{Z}/p, \mu_p)$ maps isomorphically to $\mathrm{Ext}^1(\mathbf{Z}/p, \mathbf{G}_m)$, by the exact sequence

$$\mathrm{Hom}(\mathbf{Z}/p, \mathbf{G}_m) \to \mathrm{Ext}^1(\mathbf{Z}/p, \mu_p) \to \mathrm{Ext}^1(\mathbf{Z}/p, \mathbf{G}_m) \xrightarrow{p} \mathrm{Ext}^1(\mathbf{Z}/p, \mathbf{G}_m).$$

(Here $\operatorname{Hom}(\mathbf{Z}/p, \mathbf{G}_m) = \mu_p(k) = 1$, and multiplication by p is zero on $\operatorname{Ext}^1(\mathbf{Z}/p, \mathbf{G}_m)$ because the group $\mathbf{Z}/p$ is killed by p.) The corresponding extension W of $\mathbf{Z}/p$ by $\mathbf{G}_m$ is the inverse image of $\mathbf{Z}/p \subset U$ under the surjection $R_{l/k}\mathbf{G}_m \to U$. The extension $1 \to \mathbf{G}_m \to W \to \mathbf{Z}/p \to 1$ is classified by the element of $k^*/(k^*)^p$ which is the pth power of any element of $W(k)$ that maps to $1 \in \mathbf{Z}/p$. As we have said, the element $u \in (R_{l/k}\mathbf{G}_m)(k) = l^*$ maps to $1 \in \mathbf{Z}/p$, and its pth power is t. So K_1 is the nontrivial extension K classified by $t \in k^*/(k^*)^p$, as we want.

It remains to show that H is a highly nontrivial extension of $(\mathbf{G}_a)^{p-1}$ by K. We have $H(k_s) = \mu_p(k_s(t^{1/p})) = 1$, because $k_s(t^{1/p})$ is a field of characteristic p. So the maximal smooth connected k-subgroup of H is trivial, as we want.

We now generalize the construction. Given a nontrivial extension K of $\mathbf{Z}/p$ by μ_p, we will exhibit a highly nontrivial extension $1 \to K \to H \to (\mathbf{G}_a)^{(p-1)p^{r-1}} \to 1$ over k for any $r \geq 1$. Again, let $t \in k^*$ represent the class of K in $\operatorname{Ext}^1(\mathbf{Z}/p, \mu_p) \cong (k^*)/(k^*)^p$.

Let $u = t^{1/p}$, $v = t^{1/p^{r-1}}$, and $w = t^{1/p^r}$. Our extension $U_r = \begin{pmatrix} (\mathbf{G}_a)^{(p-1)p^{r-1}} \\ \mathbf{Z}/p \end{pmatrix}$ will be

$$\begin{aligned} U_r &:= (R_{k(w)/k}\mathbf{G}_m)/(R_{k(v)/k}\mathbf{G}_m) \\ &= R_{k(v)/k}((R_{k(w)/k(v)}\mathbf{G}_m)/\mathbf{G}_m). \end{aligned}$$

The second description shows that U_r is a smooth connected commutative k-group of exponent p and dimension $(p-1)p^{r-1}$. (Indeed, $(R_{k(w)/k(v)}\mathbf{G}_m)/\mathbf{G}_m$ is essentially the $(p-1)$-dimensional unipotent group considered above, but over $k(v)$ instead of k.) This description gives equations for U_r:

$$U_r \cong \{(x_0, \ldots, x_{p-1}) \in (R_{k(v)/k}\mathbf{G}_a)^p : x_0^p + v x_1^p + \cdots + v^{p-1} x_{p-1}^p = x_{p-1}\}.$$

Define a homomorphism $f\colon U_r \to (\mathbf{G}_a)^{(p-1)p^{r-1}}$ over k by $(x_0, \ldots, x_{p-1}) \mapsto (x_0, \ldots, x_{p-2})$. (Here each x_i is in $R_{k(v)/k}\mathbf{G}_a \cong (\mathbf{G}_a)^{p^{r-1}}$.) The kernel of f is the k-subgroup $\mathbf{Z}/p$ of U_r generated by $(x_0, \ldots, x_{p-1}) = (0, \ldots, 0, 1/v)$. We noted in the case $r = 1$ that the isomorphism $(R_{l/k}\mathbf{G}_m)/\mathbf{G}_m \to U$ sends the point $u = t^{1/p}$ in $(R_{l/k}\mathbf{G}_m)(k) = l^*$ to $(0, \ldots, 0, 1/t)$ in $U(k)$. As a result, the point $(0, \ldots, 0, 1/v)$ in $U_r(k)$ is the image of the point w in $(R_{k(w)/k}\mathbf{G}_m)(k) = k(w)^*$ under the identification $U_r = (R_{k(w)/k}\mathbf{G}_m)/(R_{k(v)/k}\mathbf{G}_m)$. By counting dimensions, f is surjective, and so U is an extension $\begin{pmatrix} (\mathbf{G}_a)^{(p-1)p^{r-1}} \\ \mathbf{Z}/p \end{pmatrix}$.

The extension of U_r by $\mathbf{G}_m$ we consider is the fiber product

$$\begin{aligned} E &:= [(R_{k(w)/k}\mathbf{G}_m)/(R_{k(v)/k}\mathbf{G}_m)] \times_{(R_{k(u)/k}\mathbf{G}_m)/\mathbf{G}_m} R_{k(u)/k}\mathbf{G}_m \\ &= U_r \times_{(R_{k(u)/k}\mathbf{G}_m)/\mathbf{G}_m} R_{k(u)/k}\mathbf{G}_m, \end{aligned}$$

where the homomorphism $(R_{k(w)/k}\mathbf{G}_m)/(R_{k(v)/k}\mathbf{G}_m) \to (R_{k(u)/k}\mathbf{G}_m)/\mathbf{G}_m$ corresponds on k-rational points to taking the p^{r-1}st power. This extension comes from an extension H of U_r by μ_p,

$$H := U_r \times_{(R_{k(u)/k}\mathbf{G}_m)/\mathbf{G}_m} R_{k(u)/k}\mu_p,$$

since $(R_{k(u)/k}\mu_p)/\mu_p$ is isomorphic to $(R_{k(u)/k}\mathbf{G}_m)/\mathbf{G}_m$. Thus H is a three-step extension $\begin{pmatrix} (\mathbf{G}_a)^{(p-1)p^{r-1}} \\ \mathbf{Z}/p \\ \mu_p \end{pmatrix}$.

Let K_1 be the subgroup $\begin{pmatrix} \mathbf{Z}/p \\ \mu_p \end{pmatrix}$ in H. We want to show that K_1 is the nontrivial extension of $\mathbf{Z}/p$ by μ_p corresponding to $t \in k^*/(k^*)^p = \mathrm{Ext}^1(\mathbf{Z}/p, \mu_p)$. It is equivalent to show that the inverse image L_1 of $\mathbf{Z}/p \subset U_r$ in E is the extension of $\mathbf{Z}/p$ by $\mathbf{G}_m$ corresponding to $t \in k^*/(k^*)^p = \mathrm{Ext}^1(\mathbf{Z}/p, \mathbf{G}_m)$. As we have computed, L_1 contains the k-rational point w in $(R_{k(w)/k}\mathbf{G}_m)/(R_{k(v)/k}\mathbf{G}_m)$. The image of w under the "p^{r-1}st power homomorphism" to $(R_{k(u)/k}\mathbf{G}_m)/\mathbf{G}_m$ is clearly the image of u in $(R_{k(u)/k}\mathbf{G}_m)(k) = k(u)^*$. The pth power of u in L_1 is the point $t \in \mathbf{G}_m(k) = k^*$, which shows that the class of the extension L_1 is $t \in k^*/(k^*)^p$. So K_1 is isomorphic to the extension K of $\mathbf{Z}/p$ by μ_p classified by t, as we want.

It remains to show that the extension $1 \to K \to H \to (\mathbf{G}_a)^{p^{r-1}(p-1)} \to 1$ is highly nontrivial. We will prove the stronger statement that $H(k_s) = 1$. By definition of H, $H(k_s)$ is the fiber product

$$H(k_s) = [k_s(w)^*/k_s(v)^*] \times_{k_s(u)^*/(k_s)^*} \mu_p(k_s(u)),$$

where the left homomorphism is the p^{r-1}st power. Since $k_s(u)$ is a field of characteristic p, $\mu_p(k_s(u)) = 1$. So $H(k_s) = \{y \in k_s(w)^*/k_s(v)^* : y^{p^{r-1}} \in (k_s)^*\}$. We have $H(k_s) = 1$ because $k_s(w) \cap (k_s)^{1/p^{r-1}} = k_s(v)$ (Lemma 6.4). Thus the extension $1 \to K \to H \to (\mathbf{G}_a)^{p^{r-1}(p-1)} \to 1$ is highly nontrivial. $\square$

The following lemma is a variant of [20, Proposition 12.2.7].

Lemma 7.2. *Let A be an ordinary elliptic curve over a field k of characteristic $p > 0$. Then the p-torsion subgroup scheme $A[p]$ is an extension of a k-form of $\mathbf{Z}/p$ by a k-form of μ_p. The elliptic curve A can be defined over the subfield k^p if and only if this extension is split.*

Proof. The first statement is clear from the fact that $A_{\overline{k}}[p]$ is isomorphic to $\mu_p \times \mathbf{Z}/p$.

Write $G^{(p)}$ for the group scheme over k^p which is associated to a k-group scheme G via the isomorphism $k \xrightarrow{\cong} k^p$, $x \mapsto x^p$. Then the relative Frobenius for A is a homomorphism $F\colon A \to (A^{(p)})_k$. Define the Verschiebung $V\colon (A^{(p)})_k \to A$ to be the dual isogeny. Since $VF = p$, where $\ker(F) \subset A$ is a k-form of μ_p, $\ker(V) \subset (A^{(p)})_k$ must be a k-form of $\mathbf{Z}/p$.

If an ordinary elliptic curve A over k can be defined over k^p, then it can be written as $(B^{(p)})_k$ for some elliptic curve B over k. Then $\ker(V) \subset (B^{(p)})_k = A$ is a k-form of $\mathbf{Z}/p$. That subgroup gives a splitting of the extension $1 \to \ker(F) \to A[p] \to A[p]/\ker(F) \to 1$, as we want.

Conversely, let A be an ordinary elliptic curve over k such that that the extension $1 \to \ker(F) \to A[p] \to C \to 1$ is split, where $\ker(F)$ is a k-form of μ_p and C is a k-form of $\mathbf{Z}/p$. A splitting gives an etale k-subgroup $C \subset A$ of order p. This gives an etale k-subgroup $(C^{(p)})_k \subset (A^{(p)})_k$ of order p. But the kernel of the Verschiebung $V\colon (A^{(p)})_k \to A$ is also an etale k-subgroup of order p. By our knowledge of the p-torsion of an ordinary elliptic curve, it follows that $\ker(V) = (C^{(p)})_k \subset (A^{(p)})_k$. Therefore V gives an isomorphism $(A^{(p)}/C^{(p)})_k \xrightarrow{\cong} A$. So A comes from the elliptic curve $A^{(p)}/C^{(p)}$ over k^p. $\square$

Corollary 7.3. *Let A be an ordinary elliptic curve over a field k of characteristic p which cannot be defined over the subfield k^p. Suppose that the connected component of the identity in the p-torsion subgroup scheme $A[p]$ is isomorphic to μ_p over k. (That always holds after replacing k by some extension field of degree dividing $p-1$.) Then, for every smooth connected commutative group U of exponent p over k, there is a pseudo-abelian variety which is an extension of U by A.*

Proof. Since A cannot be defined over k^p, the p-torsion subgroup scheme of A is a nontrivial extension $1 \to \ker(F) \to A[p] \to C \to 1$ over k, by Lemma 7.2. Since we assume that $\ker(F)$ is isomorphic to μ_p over k, the quotient group C is isomorphic to $\mathbf{Z}/p$ over k, by the Weil pairing [20, section 2.8.2]. By Lemma 7.1, there is a highly nontrivial extension H of U by $A[p]$ over k. By Lemma 6.2(1), $E = (A \times H)/A[p]$ is a pseudo-abelian variety over k. It is an extension of U by A. $\square$

The proof of Lemma 7.1 suggests the following question about Ext groups in the abelian category of fppf sheaves over a field k of characteristic p, as studied by Breen [5, 6]. There are natural isomorphisms $\mathrm{Ext}^1_k(\mathbf{G}_a, \mathbf{Z}/p) \cong k^{\mathrm{perf}} = k^{1/p^\infty}$ [12, Proposition III.6.5.4] and $\mathrm{Ext}^1_k(\mathbf{Z}/p, \mathbf{G}_m) \cong (k^*)/(k^*)^p$ [12, Corollaire III.6.4.4]. So we have a product map

$$[\cdot,\cdot)\colon k^{\mathrm{perf}} \otimes_{\mathbf{Z}} k^* \to \mathrm{Ext}^2_k(\mathbf{G}_a, \mathbf{G}_m).$$

The product of an element of $\mathrm{Ext}^1_k(\mathbf{G}_a, \mathbf{Z}/p)$ with an element of $\mathrm{Ext}^1_k(\mathbf{Z}/p, \mathbf{G}_m)$ is zero in $\mathrm{Ext}^2_k(\mathbf{G}_a, \mathbf{G}_m)$ if and only if there is a three-step extension $\begin{pmatrix} \mathbf{G}_a \\ \mathbf{Z}/p \\ \mathbf{G}_m \end{pmatrix}$ such that the extensions $\begin{pmatrix} \mathbf{G}_a \\ \mathbf{Z}/p \end{pmatrix}$ and $\begin{pmatrix} \mathbf{Z}/p \\ \mathbf{G}_m \end{pmatrix}$ are the given ones. This follows from the neat description of three-step extensions in any abelian category by Grothendieck [16, Proposition IX.9.3.8]. The three-step extensions constructed in the proof of Lemma 7.1 imply the relation $[t^{1/p^r}, t) = 0$ in $\mathrm{Ext}^2_k(\mathbf{G}_a, \mathbf{G}_m)$ for all t in k^* and all $r \geq 1$. We can also check that $[s+t, s+t) = [s,s) + [t,t)$ in $\mathrm{Ext}^2_k(\mathbf{G}_a, \mathbf{G}_m)$ for all s, t in k with $s, t, s+t \neq 0$, for example using the relation to Brauer groups discussed below. So we have a homomorphism

$$\varphi\colon k^{\mathrm{perf}} \otimes_{\mathbf{Z}} k^* / \big([t^{1/p^r}, t) = 0 \text{ for all } t \in k^* \text{ and all } r \geq 1, [s+t, s+t) = [s,s) + [t,t)\big) \\ \to \mathrm{Ext}^2_k(\mathbf{G}_a, \mathbf{G}_m).$$

Question 7.4. Is φ an isomorphism, for every field k of characteristic p?

Remark 7.5. If Question 7.4 has a positive answer (about $\mathrm{Ext}^2_k(\mathbf{G}_a, \mathbf{G}_m)$ in the abelian category of fppf sheaves), then the same formula holds for Yoneda Ext in the abelian category of commutative affine k-group schemes of finite type. The point is that we have natural maps $\mathrm{Ext}^i_{k-\mathrm{group}}(G, H) \to \mathrm{Ext}^i_k(G, H)$ for commutative affine k-group schemes G and H. These maps are isomorphisms for $i \leq 1$ [12, Proposition III.4.1.9] and therefore injective for $i = 2$. (They are not always surjective for $i = 2$, by Breen [5].) The product map above lands in $\mathrm{Ext}^2_{k-\mathrm{group}}(\mathbf{G}_a, \mathbf{G}_m)$. So if the map φ to $\mathrm{Ext}^2_k(\mathbf{G}_a, \mathbf{G}_m)$ is an isomorphism, then the product map to $\mathrm{Ext}^2_{k-\mathrm{group}}(\mathbf{G}_a, \mathbf{G}_m)$ is also an isomorphism.

Question 7.4 would be a very natural calculation. By the discussion of three-step extensions, the group $\mathrm{Ext}^2_k(\mathbf{G}_a, \mathbf{G}_m)$ comes up in trying to classify the commutative group schemes over k. (One also encounters the group $\mathrm{Ext}^2_k(\mathbf{G}_a, \mu_p)$, which is isomorphic to $\mathrm{Ext}^2_k(\mathbf{G}_a, \mathbf{G}_m)$, since $\mathrm{Ext}^1_k(\mathbf{G}_a, \mathbf{G}_m) = 0$ [13, Théorème XVII.6.1.1].) Question 7.4 somewhat resembles the Milnor conjecture, or more specifically Kato's description of the p-torsion in the Brauer group of a field k of characteristic p:

$$\mathrm{Br}(k)[p] \cong k \otimes_{\mathbf{Z}} k^*/([t,t] = 0 \text{ for all } t \in k^*, [s^p, t] = [s,t) \text{ for all } s \in k, t \in k^*)$$

[19, Lemma 16, p. 674]. (There is a similar presentation of $\mathrm{Br}(k)[p]$ by Witt [38].) The analogy is explained by Breen's spectral sequence [5] (see the proof of Lemma 9.2 below), which gives an isomorphism

$$\mathrm{Ext}^2_k(\mathbf{G}_a, \mathbf{G}_m) \cong \ker(\alpha\colon \mathrm{Br}(\mathbf{A}^1_k)[p] \to \mathrm{Br}(\mathbf{A}^2_k)[p]).$$

Here $\alpha = m^* - \pi_1^* - \pi_2^*$, where m, π_1, π_2 are the morphisms $\mathbf{A}^2_k \to \mathbf{A}^1_k$ which send (x,y) to $x+y, x, y$, respectively. (This isomorphism sends a symbol $[a^{1/p^r}, b)$ in $\mathrm{Ext}^2_k(\mathbf{G}_a, \mathbf{G}_m)$ to $[ax^{p^r}, b)$ in $\mathrm{Br}(\mathbf{A}^1_k)[p] \subset \mathrm{Br}(k(x))[p]$, for $a \in k$, $r \geq 0$, and $b \in k^*$.)

8 Pseudo-abelian varieties and commutative pseudo-reductive groups

In this section, we consider the problem of classifying pseudo-abelian varieties E over a field k whose abelian subvariety is an ordinary elliptic curve which can be defined over k^p, the case not considered in Corollary 7.3. This case is very different: the possible unipotent quotient groups of E are highly restricted. Lemma 8.1 shows that the possible unipotent quotient groups in this case are essentially the same as the possible unipotent quotient groups of commutative pseudo-reductive groups. Section 9 gives positive and negative results about the possible unipotent quotient groups of commutative pseudo-reductive groups.

Lemma 8.1. *Let A be an ordinary elliptic curve over a field k of characteristic p. Suppose that A can be defined over the subfield k^p. Suppose that the subgroup scheme $\ker(F) \subset A$ is isomorphic to μ_p over k, as can always be arranged after replacing k by a field extension of degree dividing $p-1$. For a smooth connected commutative k-group U of exponent p, the following are equivalent.*

(1) There is a pseudo-abelian variety E which is an extension $1 \to A \to E \to U \to 1$ over k.

(2) There is a highly nontrivial extension $1 \to \mu_p \to H \to U \to 1$ over k.

(3) There is a commutative pseudo-reductive group G which is an extension $1 \to \mathbf{G}_m \to G \to U \to 1$.

These three equivalent properties fail for some smooth connected commutative groups U of exponent p. See section 9 for positive and negative results. Note that there exist ordinary elliptic curves over any field k of characteristic p with $\ker(F)$ isomorphic to μ_p; it suffices to apply Honda-Tate theory to produce an elliptic curve over $\mathbf{F}_p$ whose Frobenius eigenvalues are the Weil p-numbers $(-1 \pm \sqrt{1-4p})/2$ [35].

Proof. Assume (2). The obvious inclusions $\mu_p \to A$ and $\mu_p \to \mathbf{G}_m$ give commutative extensions of U by A, and of U by $\mathbf{G}_m$. The extension of U by A is a pseudo-abelian variety by Lemma 6.2(1), giving (1). The proof of Lemma 6.2(1) also works to show that the extension E of U by $\mathbf{G}_m$ is pseudo-reductive. (Given the extension $1 \to \mu_p \to H \to U \to 1$, we have $E = (\mathbf{G}_m \times H)/\mu_p$. Any smooth connected unipotent k-subgroup N of E maps trivially into $\mathbf{G}_m/\mu_p \cong \mathbf{G}_m$, and hence is contained in $H \subset E$. Since H is a highly nontrivial extension, N is trivial.) That proves (3).

Conversely, if (1) holds, then Lemma 6.2(2) shows that the extension $1 \to A \to E \to U \to 1$ comes from a highly nontrivial extension $1 \to A[p] \to L \to U \to 1$ over k. We are assuming that $\ker(F) \subset A[p]$ is isomorphic to μ_p over k. By the Weil pairing, it follows that $A[p]/\ker(F)$ is isomorphic to $\mathbf{Z}/p$ over k [20, section 2.8.2]. Since A can be defined over the subfield k^p, Lemma 7.2 shows that $A[p]$ is isomorphic to $\mu_p \times \mathbf{Z}/p$ over k. So $L/(\mathbf{Z}/p)$ is an extension $1 \to \mu_p \to L/(\mathbf{Z}/p) \to U \to 1$. Any smooth connected k-subgroup of $L/(\mathbf{Z}/p)$ must be trivial; otherwise its inverse image in L would be a smooth k-group of positive dimension, contradicting that L is a highly nontrivial extension. So $L/(\mathbf{Z}/p)$ is a highly nontrivial extension of U by μ_p, and (2) is proved. Finally, if (3) holds, then the same proof as for Lemma 6.2(2) shows that G comes from a highly nontrivial extension of U by μ_p. That is, (2) holds. □

9 Commutative pseudo-reductive groups

Conrad-Gabber-Prasad have largely reduced the classification of pseudo-reductive groups over a field k to the case of commutative pseudo-reductive groups, which seems intractable [11, Introduction]. This section gives a rough classification of the commutative pseudo-reductive groups of dimension 2 (Corollary 9.5) as well as examples showing the greater complexity of the problem in higher dimensions.

A commutative pseudo-reductive group over k is an extension of a smooth connected commutative unipotent group by a torus. So the main question is which unipotent quotient groups can occur. This is closely related to the question of which unipotent quotient groups can occur for certain pseudo-abelian varieties over k, for example those whose abelian subvariety is an ordinary elliptic curve which can be defined over k^p, by Lemma 8.1.

For any field k, $\mathrm{Ext}^1(\mathbf{G}_a, \mathbf{G}_m) = 0$ in the abelian category of commutative k-group schemes. It follows that the unipotent quotient U of a commutative pseudo-reductive group must be k-wound; that is, U does not contain the additive group $\mathbf{G}_a$ as a k-subgroup. One main result of this section is that every k-wound group of dimension 1 is the unipotent quotient of some commutative pseudo-reductive group E over k (Corollary 9.5). For k separably closed, we can take E to have dimension 2. (For a smooth connected unipotent group of dimension 1 over k, "k-wound" just means "not isomorphic to $\mathbf{G}_a$ over k".) There are many smooth connected unipotent groups of dimension 1 over an imperfect field, and so this result makes precise the idea that the class of commutative pseudo-reductive groups is big. Corollary 9.5 also gives that for every ordinary elliptic curve A over a separably closed field k, every k-wound group of dimension 1 occurs as the unipotent quotient of a pseudo-abelian variety with abelian subvariety A.

On the other hand, we give some counterexamples. First, for k not separably closed, a k-wound group of dimension 1 need not have any pseudo-reductive extension by $\mathbf{G}_m$ over k (Example 9.6). Conrad-Gabber-Prasad gave such an example in characteristic 3 [11, equation 11.3.1], and we check the required property in any characteristic at least 3.

Next, we exhibit a commutative k-wound group of dimension 2 over a separably closed field k which is not the unipotent quotient of any commutative pseudo-reductive group (Example 9.7). Finally, we exhibit a commutative k-wound group over a separably closed field k with $[k\colon k^p] = p$ which has no pseudo-reductive extension by $\mathbf{G}_m$ over k, although it does have a pseudo-reductive extension by $(\mathbf{G}_m)^2$ (Example 9.10). Question 9.11 asks whether, for a field k with $[k\colon k^p] = p$, every commutative k-wound group is the unipotent quotient of some pseudo-reductive group over k.

We now begin the proofs of these results. First we have a reduction of the problem to the case of a separably closed field.

Lemma 9.1. *Let U be a smooth connected commutative unipotent group over a field k. Then U is the unipotent quotient of some commutative pseudo-reductive group over k if and only if U_{k_s} is the unipotent quotient of some commutative pseudo-reductive group over the separable closure k_s.*

Proof. In one direction, let $1 \to T \to E \to U \to 1$ be a commutative pseudo-reductive extension of U by a torus T over k. Then E_{k_s} is an extension $1 \to T_{k_s} \to E_{k_s} \to U_{k_s} \to 1$, and E_{k_s} is pseudo-reductive, because the maximal smooth connected affine normal k_s-subgroup of E_{k_s} is Galois-invariant and hence defined over k [11, Proposition 1.1.9].

Conversely, suppose that U_{k_s} is the unipotent quotient of some commutative pseudo-reductive group over k_s. Then there is a finite separable extension F of k and an extension $1 \to T \to E \to U_F \to 1$ of U_F by a torus T over F such that E is pseudo-reductive. The Weil restriction $R_{F/k}E$ is an extension

$$1 \to R_{F/k}T \to R_{F/k}E \to R_{F/k}(U_F) \to 1.$$

Here $R_{F/k}E$ is pseudo-reductive, by the universal property of Weil restriction [11, Proposition 1.1.10]. Also, $R_{F/k}T$ is a torus because F is separable over k. Finally, U is a subgroup of $R_{F/k}(U_F)$ by the universal property of Weil restriction. The inverse image of U in $R_{F/k}E$ is a pseudo-reductive extension of U by $R_{F/k}T$. $\square$

We now begin to analyze extensions of unipotent groups by the multiplicative group. The group $\mathrm{Ext}^1(A, \mathbf{G}_m)$ of commutative extensions of an abelian variety A by the multiplicative group can be identified with the group $\mathrm{Pic}^0(A)$ of isomorphism classes of numerically trivial line bundles on A [32, Theorem VII.6]. For smooth connected commutative unipotent groups U, it was known that $\mathrm{Ext}^1(U, \mathbf{G}_m)$ is a subgroup of $\mathrm{Pic}(U)$ [18, Lemma 6.13.1], but the following lemma gives an explicit description of that subgroup, analogous to what happens for abelian varieties.

Lemma 9.2. *Let U be a smooth connected commutative unipotent group over a field k. Then $Ext^1(U, \mathbf{G}_m)$ is the subgroup of elements $L \in Pic(U)$ such that the translation T_aL is isomorphic to L for all separable extension fields F of k (not necessarily algebraic) and all $a \in U(F)$. In short: $Ext^1(U, \mathbf{G}_m) = Pic(U)^U$.*

The group $Ext^1(U, \mathbf{G}_m)$ *can also be described as the subgroup of primitive elements in* $Pic(U)$*, meaning that*

$$Ext^1(U, \mathbf{G}_m) = \{y \in Pic(U) : m^*(y) = \pi_1^*(y) + \pi_2^*(y) \in Pic(U \times U)\},$$

where $m\colon U \times U \to U$ *is the group operation and* $\pi_1, \pi_2\colon U \times U \to U$ *are the two projections.*

Proof. Denote the group operation on U by addition. We will use Breen's spectral sequence for computing Ext groups in the abelian category of fppf sheaves over k [5]. One can also give a more elementary but less efficient proof by imitating Serre's proof of the analogous statement for abelian varieties [32, Theorem VII.5].

For any commutative k-group schemes B and C, Breen's spectral sequence has the form

$$E_1^{i,j} = H^j_{\text{fppf}}(X_i(B), C) \Rightarrow \text{Ext}^{i+j}(B, C),$$

where the k-schemes $X_i(B)$ are explicit disjoint unions of powers of B, starting with $X_0(B) = B$, $X_1(B) = B^2 = B \times_k B$, and $X_2(B) = B^3 \coprod B^2$. The differential d_1 is an explicit alternating sum of pullback maps. In particular, d_1 on the 0th column is the homomorphism $\alpha\colon H^j(B, C) \to H^j(B^2, C)$ given by $\alpha = m^* - \pi_1^* - \pi_2^*$.

We apply the spectral sequence to compute $\text{Ext}^1(U, \mathbf{G}_m)$ for U a smooth connected commutative unipotent group U over a field k, with the following E_1 term. Since $\mathbf{G}_m$ is smooth over k, the fppf cohomology groups shown can also be viewed as etale cohomology groups [25, Theorem III.3.9].

$$\begin{tikzcd}
\cdots & & & \\
H^1(U,\mathbf{G}_m) \arrow[r] \arrow[drr, dashed] & H^1(U^2,\mathbf{G}_m) \arrow[rr] & & \cdots \\
H^0(U,\mathbf{G}_m) \arrow[r] & H^0(U^2,\mathbf{G}_m) \arrow[r] & H^0(U^3,\mathbf{G}_m) \oplus H^0(U^2,\mathbf{G}_m) \arrow[r] & {}
\end{tikzcd}$$

Since U becomes isomorphic to affine space as a scheme over the algebraic closure $\overline{k}$ [13, Corollaire XVII.4.1.3], we have $O(U^r)^* = k^*$ for every $r \geq 0$. It follows that the d_1 differential on the zeroth row of the spectral sequence is exact, by comparing with the spectral sequence computing $\text{Ext}^*(0, \mathbf{G}_m) = 0$. In particular, the d_2 differential shown as a dotted arrow maps into the zero group. Therefore, the spectral sequence gives an isomorphism

$$\text{Ext}^1(U, \mathbf{G}_m) = \ker(\alpha\colon \text{Pic}(U) \to \text{Pic}(U \times U)),$$

as we want. The right side is called the group of primitive line bundles on U.

We now prove the other description of $\text{Ext}^1(U, \mathbf{G}_m)$. For a primitive line bundle L on U, fix a trivialization of L at the origin in U. Then there is an isomorphism $m^*L \cong \pi_1^*L \otimes \pi_2^*L$, which is uniquely determined if we require it to be compatible with the trivialization of L at $(0, 0)$ in $U \times U$. (That isomorphism gives a canonical isomorphism $L_{a+b} \cong L_a \otimes L_b$ for all $a, b \in U(F)$ and all extension fields F of k.) Restricting that isomorphism to U times an F-rational point of U gives an isomorphism $T_aL \cong L$ on U_F for all $a \in U(F)$, and all extension fields F of k.

Conversely, suppose that $T_aL \cong L$ for all $a \in U(F)$ and all separable extension fields F of k. We apply this to the function field $F = k(U)$ and $a \in U(F)$ the generic point. Here F is separable over k since U is smooth over k. We can rewrite the isomorphism $T_aL \cong L$ on U_F as $T_aL \cong L_a \otimes L$, since L_a is just a 1-dimensional F-vector space. This means that the line bundle $M := m^*(L) \otimes \pi_1^*(L^*) \otimes \pi_2^*(L^*)$ on $U \times U$ is trivial on $U \times (U - S)$ for some codimension-1 closed subset S of U. Therefore M is linearly equivalent on $U \times U$ to $\pi_2^* D$ for some divisor $D \subset U$ supported on S. Restricting to $0 \times U$, where M is trivial, shows that D is linearly equivalent to 0 on U. So M is trivial on $U \times U$. That is, L is primitive. $\square$

Lemma 9.3. *Let U be a k-wound group of dimension 1 over a field k. Then $Pic(U) \neq 0$.*

Lemma 9.3 was proved by Kambayashi-Miyanishi-Takeuchi [18, Theorem 6.5(i)]. We give a proof here for clarity.

Proof. Let C be the unique regular compactification of U over k. Then $C - U$ is a single closed point, because U becomes isomorphic to $\mathbf{A}^1$ over the algebraic closure $\overline{k}$. The group $\mathrm{Pic}(U)$ is the quotient of $\mathrm{Pic}(C)$ by the class of the closed point $C - U$. I claim that the closed point $C - U$ has degree a multiple of p over k (in fact, a power of p greater than 1). It suffices to prove this after passing to the separable closure k_s; then U_{k_s} remains k-wound and C_{k_s} remains regular [4, Prop. X.6.5]. All finite field extensions of k_s have degree a power of p, so it suffices to show that $(C - U)(k_s) = \emptyset$. So suppose that there is a k_s-rational point w in $C - U$. Since C is regular, it is smooth over k_s near w. This gives a point of $U(k_s((t)))$ that does not extend to $U(k_s[[t]])$, contradicting a property of k-wound groups [26, Proposition V.8].

Therefore, the degree homomorphism $\deg\colon \mathrm{Pic}(C) \to \mathbf{Z}$ passes to a well-defined homomorphism $\mathrm{Pic}(U) \to \mathbf{Z}/p$. The homomorphism $\mathrm{Pic}(U) \to \mathbf{Z}/p$ is surjective, since the line bundle $O(0)$ on U has degree 1, where $0 \in U(k)$ is the identity element. $\square$

Lemma 9.4. *Let U be a k-wound group of dimension 1 over a separably closed field k. Then $Ext^1(U, \mathbf{G}_m) \neq 0$.*

This can fail for k not separably closed, by Example 9.6.

Proof. Let C be the regular compactification of U over k. Let $\mathrm{Pic}_{C/k}$ be the Picard scheme [21, Theorem 9.4.8]. Then $\mathrm{Pic}_{C/k}$ is a k-group scheme, locally of finite type, with $\mathrm{Pic}(C_F) \cong \mathrm{Pic}_{C/k}(F)$ for every field extension F of k (using that $H^0(C, O) = k$ and C has a k-rational point). Since C is a geometrically irreducible projective curve, the kernel $\mathrm{Pic}^0_{C/k}$ of the degree homomorphism $\mathrm{Pic}_{C/k} \to \mathbf{Z}$ is smooth, connected, and of finite type over k [3, Theorem 8.2.3 and Proposition 8.4.2]. The curve C becomes rational over the algebraic closure $\overline{k}$, and so $\mathrm{Pic}^0_{C/k}$ is affine (as the abelian variety quotient of $(\mathrm{Pic}^0_{C/k})_{\overline{k}}$ is the Jacobian of the normalization of $C_{\overline{k}}$ [32, section V.17], [3, Proposition 9.2.10]). Because $U_{\overline{k}}$ is isomorphic to $\mathbf{A}^1_{\overline{k}}$, the point $C_{\overline{k}} - U_{\overline{k}}$ corresponds to a single point on the normalization $\mathbf{P}^1_{\overline{k}}$ of $C_{\overline{k}}$, and so $(\mathrm{Pic}^0_{C/k})_{\overline{k}}$ is unipotent [32, section V.17], [3, Proposition 9.2.9]. It follows that $\mathrm{Pic}^0_{C/k}$ is unipotent.

The action of U on itself by translation extends to an action of U on C, by the uniqueness of the regular compactification C and the smoothness of U. By the proof of Lemma 9.3, $\mathrm{Pic}(U)$ is an extension of a finite cyclic group by the group $\mathrm{Pic}^0_{C/k}(k)$. The action of $U(k)$ by translations on $\mathrm{Pic}(U)$ clearly restricts to the action of $U(k)$ on $\mathrm{Pic}^0_{C/k}(k)$ by translations.

If $\mathrm{Pic}^0_{C/k}$ is zero, then $\mathrm{Pic}_{C/k}$ is isomorphic to $\mathbf{Z}$ by the degree. Then the action of U on $\mathrm{Pic}_{C/k}$ is trivial, since U is connected. In this case, $\mathrm{Pic}(U_F)$ is a finite cyclic group for all separable extension fields F of k, and $U(F)$ acts trivially on $\mathrm{Pic}(U_F)$ since $\mathrm{Pic}(C_F) \to \mathrm{Pic}(U_F)$ is surjective. So $\mathrm{Ext}^1(U, \mathbf{G}_m) = \mathrm{Pic}(U)$ in this case (using Lemma 9.2) and this is a nonzero cyclic group by Lemma 9.3. (For this case, we did not need k to be separably closed.)

Otherwise, $\mathrm{Pic}^0_{C/k}$ is not zero. In this case, we will show that the subgroup $\mathrm{Pic}^0(C)^U$ of $\mathrm{Ext}^1(U, \mathbf{G}_m)$ is not zero, using the notation of Lemma 9.2. Since $P := \mathrm{Pic}^0_{C/k}$ is a smooth connected commutative unipotent k-group, the semidirect product $U \ltimes P$ is unipotent, and therefore is a nilpotent group by the results listed in section 1. That implies that the action of U on P must be nilpotent. In more detail, write $(u-1)q$ to mean $uq - q$ for any extension field F of k, $u \in U(F)$, and $q \in P(F)$, where the group operation on P is written additively. If we define P^m for each natural number m as the closed subgroup of P generated by elements $(u_1 - 1) \cdots (u_m - 1)q$ for $u_i \in U(k_s)$ and $q \in P(k_s)$, then the subgroups $P = P^0 \supset P^1 \supset P^2 \supset \cdots$ are closed and connected, eventually equal to zero because the group $U \ltimes P$ is nilpotent. Also, the group U acts trivially on each P^m/P^{m+1}.

In particular, the last P^m not equal to zero is a nontrivial smooth connected subgroup of $\mathrm{Pic}^0_{C/k}$ such that U acts trivially on P^m. Thus $P^m(k) \subset \mathrm{Ext}^1(U, \mathbf{G}_m)$ by Lemma 9.2. Since k is separably closed, $P^m(k) \neq 0$. □

Corollary 9.5. *Let U be a k-wound group of dimension 1 over a field k. Then U is the unipotent quotient of some commutative pseudo-reductive group E over k.*

Suppose in addition that k is separably closed. Then we can take E to be an extension of U by $\mathbf{G}_m$. Also, for any ordinary elliptic curve A over k, there is an extension of U by A which is a pseudo-abelian variety.

Recall that Corollaries 6.5 and 7.3 give a larger class of pseudo-abelian varieties when the abelian subvariety is a supersingular elliptic curve, or an ordinary elliptic curve which cannot be defined over the subfield k^p.

Proof. By Lemma 9.1, we can assume that k is separably closed. By Lemma 9.4, there is a nontrivial extension

$$1 \to \mathbf{G}_m \to E \to U \to 1$$

of commutative k-groups. If N is a nontrivial smooth connected unipotent k-subgroup of E, then $N \cap \mathbf{G}_m = 1$ as a group scheme, and so N projects isomorphically to a subgroup of U. Since U has dimension 1, N projects isomorphically to U, contradicting that the extension is nontrivial. So E must be pseudo-reductive.

An ordinary elliptic curve A over k has $\ker(F)$ isomorphic to μ_p, since k is separably closed. The existence of the pseudo-reductive extension E implies that there is a pseudo-abelian extension of U by A, by Corollary 7.3 and Lemma 8.1. □

Example 9.6. Let k_0 be a field of characteristic $p \geq 3$ and let k be the rational function field $k_0(t)$. Let U be the subgroup $\{(x, y) : y^p = x - tx^p\}$ of $(\mathbf{G}_a)^2$ over k. Then U is a k-wound group of dimension 1 with $\text{Ext}^1(U, \mathbf{G}_m) = 0$. Therefore, U has no extension by $\mathbf{G}_m$ over k which is pseudo-reductive.

Conrad-Gabber-Prasad observed that $\text{Ext}^1(U, \mathbf{G}_m) = 0$ in this example when $p = 3$ [11, equation 11.3.1]. Note that U does have an pseudo-reductive extension by some torus over k, by Corollary 9.5.

Proof. Over $\overline{k}$, U becomes isomorphic to $\mathbf{G}_a$ by a simple change of variables. So U is connected and smooth over k. If U were isomorphic to $\mathbf{G}_a$, then the projective closure $X = \{[x, y, z] \in \mathbf{P}^2 : y^p = xz^{p-1} - tx^p\}$ of U would have normalization isomorphic to $\mathbf{P}^1$ over k, and the image of $\infty \in \mathbf{P}^1$ would be a k-rational point in $X - U$. But there is no such point, and so U is k-wound.

By Kambayashi-Miyanishi-Takeuchi [18, 6.13.3],

$$\text{Ext}^1(U, \mathbf{G}_m) \cong \left\{ (c_0, \ldots, c_{p-2}) \in k^{p-1} : c_{p-2} = \sum_{0 \leq j \leq p-2} c_j^p t^j \right\}.$$

We will show that this equation has no nonzero solutions in $k = k_0(t)$. We can assume that k_0 is algebraically closed.

Suppose that $(c_0, \ldots, c_{p-2})$ is a nonzero element of $\text{Ext}^1(U, \mathbf{G}_m)$. If $c_{p-2} = 0$, then the equation gives that $1, t, \ldots, t^{p-3}$ are linearly dependent over the field k^p, which is false. So $c_{p-2} \neq 0$.

Viewing $c_0, \ldots, c_{p-2}$ as rational functions over k_0, we can differentiate the equation $p - 2$ times to get $c_{p-2}^{(p-2)} = (p-2)!\, c_{p-2}^p$. By considering the pole order of c_{p-2} at each point $a \in k_0$, we deduce from this equation that c_{p-2} is regular at each point $a \in k_0$. Since k_0 is algebraically closed, that means that c_{p-2} is a polynomial over k_0. Let d be its degree. Then c_{p-2}^p is nonzero of degree pd while $c_{p-2}^{(p-2)}$ has lower degree, a contradiction. We have shown that $\text{Ext}^1(U, \mathbf{G}_m) = 0$ over $k = k_0(t)$. $\square$

Example 9.7. Let k_0 be a field of characteristic $p > 0$, and let k be the rational function field $k_0(a, b)$. Let U be the subgroup

$$\{(x, y, z) : x + ax^p + by^p + z^p = 0\}$$

of $(\mathbf{G}_a)^3$. Then U is a commutative k-wound group of dimension 2 with $\text{Ext}^1(U_{k_s}, \mathbf{G}_m) = 0$. It follows that, even over the separable closure k_s, U is not the unipotent quotient of any commutative pseudo-reductive group.

Proof. Let X be the projective closure of U,

$$X = \{[x, y, z, w] \in \mathbf{P}^3_k : xw^{p-1} + ax^p + by^p + z^p = 0\}.$$

Then X has no k-points at infinity (meaning points with $w = 0$). It follows that U is k-wound.

Since $\text{Ext}^1(U_{k_s}, \mathbf{G}_m)$ is a subgroup of $\text{Pic}(U_{k_s})$ (Lemma 9.2), it suffices to show that $\text{Pic}(U_{k_s}) = 0$. We start by finding the non-regular locus of the surface X. To do so, we compute the zero locus of all derivatives of the equation with respect to x, y, z, w and also a, b: this gives that $w^{p-1} = 0$, $x^p = 0$, $y^p = 0$, and hence $z^p = 0$, which defines the empty set in $\mathbf{P}^3_k$. So X is regular, and it follows that X_{k_s} is regular [4, Prop. X.6.5]. Also, $X-U$ is the plane curve $D = \{[x,y,z] \in \mathbf{P}^2_k : ax^p+by^p+z^p = 0\}$, which is regular over k_s and hence irreducible over k_s. It follows that

$$\mathrm{Pic}(U_{k_s}) \cong \mathrm{Pic}(X_{k_s})/\mathbf{Z}\cdot[D_{k_s}] = \mathrm{Pic}(X_{k_s})/\mathbf{Z}\cdot O(1).$$

So it suffices to show that $\mathrm{Pic}(X_{k_s}) = \mathbf{Z}\cdot O(1)$.

Lemma 9.8. *Let Y be a scheme of finite type over a field F such that $H^0(Y,O) = F$. Then the homomorphism $\mathrm{Pic}(Y) \to \mathrm{Pic}(Y_E)$ is injective for any extension field E of F.*

Proof. We have $H^0(Y_E,O) = H^0(Y,O)\otimes_F E = E$. Let L be a line bundle on Y which becomes trivial over E. Then L and the dual line bundle L^* have 1-dimensional spaces of sections over Y, since that is true over Y_E. Let $s \in H^0(Y,L)$ and $t \in H^0(Y,L^*)$ be nonzero sections. Then the product $st \in H^0(Y,O) = F$ is not zero since that is true over E. This means that the compositions $O_Y \xrightarrow{s} L \xrightarrow{t} O_Y$ and $L \xrightarrow{t} O_Y \xrightarrow{s} L$ are isomorphisms. So L is trivial. □

A referee pointed out that one can prove Lemma 9.8 under the weaker assumption that the ring $O(Y)$ has trivial Picard group. Consider the morphism $f\colon Y \to S := \mathrm{Spec}\, O(Y)$. Then the Leray spectral sequence for fppf cohomology gives (since $f_*\mathbf{G}_m = \mathbf{G}_m$) that $\mathrm{Pic}(Y)/\mathrm{Pic}(S)$ injects into $H^0_{\mathrm{fppf}}(S, R^1 f_*\mathbf{G}_m)$, which gives the result.

Since X is a surface in $\mathbf{P}^3$, we have $H^0(X_{k_s},O) = k_s$ by the exact sequence of sheaves $0 \to O_{\mathbf{P}^3}(-X) \to O_{\mathbf{P}^3} \to O_X \to 0$. By Lemma 9.8, we have $\mathrm{Pic}(X_{k_s}) = \mathbf{Z}\cdot O(1)$ as we want if we can show that $\mathrm{Pic}(X_{\overline{k}}) = \mathbf{Z}\cdot O(1)$. We have

$$\begin{aligned} X_{\overline{k}} &\cong \{[x,y,z,w] \in \mathbf{P}^3 : xw^{p-1} + x^p + y^p + z^p = 0\} \\ &\cong \{[x,y,z,w] \in \mathbf{P}^3 : xw^{p-1} + y^p = 0\} \end{aligned}$$

Thus $X_{\overline{k}}$ is the projective cone over the plane curve $xw^{p-1} + y^p = 0$.

Let Y be a projective scheme over a field k such that $H^0(Y,O_Y) = k$, and let $O_Y(1)$ be an ample line bundle on Y. Let R be the homogeneous coordinate ring $\oplus_{j\geq 0} H^0(Y,O_Y(j))$ as a graded ring, and define the *projective cone* over Y to be $X = \mathrm{Proj}\, R[x]$, where x has degree 1. For a closed subscheme $Y \subset \mathbf{P}^n$ over k, there is a finite morphism from X to the *classical projective cone* over Y in $\mathbf{P}^{n+1}$, which is an isomorphism away from the vertex [23, section 2.56]. This morphism is an isomorphism if the k-algebra $\oplus_{j\geq 0} H^0(Y,O_Y(j))$ is generated by $H^0(\mathbf{P}^n, O(1))$, but in general the projective cone as defined here has better properties.

Lemma 9.9. *Let Y be a projective scheme over a field k such that $H^0(Y,O_Y) = k$, and let $O_Y(1)$ be an ample line bundle on Y. Let X be the projective cone over Y. Then $\mathrm{Pic}(X) = \mathbf{Z}\cdot O_X(1)$.*

Proof. Let Z be the $\mathbf{P}^1$-bundle $P(O_Y \oplus O_Y(1))$ over Y. By the calculation of the K-theory of projective bundles [1, Theorem VI.1.1], $\mathrm{Pic}(Z) \cong \mathrm{Pic}(Y) \oplus \mathbf{Z}$ for any connected scheme Y. (Here the summand $\mathbf{Z}$ is generated by the natural line bundle $O_Z(1)$ on the projective bundle Z. The statement means that every line bundle on Z is, in a unique way, a pullback from Y tensored with $O_Z(j)$ for some integer j.) Since

$H^0(Y, O_Y) = k$, Y is connected and so $\mathrm{Pic}(Z) = \mathrm{Pic}(Y) \oplus \mathbf{Z}$. Since Y is projective over k with $H^0(Y, O_Y) = k$, there is a surjection $f\colon Z \to X$ which contracts a copy of Y (the section corresponding to the first projection $O_Y \oplus O_Y(1) \twoheadrightarrow O_Y$ over Y) to a point [14, Proposition 8.6.2]. For any line bundle L on X, the pullback f^*L is trivial on Y, and so the image of $f^*\colon \mathrm{Pic}(X) \to \mathrm{Pic}(Z)$ is contained in $\mathbf{Z} \cdot O_Z(1)$. (By restricting to a fiber of the $\mathbf{P}^1$-bundle $Z \to Y$, we see that $f^*O_X(1) \cong O_Z(1)$.) It remains to show that $f^*\colon \mathrm{Pic}(X) \to \mathrm{Pic}(Z)$ is injective.

The natural map $O_X \to f_*O_Z$ is an isomorphism [14, Proposition 8.8.6]. So, for any line bundle L on X, the natural map $L \to f_*f^*(L)$ is an isomorphism. If L is a line bundle on X whose pullback to Z is trivial, then $H^0(X, L) = H^0(X, f_*f^*L) \cong H^0(Z, f^*(L)) \cong H^0(Z, O_Z) = k$. Likewise, $H^0(X, L^*) \cong k$. It follows that L is trivial, as in the proof of Lemma 9.8. $\square$

We now return to Example 9.7. The surface $X_{\overline{k}}$ is the classical projective cone over the plane curve $Y = \{xw^{p-1} + y^p = 0\}$ over $\overline{k}$. For a curve Y of any degree d in $\mathbf{P}^2$, the k-algebra $\oplus_{j \geq 0} H^0(Y, O_Y(j))$ is generated in degree 1, by considering the exact sequence of sheaves on $\mathbf{P}^2$, $0 \to O_{\mathbf{P}^2}(j-d) \to O_{\mathbf{P}^2}(j) \to O_Y(j) \to 0$. So $X_{\overline{k}}$ is the projective cone over Y in the sense defined above. By Lemma 9.9, $\mathrm{Pic}(X_{\overline{k}}) = \mathbf{Z} \cdot O_X(1)$. As we have said, it follows that $\mathrm{Pic}(U_{k_s}) = 0$. Example 9.7 is proved. $\square$

The following example, supplied by a referee, answers a question in the original version of this paper. Note that k can be separably closed in the following example.

Example 9.10. Let k be a field of characteristic $p > 0$ with $[k\colon k^p] = p$. Let $k_1 = k^{1/p}$, which is an extension of degree p of k. Let U be the smooth connected commutative unipotent k-group $(R_{k_1/k}\mathbf{G}_m)/\mathbf{G}_m$ of dimension $p-1$. Then $U \times U$ is k-wound, but $U \times U$ has no extension by $\mathbf{G}_m$ over k which is pseudo-reductive. It does have an extension by $(\mathbf{G}_m)^2$ which is pseudo-reductive.

Proof. We first consider a more general situation. Let k be any field of characteristic $p > 0$. For any smooth connected commutative affine k-group G with maximal torus T, let K be the field of definition over k of the geometric unipotent radical of G. Thus G_K is the product of T_K with a smooth connected unipotent K-group; in particular, we have a unique splitting $G_K \to T_K$ of the inclusion. By the universal property of Weil restriction, this gives a homomorphism $f\colon G \to R_{K/k}(T_K)$ which restricts to the obvious inclusion $T \to R_{K/k}(T_K)$. Moreover, f does not factorize through $R_{L/k}(T_L)$ for any proper subextension L of K/k. Let k_1 denote the extension field $k^{1/p}$. Suppose that p kills G/T; then the image in $T(K)$ of a point in $G(k)$ has pth power in $T(k)$, and so that image lies in $T(k_1)$. It follows that f factors through $R_{L/k}(T_L)$ for $L = K \cap k_1$, and so K is contained in k_1.

We now return to the notation of this Example, so that k is a field with $[k\colon k^p] = p$. Then k_1 is equal to $k(t^{1/p})$ for any element $t \in k^*$ which is not a pth power. We know that U is k-wound, because $R_{k_1/k}\mathbf{G}_m$ is pseudo-reductive. (Use the universal property of Weil restriction: a homomorphism $\mathbf{G}_a \to R_{k_1/k}\mathbf{G}_m$ over k is equivalent to a homomorphism $\mathbf{G}_a \to \mathbf{G}_m$ over k_1, which must be trivial.) The product $(R_{k_1/k}\mathbf{G}_m)^2$ is an extension of $U \times U$ by $(\mathbf{G}_m)^2$ which is pseudo-reductive.

By Oesterlé, as we used in the proof of Lemma 7.1, U is isomorphic to

$$\{(x_0, \ldots, x_{p-1}) \in (\mathbf{G}_a)^p : x_0^p + tx_1^p + \cdots + t^{p-1}x_{p-1}^p = x_{p-1}\}$$

[26, Proposition VI.5.3]. The Lie algebra of U is a restricted Lie algebra with pth power operation equal to zero (since that is true for $(\mathbf{G}_a)^p$, for example). So every nonzero element of the Lie algebra of U gives an α_p subgroup of U. The intersections of U with one-dimensional k-linear subspaces of $(\mathbf{G}_a)^p$ give exactly the k-subgroup schemes of order p in U. Therefore the quotient of U by any k-subgroup scheme of order p (in particular, any α_p subgroup) is isomorphic to $(\mathbf{G}_a)^{p-1}$ over k. So any homomorphism $U \to U$ over k is either zero or induces an isomorphism on Lie algebras, using that U is k-wound.

Now let G be a commutative extension of $U \times U$ by $\mathbf{G}_m$ over k. Since $U \times U$ is killed by p, we showed above that the geometric unipotent radical of G is defined over k_1. As above, this gives a homomorphism $G \to R_{k_1/k}\mathbf{G}_m$ which is the identity on the subgroup $\mathbf{G}_m$. On the quotients by $\mathbf{G}_m$, this gives a homomorphism $h\colon U \times U \to U$, and the extension G of $U \times U$ by $\mathbf{G}_m$ is pulled back via h. By the previous paragraph, either h is zero or h induces a surjection on Lie algebras. In both cases, $\ker(h)$ is a smooth k-subgroup of positive dimension. Since the extension G of $U \times U$ by $\mathbf{G}_m$ splits over $\ker(h)$, G is not pseudo-reductive. □

The following question is suggested by Corollary 9.5, Example 9.7, and Example 9.10.

Question 9.11. If k is a field with $[k\colon k^p] = p$, is every k-wound commutative unipotent group the unipotent quotient of some commutative pseudo-reductive group over k?

In view of Example 9.10, the maximal torus of the pseudo-reductive group will in general have dimension greater than 1. By Lemma 9.1, it suffices to answer Question 9.11 for k separably closed.

We know that the unipotent quotient of a commutative pseudo-reductive group is k-wound. So Question 9.11 would describe exactly which groups occur as the unipotent quotients of commutative pseudo-reductive groups over a field k with $[k\colon k^p] = p$. (For example, that would apply to the function field of a curve over a finite field.)

References

[1] P. Berthelot, A. Grothendieck, and L. Illusie. *Théorie des intersections et théorème de Riemann-Roch (SGA 6).* Lecture Notes in Mathematics 225, Springer (1971).

[2] A. Borel. *Linear algebraic groups.* Springer (1991).

[3] S. Bosch, W. Lütkebohmert, and M. Raynaud. *Néron models.* Springer (1990).

[4] N. Bourbaki. *Algèbre commutative. Chapitre 10.* Springer (2007).

[5] L. Breen. On a nontrivial higher extension of representable abelian sheaves. *Bull. Amer. Math. Soc.* **75** (1969), 1249–1253.

[6] L. Breen. Un théorème d'annulation pour certains Ext^i de faisceaux abéliens. *Ann. Sci. Ecole Normale Supérieure* **8** (1975), 339–352.

[7] M. Brion. Anti-affine algebraic groups. *J. Alg.* **321** (2009), 934–952.

[8] C. Chevalley. Une démonstration d'un théorème sur les groupes algébriques. *J. Math. Pure Appl.* **39** (1960), 307–317.

[9] B. Conrad. A modern proof of Chevalley's theorem on algebraic groups. *J. Ramanujan Math. Soc.* **17** (2002), 1–18.

[10] B. Conrad. Chow's K/k-image and K/k-trace, and the Lang-Néron theorem. *Enseign. Math.* **52** (2006), 37–108.

[11] B. Conrad, O. Gabber, and G. Prasad. *Pseudo-reductive groups.* Cambridge (2010).

[12] M. Demazure and P. Gabriel. *Groupes algébriques.* Masson (1970).

[13] M. Demazure and A. Grothendieck. *Schémas en groupes* I, II, III (SGA 3). Lecture Notes in Mathematics 151, 152, 153, Springer (1970). Revised version edited by P. Gille and P. Polo, vols. I and III, Soc. Math. de France (2011).

[14] A. Grothendieck. Eléments de géométrie algébrique. II. Etude globale élémentaire de quelques classes de morphismes. *Publ. Math. IHES* **8** (1961), 5–222.

[15] A. Grothendieck. Eléments de géométrie algébrique. III. Etude cohomologique des faisceaux cohérents. Seconde partie. *Publ. Math. IHES* **17** (1963), 5–91.

[16] A. Grothendieck. *Groupes de monodromie en géométrie algébrique* I (SGA 7). Lecture Notes in Mathematics 288, Springer (1972).

[17] R. Hartshorne. *Algebraic geometry.* Springer (1977).

[18] T. Kambayashi, M. Miyanishi, and M. Takeuchi. *Unipotent algebraic groups.* Lecture Notes in Mathematics 414, Springer (1974).

[19] K. Kato. A generalization of local class field theory by using K-groups. II. *J. Fac. Sci. Univ. Tokyo Sect. 1A Math.* **27** (1980), 603–683.

[20] N. Katz and B. Mazur. *Arithmetic moduli of elliptic curves.* Princeton (1985).

[21] S. Kleiman. The Picard scheme. *Fundamental algebraic geometry,* 235–321. Amer. Math. Soc. (2005).

[22] J. Kollár. *Rational curves on algebraic varieties.* Springer (1996).

[23] J. Kollár. *Singularities of the minimal model program.* Cambridge (2013).

[24] S. Lang. *Fundamentals of Diophantine geometry.* Springer (1983).

[25] J. Milne. *Etale cohomology.* Princeton (1980).

[26] J. Oesterlé. Nombres de Tamagawa et groupes unipotents en caractéristique p. *Invent. Math.* **78** (1984), 13–88.

[27] M. Raynaud. *Faisceaux amples sur les schémas en groupes et les espaces homogènes.* Lecture Notes in Mathematics 119, Springer (1970).

[28] M. Rosenlicht. Some basic theorems on algebraic groups. *Amer. J. Math.* **78** (1956), 401–443.

[29] M. Rosenlicht. Some rationality questions on algebraic groups. *Ann. Mat. Pura Appl.* (4) **43** (1957), 25–50.

[30] C. Sancho de Salas and F. Sancho de Salas. Principal bundles, quasi-abelian varieties and structure of algebraic groups. *J. Alg.* **322** (2009), 2751–2772.

[31] S. Schröer. On genus change in algebraic curves over imperfect fields. *Proc. Amer. Math. Soc.* **137** (2009), 1239–1243.

[32] J.-P. Serre. *Algebraic groups and class fields.* Springer (1988).

[33] K.-O. Stöhr. Hyperelliptic Gorenstein curves. *J. Pure Appl. Algebra* **135** (1999), 93–105.

[34] J. Tate. Genus change in inseparable extensions of function fields. *Proc. Amer. Math. Soc.* **3** (1952), 400–406.

[35] J. Tate. Classes d'isogénie des variétés abéliennes sur un corps fini (d'après T. Honda). *Séminaire Bourbaki,* vol. 1968/69, Exposés 347–363, Springer (1971), 95–110.

[36] J. Tits. Théorie des groupes. *Annuaire du Collège de France,* 1991–1992, 115–133.

[37] J. Tits. Théorie des groupes. *Annuaire du Collège de France,* 1992–1993, 113–131.

[38] E. Witt. p-Algebren und Pfaffsche Formen. *Abh. Math. Sem. Univ. Hamburg* **22** (1958), 308–315. Also in E. Witt, *Collected papers. Gesammelte Abhandlungen*, Springer (1998).

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