Terminal 3-folds that are not Cohen-Macaulay

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An important ingredient of the minimal model program is that Kawamata log terminal singularities in characteristic zero are rational, and in particular Cohen-Macaulay. In the special case of cone singularities, this fact is related to the Kodaira vanishing theorem restricted to Fano varieties. It turns out that Kodaira vanishing fails even for Fano varieties, in every characteristic $p > 0$ [29]. This led to examples of klt, and even terminal, singularities that are not Cohen-Macaulay [21, 29, 31]. (Terminal singularities are the smallest class of singularities that can be allowed on minimal models.)

The most notable example was a terminal singularity *of dimension 3* that is not Cohen-Macaulay. Namely, let X be the quotient $(A^1 - 0)^3/G$ over the field $\mathbf{F}_2$, where the generator σ of the group $G = \mathbf{Z}/2$ acts by

$$\sigma(x_1, x_2, x_3) = \left(\frac{1}{x_1}, \frac{1}{x_2}, \frac{1}{x_3}\right).$$

Then X is terminal but not Cohen-Macaulay [29, Theorem 5.1]. This is the lowest possible dimension, because every terminal (or just normal) surface is Cohen-Macaulay. Cohen-Macaulayness and stronger properties such as F-regularity help to construct contractions of varieties. Partly for this reason, the MMP for 3-folds is known only in characteristics at least 5 [15, 8, 14]. By Arvidsson–Bernasconi–Lacini, klt singularities in characteristic greater than 5 are Cohen-Macaulay, whereas there are klt singularities that are not Cohen-Macaulay in characteristics 2, 3, and 5 [4, 6, 10].

In this paper, we construct terminal 3-fold singularities that are not Cohen-Macaulay in five new cases: mixed characteristic $(0, 2)$, characteristic 3, mixed characteristic $(0, 3)$, characteristic 5, and mixed characteristic $(0, 5)$ (Theorems 0.1, 6.1, 7.1, 8.1, and 9.1). This is optimal, in view of the result of Arvidsson–Bernasconi–Lacini. Indeed, the MMP for schemes of dimension 3 was developed in mixed characteristic when the residue characteristic is greater than 5 [7, 28]. This raised the question of whether vanishing theorems for 3-folds hold in mixed characteristic. Given our counterexample over $\mathbf{F}_2$, one might expect an example of dimension 4, flat over the 2-adic integers $\mathbf{Z}_2$, with fiber over $\mathbf{F}_2$ being the 3-fold singularity above. In fact, each of our examples has dimension 3 as a scheme. For example, over $\mathbf{Z}_2$ we have:

Theorem 0.1. *Let* $Y = \{(x, y, i) \in A^3_{\mathbf{Z}_2} : x \neq 0, y \neq 0, i^2 = -1\}$. *Let the group* $G = \mathbf{Z}/2 = \{1, \sigma\}$ *act on* Y *by*

$$\sigma(x, y, i) = (1/x, 1/y, -i).$$

Then the scheme Y/G *is terminal, not Cohen-Macaulay, of dimension 3, and flat over* $\mathbf{Z}_2$.

Note that an action of a p-group with an isolated fixed point on a positive-dimensional smooth variety in characteristic p is never formally isomorphic to a linear action, because a nonzero representation of a p-group G in characteristic p has nonzero G-fixed subspace. In fact, there are continuous families of inequivalent actions on smooth varieties in characteristic p, and likewise on regular schemes of mixed characteristic. For an action of $G = \mathbf{Z}/p$ on a regular scheme W of dimension 3 with an isolated fixed point in characteristic p (as in Theorem 0.1), it is common for W/G not to be Cohen-Macaulay, essentially because the cohomology of G contributes to the local cohomology of W/G. The difficulty is to construct an example with W/G terminal. For a more complicated action of G, the quotient scheme would usually not be terminal or even log canonical. To find the examples in this paper, the idea was to look for the simplest possible actions of $G = \mathbf{Z}/p$ on a regular 3-dimensional scheme with an isolated fixed point of residue characteristic p.

Our examples build on Artin's examples of the simplest $\mathbf{Z}/p$-actions on smooth surfaces in characteristic p with isolated fixed points [3]. Namely, he constructed a $\mathbf{Z}/2$-action in characteristic 2 with quotient a du Val singularity of type D_4, a $\mathbf{Z}/3$-action in characteristic 3 with quotient an E_6 singularity, and a $\mathbf{Z}/5$-action in characteristic 5 with quotient an E_8 singularity. These special group actions arise globally from actions on del Pezzo surfaces, for example $\mathbf{Z}/5$ acting on the quintic del Pezzo surface (as discussed in section 8).

To show that our 3-dimensional quotients W/G are terminal, the obvious approach would be to resolve the singularities of W/G and make a calculation. Resolving these singularities is hard, however. We can greatly simplify the work by stopping at a partial resolution of W/G that has toric singularities (specifically, μ_p-quotient singularities, which we call tame quotient singularities); those are easy to analyze in combinatorial terms. (Recent advances suggest that an efficient substitute for resolving singularities in any characteristic would be to seek a resolution by a tame stack, rather than by a regular scheme [2, 23, 1].) Our key technical tool is Theorem 2.2, which gives a sufficient condition for a quotient scheme U/G (where $G = \mathbf{Z}/p$, in positive or mixed characteristic) to have toric singularities.

These examples should lead to other failures of vanishing theorems. In particular, by Baudin, Bernasconi, and Kawakami, these examples imply that Frobenius-stable Grauert-Riemenschneider vanishing fails for terminal 3-folds in characteristic 2, 3, or 5 [5, Theorem 1.1].

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1 Notation

We use the notation "$x = y + I$", for elements x and y of a ring and an ideal I, to mean that there is an $i \in I$ such that $x = y + i$. We also use variants of this notation such as "$x = (y+I)(z+J)$". Another variant (modeled on big-O notation in analysis) is to write "$x = y + O(z)$" for "$x = y + (z)$".

We write $R\{x_1, \ldots, x_n\}$ for the free module over a ring R with basis elements $x_1, \ldots, x_n$.

For a closed point P in a regular scheme U with residue field k_U, we say that $f_1, \dots, f_n$ are *coordinates* for U at P (or a *regular system of parameters*) if $f_1, \dots, f_n$ are elements of the maximal ideal $\mathfrak{m}$ of $O(U)$ (the regular functions vanishing at P) that map to a basis for the k_U-vector space $\mathfrak{m} / \mathfrak{m}^2$.

For a group G acting on a scheme X, G acts on the ring of regular functions $O(X)$ by $(g(f))(x) = f(g^{-1}x)$, or equivalently $g(f) = (g^{-1})^*(f)$. The inverse is needed because of our convention of writing group actions on the left. Throughout the paper, we write $G = \mathbf{Z}/p = \langle \sigma : \sigma^p = 1 \rangle$ for the cyclic group of prime order p. We fix the name $\tau = \sigma^{-1}$, because of the inverse that comes up in writing the G-action on functions. Write $I(f) = \sigma(f) - f$ for a function f on a G-scheme.

See section 5 for the definition of the canonical divisor and terminal singularities on general schemes, following [20, section 2.1].

For a positive integer r, let μ_r be the group scheme (over any base scheme) of rth roots of unity. The Reid-Tai criterion is the following description of which cyclic quotient singularities are canonical or terminal [25, Theorem 4.11]. This is often stated over a field, but it works even in mixed characteristic for μ_r-quotient singularities. The point is that Kato's theory of log regular schemes provides a mixed-characteristic analog of toric singularities, which includes the case of μ_r-quotient singularities [17]. For such schemes, resolutions of singularities and the canonical divisor can be described in purely combinatorial terms. In the following criterion, for a an integer and b a positive integer, consider $a \bmod b$ as an integer in the set $\{0, \dots, b-1\}$.

Theorem 1.1. *For a positive integer r, let μ_r act on a regular scheme X, fixing a closed point P with maximal ideal $\mathfrak{m}$. Suppose that μ_r acts on a basis for $\mathfrak{m} / \mathfrak{m}^2$ by $\zeta(t_1, \dots, t_n) = (\zeta^{b_1} t_1, \dots, \zeta^{b_n} t_n)$, for some $b_1, \dots, b_n \in \mathbf{Z}/r$. (The quotient X/μ_r is said to be a μ_r-quotient singularity of type $\frac{1}{r}(b_1, \dots, b_n)$.) Assume that the action is well-formed in the sense that $\gcd(r, b_1, \dots, \widehat{b_j}, \dots, b_n) = 1$ for all $j = 1, \dots, n$. Then X/μ_r is canonical (resp. terminal) near the image of P if and only if*

$$\sum_{j=1}^{n} ib_j \bmod r \geq r$$

(resp. $> r$) for all $i = 1, \dots, r-1$.

We sometimes need the extension of the Reid-Tai criterion that describes when a toric pair is terminal, as follows. The proof is the same as Reid's: X/μ_r has a toric resolution of singularities, and so it suffices to compute discrepancies for toric divisors over X/μ_r.

Theorem 1.2. *Under the assumptions of Theorem 1.1, let $t_1, \dots, t_n$ be coordinates for X at P that are μ_r-eigenfunctions with weights $b_1, \dots, b_n$. For $i = 1, \dots, n$, let D_i be the irreducible divisor on X/μ_r that is the image of $\{t_i = 0\}$ in X. Let $c_1, \dots, c_n$ be real numbers. Then the pair $(X/\mu_r, \sum c_j D_j)$ is terminal near the image of P if and only if $c_j < 1$ for each j, $c_j + c_k < 1$ for each $j \neq k$, and*

$$\sum_{j=1}^{n} (1 - c_j)(ib_j \bmod r) > r$$

for all $i = 1, \dots, r-1$.

2 Recognizing μ_p-quotient singularities

Király and Lütkebohmert analyzed when a quotient scheme by $\mathbf{Z}/p$ is regular, the hard case being where the residue characteristic is p [18]. More generally, we now give a sufficient condition for a quotient by $\mathbf{Z}/p$ to be a toric singularity, or in particular to be a μ_p-quotient singularity. (Actions of μ_p are far simpler than actions of $\mathbf{Z}/p$: they are linearizable near a fixed point, like finite group actions in characteristic zero, because μ_p is linearly reductive. Also, μ_p-quotient singularities are always Cohen-Macaulay.)

It would be very appealing to find a broader sufficient condition for a quotient by $\mathbf{Z}/p$ to have toric singularities, perhaps necessary and sufficient. Theorem 2.2 is adapted to the situation where the fixed point scheme is a Cartier divisor $E_1 = \{e = 0\}$ except on a closed subset of E_1. When it applies, the theorem can be described as a $\mathbf{Z}/p - \mu_p$ switch.

Here is Király and Lütkebohmert's main result [18, Theorem 2]. (See Theorem 2.6 for a more detailed statement.)

Theorem 2.1. *Let G be a cyclic group of prime order which acts on a regular scheme X. If the fixed point scheme X^G is a Cartier divisor in X, then the quotient space X/G is regular.*

Here is our sufficient condition for toric singularities. Recall that $I(f)$ means $\sigma(f) - f$, for a function f on a scheme with an action of the group $G = \mathbf{Z}/p = \langle \sigma : \sigma^p = 1\rangle$.

Theorem 2.2. *Let U be a regular scheme with an action of the group $G = \mathbf{Z}/p = \langle \sigma : \sigma^p = 1\rangle$, for a prime number p. Suppose that G fixes a closed point P with perfect residue field k_U of characteristic p. Write $\mathfrak{m}$ for the maximal ideal in the local ring $O_{U,P}$. Suppose that there are $e, s \in \mathfrak{m} - \{0\}$ and coordinates $x_1, \ldots, x_n$ for U at P such that*

$$I(s) = es(\textit{unit})$$

and

$$I(x_i) \in (e)$$

for $i = 1, \ldots, n$. Suppose that $p \in e^{p-1}\,\mathfrak{m}$. (For example, that holds if $p = 0$ on U.) Then U/G is regular or has a μ_p-quotient singularity at the image of P.

More precisely, if $I(x_i)/e$ is nonzero at P for some i, then U/G is regular near P. Otherwise, let $\varphi(y) = I(y)/e$, which gives a linear map from $\mathfrak{m}/\mathfrak{m}^2$ to itself. Then φ is diagonalizable, and after multiplying it by some nonzero scalar, its eigenvalues $b_1, \ldots, b_n$ are in $\mathbf{F}_p \subset k_U$. In this case, U/G has a singularity of the form $\frac{1}{p}(b_1, \ldots, b_n)$. That is, near the image of P, U/G is the quotient of a regular scheme by an action of μ_p with weights $b_1, \ldots, b_n$ at a fixed point.

Theorem 4.1 is a refinement of Theorem 2.2, showing that certain divisors in U can be viewed as toric divisors in U/G.

We will often use the special case of Theorem 2.2 where $e = s = x_1$ for one of the coordinates x_1; in that case, we are assuming that $I(x_1) = x_1^2(\text{unit})$ and $I(x_i) \in (x_1)$ for $i = 1, \ldots, n$. Then the fixed point scheme U^G is the regular divisor $\{x_1 = 0\}$ except on a closed subset of E_1. For other applications in this paper, we need the greater generality of Theorem 2.2, where $(U^G)_{\text{red}}$ may be a singular divisor.

One might ask whether the assumption that $p \in e^{p-1}\,\mathfrak{m}$ in Theorem 2.2 can be omitted. Fortunately, this assumption is automatic in characteristic p, and it will be easy to check in our mixed-characteristic examples.

Proof. The assumption that P is fixed by G means that G preserves $\mathfrak{m} = \mathfrak{m}_U$ and acts as the identity on the residue field $k_U := O_{U,P}/\,\mathfrak{m}$, which has characteristic p. For each regular function f in the local ring $O_{U,P}$, its norm $N(f) := \prod_{i=0}^{p-1} \sigma^i(f)$ is G-invariant, and it maps to f^p in k_U by triviality of the G-action there. So the residue field $k_{U/G} \subset k_U$ at the image of the point P contains k_U^p. Since we assume that k_U is perfect, $k_{U/G}$ is equal to k_U. In what follows, we replace U by a G-invariant open neighborhood of P as needed, since we are only trying to describe U/G near the image of P (which we also call P). In particular, we can assume that U is affine.

The fixed point scheme U^G is defined as the closed subscheme of U cut out by the ideal generated by $I(O(U))$. The following formulas hold for any action of G on a commutative ring [18, Remark 3]:

Lemma 2.3. *(1) $I(xy) = I(x)\sigma(y) + xI(y)$.*
(2) For $m \geq 0$, $I(x^m) = I(x)\sum_{i=1}^{m} \sigma(x)^{i-1}x^{m-i}$.

We are given coordinates $x_1, \ldots, x_n$ for U near P. Under our assumptions (U regular and $k_{U/G} = k_U$), the ideal generated by $I(O_{U,P})$ in $O_{U,P}$ is generated by $I(x_1), \ldots, I(x_n)$ [18, Proposition 6]. That is, after shrinking U around P if necessary, the fixed point scheme U^G is the closed subscheme defined by $I(x_1), \ldots, I(x_n)$.

In particular, if $I(x_i)/e$ is a unit for some $i = 1, \ldots, n$, then the fixed point scheme U^G is the Cartier divisor $\{e = 0\}$, and then Theorem 2.1 gives that U/G is regular. *So we can assume from now on that $I(x_i)$ is in $e\,\mathfrak{m}$ for each i.* Equivalently, after shrinking U around P, $I(O(U))$ is contained in $e\,\mathfrak{m}$. In this case, we will show that U/G has a μ_p-quotient singularity at P.

The following lemma is implicit in the statement of Theorem 2.2.

Lemma 2.4. *Let U be a regular scheme with an action of the group $G = \mathbf{Z}/p = \langle \sigma : \sigma^p = 1 \rangle$. Suppose that G fixes a closed point P with perfect residue field k_U of characteristic p. Write $\mathfrak{m}$ for the maximal ideal in the local ring $O_{U,P}$. Let $e \in \mathfrak{m}$, $e \neq 0$, such that $I := \sigma - 1$ satisfies $I(\mathfrak{m}) \subset e\,\mathfrak{m}$. Then $\varphi(y) := I(y)/e$ is a well-defined k_U-linear map from $\mathfrak{m}/\mathfrak{m}^2$ to itself.*

Proof. Since the local ring $O_{U,P}$ is regular, it is a domain, and so $I(y)/e$ is well-defined for each element y in $\mathfrak{m}$. Since G fixes the point P, G maps $\mathfrak{m}$ into itself. Since $I(\mathfrak{m}) \subset e\,\mathfrak{m}$, Lemma 2.3 gives that $I(\mathfrak{m}^2) \subset e\,\mathfrak{m}^2$. Since I is additive, it follows that φ is a well-defined additive function from $\mathfrak{m}/\mathfrak{m}^2$ to itself. By Lemma 2.3, φ is linear over the ring of invariants $(O_{U,P})^G$. Since that ring has residue field $k_{U/G} = k_U$, φ is k_U-linear. □

We are given a function $s \in \mathfrak{m} - \{0\}$ such that $I(s) = es(\text{unit})$. After multiplying e by a unit, we can assume that $I(s) = es$; this does not change the other assumption that $I(x_i) \in (e)$ for $i = 1, \ldots, n$. *Thus, from now on, we have $I(s) = es$.* This changes the endomorphism φ of $\mathfrak{m}/\mathfrak{m}^2$ (which is defined in terms of e) by a nonzero scalar. Having made this change, we will show that φ is diagonalizable, with eigenvalues $b_1, \ldots, b_n$ in $\mathbf{F}_p \subset k_U$, and that U/G has a μ_p-quotient singularity of the form $\frac{1}{p}(b_1, \ldots, b_n)$.

Let $v = \sigma(s)/s$. By our assumptions, v is a unit on U, and $v = 1 + e$. Since $v = \sigma(s)/s$, v has norm 1 for the action of G. Write $(\sigma/\operatorname{id})(x)$ for $\sigma(x)/x$; this is the multiplicative action of $\sigma - 1 \in \mathbf{Z}G$ on a commutative ring. In these terms, define $f = (\sigma/\operatorname{id})^{p-2}(v)$. In the group ring $\mathbf{Z}G$, we have $(\sigma - 1)^{p-1} \equiv \sigma^{p-1} + \cdots + \sigma + 1 \pmod{p}$, and so we can define an element $\alpha \in \mathbf{Z}G$ by

$$(\sigma - 1)^{p-1} = \sigma^{p-1} + \cdots + \sigma + 1 - p\alpha. \qquad (*)$$

It follows that $\sigma(f)/f = N(v)/g^p$, with $g := \alpha(v)$ (where α acts multiplicatively). Since v has norm 1, we have $\sigma(f)/f = 1/g^p$. This formula will be exactly what we need to construct a μ_p-torsor $W \to U$ with a commuting action of G.

We first analyze the function g in more detail. By equation (*), the sum of the coefficients of α in $\mathbf{Z}G$ is 1. Therefore, $g = \alpha(v)$ is a product of integer powers of the functions $\sigma^j(v) = 1 + e(1 + \mathfrak{m})$ with total exponent 1, and so $g = 1 + e(1 + \mathfrak{m})$.

Lemma 2.5. *For any pth root of unity ζ in $O(U)$, $1 - \zeta$ is in $e\,\mathfrak{m}$.*

Proof. The statement is clear if $\zeta = 1$. So assume that $\zeta \neq 1$. Since $O(U)$ is a domain and $(1-\zeta)(1+\zeta+\cdots+\zeta^{p-1}) = 1-\zeta^p = 0$, we must have $1+\zeta+\cdots+\zeta^{p-1} = 0$. That is, we have a homomorphism from the ring of integers $\mathbf{Z}[\zeta_p]$ in $\mathbf{Q}(\zeta_p)$ to $O(U)$, taking the primitive pth root of unity ζ_p to ζ. In $\mathbf{Z}[\zeta_p]$, p is $(1 - \zeta_p)^{p-1}$ times a unit [22, section IV.1], and so the same is true in $O(U)$. We are given that p is in $e^{p-1}\,\mathfrak{m}$. Since U is regular, $O_{U,P}$ is a unique factorization domain, and so $1 - \zeta$ must be a multiple of e; write $1 - \zeta = ea$ for some $a \in O_{U,P}$. If a is a unit, then p would be e^{p-1} times a unit, a contradiction. So, as an element of $O_{U,P}$, $1 - \zeta$ is in $e\,\mathfrak{m}$. After shrinking U around P if necessary, this gives the same conclusion in $O(U)$. ☐

The reason for constructing units f and g with $\sigma(f)/f = 1/g^p$ is to define a μ_p-torsor over U with a commuting action of G. Namely, define a μ_p-torsor $W \to U$ by $w^p = f$. Here μ_p acts on $W \subset A^1 \times U$ by $\zeta(w, x) = (\zeta w, x)$, for $\zeta \in \mu_p$. Write $\tau = \sigma^{-1}$ in G. Since $\sigma(f)/f = 1/g^p$, W has an action of G that commutes with the action of μ_p, by $\tau(w, x) = (w/g(x), \tau(x))$. (We check these properties in the next paragraph.) In particular, $\sigma(w) = \tau^*(w) = w/g$, by definition of the G-action on functions. We will show that the scheme $Q := W/G$ is regular. Then $U/G = Q/\mu_p$ will be a quotient of a regular scheme by μ_p, as we want.

$$\begin{array}{ccc} W & \xrightarrow{\;G\;} & Q \\ \big\downarrow{\scriptstyle \mu_p} & & \big\downarrow{\scriptstyle \mu_p} \\ U & \xrightarrow{\;G\;} & U/G. \end{array}$$

For convenience, we write P for the closed point of interest in each of these schemes. (There is a unique closed point in W over P in U, and it maps to a closed point in Q and in $U/G = Q/\mu_p$.) We have seen that the points $P \in U$ and $P \in U/G$ have the same residue field k_U. Since $P \in W$ is given by the equations $w = 1$ and $x_1 = \cdots = x_n = 0$, we see that $P \in W$ has the same residue field k_U. Since $k_{U/G} \subset k_Q \subset k_W$, $P \in Q$ also has the same residue field.

For clarity, let us first check that the formulas above give an action of G on W that commutes with the action of μ_p. First, to show that σ as above maps $W = \{w^p = f\}$ into itself, we have to show that if $w^p = f(x)$, then $(w/g(x))^p = f(\sigma^{-1}(x))$, or equivalently that $w^p/g^p = \sigma(f)$; this follows from the fact that $\sigma(f)/f = 1/g^p$. Next, let us show that $\sigma^p = 1$ on W. By induction, we have $\sigma^{-i}(w,x) = (w/(g(x)g(\sigma^{-1}x)\cdots g(\sigma^{1-i}x)), \sigma^{-i}x)$ for each natural number i. Therefore, to show that σ^p is the identity on W (hence that we have an action of G), it suffices to show that g has norm 1. But that is true, because v has norm 1 and g is a product of integer powers of $\sigma^j(v)$ for integers j. So we have an action of G on W. Finally, to show that G and μ_p commute on W: for $\zeta \in \mu_p$, we have $\zeta\sigma^{-1}(w,x) = \zeta(w/g(x), \sigma^{-1}(x)) = (\zeta w/g(x), \sigma^{-1}(x))$ while $\sigma^{-1}\zeta(w,x) = \sigma^{-1}(\zeta w, x) = (\zeta w/g(x), \sigma^{-1}(x))$. These are equal as regular functions on the scheme $\mu_p \times W$. So we have shown that G and μ_p commute on W.

We need to show that f is not a pth power in $O(U)$. Suppose it is, say $f = u^p$ for a regular function u on U. Since f is a unit, so is u. Then $1/g^p = \sigma(f)/f = (\sigma(u)/u)^p$, and so $\zeta/g = \sigma(u)/u$ for some pth root of unity ζ in $O(U)$. Here $\zeta = 1+(\zeta-1) = 1+e\,\mathfrak{m}$ by Lemma 2.5, and $1/g = 1-e(1+\mathfrak{m})$, so $\sigma(u)/u = 1-e(1+\mathfrak{m})$. Here $\sigma(u)/u = 1 + I(u)/u$, so $I(u) = -ue(1+\mathfrak{m}) = e(\text{unit})$. This contradicts our assumption that $I(O(U))$ is contained in $e\,\mathfrak{m}$. So in fact f is not a pth power.

From there, we can show that the scheme W is integral (after shrinking U around P, if necessary). Namely, since f is not a pth power in $O(U)$, f is also not a pth power in the function field $k(U)$, and so $k(U)[f^{1/p}]$ is a degree-p field extension of $k(U)$. Write α for the μ_p-torsor $W \to U$. Since $W = \{w^p = f\}$, there is a nonempty open subset $V \subset U$ with $\alpha^{-1}(V)$ integral. Since $W \to U$ is finite and flat, W is Cohen-Macaulay and equidimensional. By equidimensionality, every irreducible component of W must dominate U. Since $\alpha^{-1}(V)$ is irreducible, it follows that W is irreducible. Since $\alpha^{-1}(V)$ is reduced, W is reduced in codimension 0; since W is Cohen-Macaulay, it follows that W is reduced [27, Tag 031R]. Since W is reduced and irreducible, it is integral.

It is not needed for what follows, but for clarity, let us analyze the singularities of W in the special case where $s = e(1+\mathfrak{m})$ (for example when $s = e$), so that $I(e) = e^2(1+\mathfrak{m})$. The equation of W is $w^p = f$. Since $v = 1+e$, one can show by induction from the formula for $I(e)$ that $f = (\sigma/\operatorname{id})^{p-2}(v) = 1 + e^{p-1}q$ for some unit q on U, and so we can rewrite the equation of W as $w^p = 1 + e^{p-1}q$. Let $w_0 = w - 1$, so that w_0 vanishes at the point P of interest in W. In terms of w_0, the equation of W becomes $(1+w_0)^p = 1 + e^{p-1}q$, that is,

$$w_0^p = e^{p-1}q - \binom{p}{1}w_0 - \cdots - \binom{p}{p-1}w_0^{p-1}.$$

We are given that p is in $e^{p-1}\,\mathfrak{m}$, and so this equation has the form $w_0^p = e^{p-1}s$ for some unit s on W. If $p = 2$ and $e \notin \mathfrak{m}^2$, it follows that W is regular. However, if $p > 2$, then W is not normal. For example, if $e \notin \mathfrak{m}^2$, then the singularity of W looks roughly like the cuspidal curve $\{w_0^p = x_1^{p-1}\}$ times a smooth variety.

We will need the following version of Király and Lütkebohmert's results.

Theorem 2.6.

(1) Let B be a local domain with residue field k_B. Let p be a prime number, and let $G = \mathbf{Z}/p$ act nontrivially on B. Suppose that the ideal $B \cdot I(B)$ that defines the fixed point scheme in $\operatorname{Spec} B$ *is generated by one element. Then B is free*

of rank p over the ring of invariants B^G. More precisely, for any element t such that $I(t)$ generates the ideal $B \cdot I(B)$, we have $B = A\{1, t, \ldots, t^{p-1}\}$.

(2) *In addition to the assumptions of (1), assume that B is regular. Then B^G is regular.*

(3) *In addition to the assumptions of (1), assume that B is noetherian and the inclusion $k_{B^G} \subset k_B$ is an equality. Then there is a minimal set of generators $y_1, \ldots, y_r$ for $\mathfrak{m}_B$ such that $I(y_1)$ generates the ideal $B \cdot I(B)$ and $y_2, \ldots, y_r$ are G-invariants.*

Proof. Statement (1) is due to Király and Lütkebohmert for B normal, but their proof works without change for B a domain [18, Theorem 2 and Proposition 5]. They also prove statement (2). They prove statement (3) when B is regular. We now extend the proof of (3) for B only a domain.

Since the inclusion $k_{B^G} \subset k_B$ is an equality, we have $B = B^G + \mathfrak{m}_B$, and so $I(B) = I(\mathfrak{m}_B)$. Since the ideal $B \cdot I(B)$ is generated by one element, there is an element $y_1 \in \mathfrak{m}_B$ such that $I(y_1)$ generates this ideal. By Lemma 2.3, we have $I(\mathfrak{m}_B^2) \subset \mathfrak{m}_B I(\mathfrak{m}_B)$. Here $I(\mathfrak{m}_B)$ is not zero (because the G-action is nontrivial), and so y_1 is not in $\mathfrak{m}_B^2$.

Since B is noetherian, the ideal $\mathfrak{m}_B$ is finitely generated. Choose elements $z_2, \ldots, z_r$ in $\mathfrak{m}_B$ such that $y_1, z_2, \ldots, z_r$ form a basis for $\mathfrak{m}_B / \mathfrak{m}_B^2$. By part (1), we know that $B = B^G\{1, y_1, \ldots, y_1^{p-1}\}$. For $2 \leq i \leq r$, let y_i be the projection of z_i to B^G with respect to this decomposition. Then $y_i \equiv z_i \pmod{(y_1) + \mathfrak{m}_B^2}$, and so $y_1, \ldots y_r$ map to a basis for $\mathfrak{m}_B / \mathfrak{m}_B^2$. Thus $y_1, \ldots, y_r$ are a minimal set of generators for $\mathfrak{m}_B$ (by Nakayama's lemma), and $y_2, \ldots, y_r$ are G-invariant. □

Let us write out the action of G on W. The maximal ideal of P in W is generated by $w_0, x_1, \ldots, x_n$. We have $I(w_0) = I(w) = w(\frac{1}{g} - 1) = (1 + w_0)(\frac{1}{g} - 1)$. Since $1/g = 1 + eu$ for some unit u on U, we have $I(w_0) = eu(1 + w_0) = e(\text{unit})$. We also have $I(x_i)$ in the ideal (e) for $i = 1, \ldots, n$; so the fixed point scheme W^G is defined by the single equation $e = 0$ in W. As a result, even though W is typically not normal, Theorem 2.6 gives that the morphism $W \to W/G = Q$ is finite and faithfully flat of degree p. It follows that Q is noetherian [27, Tag 033E]. (Beware that for a general noetherian scheme X with an action of a finite group G, X/G need not be noetherian, and the morphism $X \to X/G$ need not be finite [24, Proposition 0.10]. These properties do hold if X is of finite type over a noetherian ring A and G acts A-linearly [12, Theorem and Corollary 4].)

The action of μ_p on the affine scheme Q gives a grading of $O(Q)$ by $\mathbf{Z}/p$. For each $j \in \mathbf{Z}/p$, since $O(Q)$ is noetherian, the ideal in $O(Q)$ generated by the jth graded piece $O(Q)_j$ is finitely generated, and so $O(Q)_j$ is a finitely generated module over $O(Q)_0 = O(Q/\mu_p)$. So the whole ring $O(Q)$ is finite over $O(Q/\mu_p)$; that is, $Q \to Q/\mu_p$ is finite. Also, $O(Q/\mu_p)$ is a pure subring (because it is a summand) of the noetherian ring $O(Q)$, so it is noetherian [16, Proposition 6.15]; that is, $Q/\mu_p = U/G$ is noetherian. Finally, the composition $W \to Q \to Q/\mu_p = U/G$ is finite, and $O(U)$ is a sub-$O(U/G)$-module of $O(W)$, so $O(U)$ is a finitely generated $O(U/G)$-module; that is, $U \to U/G$ is finite.

Let $h_1, \ldots, h_r$ be a minimal set of generators for the maximal ideal at P of $O(Q)$. (So r is at least the dimension n of Q.)

Lemma 2.7. *The ideals $(h_1, \ldots, h_r)$ and $(x_1, \ldots, x_n)$ in $O(W)$ are equal. (That is: the fiber in W over the closed point $P \in Q$ is equal to the fiber in W over the closed point $P \in U$, as a closed subscheme.)*

Proof. We have seen that the degree-p morphism $W \to W/G = Q$ is finite and flat. So the fiber in W over the point P in Q has degree p over the residue field of $P \in Q$, which we have seen is k_U. As a set, this fiber is one point $P \in W$, with the same residue field k_U. So the quotient ring $O(W)/(h_1, \ldots, h_r)$ is an artinian local ring of length p.

The ideal $(x_1, \ldots, x_n)$ in $O(W)$ defines the fiber in W over the point P in U. Since $W \to U$ is a μ_p-torsor, this fiber has degree p over the residue field of $P \in U$, which is the same field k_U. Again, this fiber is one point $P \in W$ as a set, with the same residue field; so $O(W)/(x_1, \ldots, x_n)$ is an artinian local ring of length p. So if we can show that the ideal $(h_1, \ldots, h_r)$ in $O(W)$ is contained in $(x_1, \ldots, x_n)$ in $O(W)$, then they are equal.

It suffices to show (*) that every function y on W that vanishes at the point P in W but has nonzero image in $O(W)/(x_1, \ldots, x_n)$ has $I(y) \neq 0$ (that is, it is not G-invariant). (Namely, this would imply that the G-invariant functions $h_1, \ldots, h_n$ on W lie in the ideal $(x_1, \ldots, x_n)$, as we want.) By the formula for the G-action on W, in particular that $\sigma(w) = w/g$ where $g = 1 + e(1 + \mathfrak{m})$, we see that G fixes the closed subscheme $\{e = 0\}$ in W. That is, I maps $O(W)$ into the ideal (e) in $O(W)$. Also, we know that $I(x_i) \in e\,\mathfrak{m}_U = e(x_1, \ldots, x_n) \subset O(U)$ for $i = 1, \ldots, n$. So $\varphi(y) := I(y)/e$ is a well-defined linear map from $O(W)/(x_1, \ldots, x_n)$ to itself. Explicitly, by the equation of W, $O(W)/(x_1, \ldots, x_n)$ is a k_U-vector space with basis $1, w, \ldots, w^{p-1}$. Equivalently, in terms of $w_0 = w - 1$ (which vanishes at P in W), a basis for $O(W)/(x_1, \ldots, x_n)$ is given by $1, w_0, \ldots, w_0^{p-1}$.

The claim (*) will follow if the map φ restricted to $k\{w_0, w_0^2, \ldots, w_0^{p-1}\}$ is injective. Since $g = 1 + e(1 + \mathfrak{m}_U)$, we have $1/g = 1 - e(1 + \mathfrak{m}_U)$, and hence $I(w_0) = I(w) = (w/g) - w = -ew(1 + \mathfrak{m}_U) = -e(1 + w_0)(1 + \mathfrak{m}_U)$. So $\varphi(w_0) = -(1 + w_0)$. By Lemma 2.3, for $m \geq 0$,

$$\begin{aligned} I(w_0^m) &= I(w_0) \sum_{j=1}^{m} \sigma(w_0)^{j-1} w_0^{m-j} \\ &= -e(1 + w_0)(1 + \mathfrak{m}_U) \sum_{j=1}^{m} (w_0 - (1 + w_0)e(1 + \mathfrak{m}_U))^{j-1} w_0^{m-j}. \end{aligned}$$

Since φ takes values in $O(W)/(x_1, \ldots, x_n)$ (where e is zero), it follows that $\varphi(w_0^m) = -m w_0^m (1 + w_0)$. It is clear that these elements are linearly independent over k_U for $m = 1, \ldots, p-2$; to show that they are linearly independent for $m = 1, \ldots, p-1$, it will suffice to show that w_0^p is zero in $O(W)/(x_1, \ldots, x_n) = k_U\{1, w_0, \ldots, w_0^{p-1}\}$. (This comes up because w_0^p appears in our formula for $I(w_0^{p-1})$.)

Namely, we have $w_0^p = (w-1)^p = w^p - 1$ plus a multiple of p in $O(W)$. Here $w^p - 1 = f - 1 = e^{p-1}(1 + \mathfrak{m}_U)$, and p is in $e^{p-1}\mathfrak{m}_U$, as we assumed; so $w_0^p = e^{p-1}(1 + \mathfrak{m}_U)$, which is zero in $O(W)/(x_1, \ldots, x_n)$, as we want. Lemma 2.7 is proved. ∎

The number r of generators $h_1, \ldots, h_r$ for the maximal ideal $\mathfrak{m}_Q$ in the local ring $O_{Q,P}$ is at least $n = \dim(Q)$. On the other hand, Lemma 2.7 implies that the extended ideal $(h_1, \ldots, h_r)$ in $O_{W,P}$ can be generated by only n elements, so the vector space $(h_1, \ldots, h_r) \otimes_{O_{W,P}} k_W$ has dimension at most n. So this vector space is spanned by n of the h_i's, which we can assume are $h_1, \ldots, h_n$. By Nakayama's lemma, it follows that the extended ideal $(h_1, \ldots, h_r)$ is equal to the extended ideal $(h_1, \ldots, h_n)$ in $O_{W,P}$. Since $W \to Q$ is faithfully flat, extending and contracting an ideal in $O_{Q,P}$ gives the same ideal [27, Tag 05CK]. As a result, we have $(h_1, \ldots, h_n) = (h_1, \ldots, h_r)$ in $O_{Q,P}$. That is, the maximal ideal $\mathfrak{m}_Q$ can be generated by $n = \dim(Q)$ elements, which means that Q is regular. (This is somewhat surprising, since W is typically not regular or even normal.)

It remains to show that the point P in Q is a fixed point for μ_p, with weights given by the eigenvalues of φ. First, let us show that μ_p fixes the point P in Q. (This does not seem obvious, since μ_p does not fix the point P in W; in fact, μ_p acts freely on W.) The functions $x_1, \ldots, x_n$ on W are pulled back from U, hence fixed by μ_p. As a result, the ideal $(x_1, \ldots, x_n)$ in $O(W)$ is preserved by μ_p. Equivalently, by Lemma 2.7, the ideal $(h_1, \ldots, h_n)$ in $O(W)$ is preserved by μ_p. We have seen that the morphism $W \to W/G = Q$ is faithfully flat. As a result, the ideal $(h_1, \ldots, h_n)$ in $O(Q)$ is equal to the intersection of the extended ideal $(h_1, \ldots, h_n)$ in $O(W)$ with $O(Q)$. Since $W \to Q$ is μ_p-equivariant, it follows that the ideal $(h_1, \ldots, h_n) = \mathfrak{m}_Q$ in $O(Q)$ is preserved by μ_p. Also, the residue field of Q/μ_p at P maps isomorphically to the residue field of Q at P, and so μ_p acts trivially on the latter field. That is, μ_p fixes the point P in Q, as we want.

We now change our choice of the functions $h_1, \ldots, h_n$. Since μ_p is linearly reductive, we can choose coordinates $h_1, \ldots, h_n$ for Q near P that are μ_p-eigenfunctions. That is, each h_i has some weight $b_i \in \mathbf{Z}/p$ for the action of μ_p. In these terms, $Q/\mu_p = U/G$ is a toric singularity of type $\frac{1}{p}(b_1, \ldots, b_n)$. It remains to show that the endomorphism φ of $\mathfrak{m}_U / \mathfrak{m}_U^2$ is diagonalizable, with eigenvalues in $\mathbf{F}_p \subset k_U$, and that these eigenvalues are equal to $b_1, \ldots, b_n$.

Consider $h_1, \ldots, h_n$ as G-invariant functions on W. Here W is a μ_p-torsor over U defined by $w^p = f$; so $O(W) = O(U)\{1, w, \ldots, w^{p-1}\}$, and this grading by $\mathbf{Z}/p$ describes the action of μ_p on $O(W)$. For $i = 1, \ldots, n$, h_i has weight b_i for the action of μ_p, and so we can write $h_i = g_i w^{b_i}$ for some regular function g_i on U. (Here we think of b_i as an integer in $\{0, \ldots, p-1\}$.) Clearly the functions $g_1, \ldots, g_n$ vanish at P (using that w is a unit). Also, $(g_1, \ldots, g_n)$ is equal to $(h_1, \ldots, h_n)$ as an ideal in $O(W)$, and we showed that the latter ideal is equal to $(x_1, \ldots, x_n)$ in $O(W)$. Since $W \to U$ is faithfully flat, it follows that $(g_1, \ldots, g_n)$ is equal to $(x_1, \ldots, x_n)$ as an ideal in $O(U)$. That is, $g_1, \ldots, g_n$ form coordinates on U near P.

For $1 \leq i \leq n$, we have $I(g_i) \equiv e\varphi(g_i) \in e(\mathfrak{m}_U / \mathfrak{m}_U^2)$, by definition of the endomorphism φ of $\mathfrak{m}_U / \mathfrak{m}_U^2$. We showed above that $I(w_0) = -e(1 + w_0)(1 + \mathfrak{m}_U)$, and so $I(w) = I(w_0) = -ew(1 + \mathfrak{m}_U)$. For each $b \geq 0$, Lemma 2.3 gives that

$$\begin{aligned} I(w^b) &= I(w) \sum_{m=1}^{b} \sigma(w)^{m-1} w^{b-m} \\ &= -bew^b(1 + \mathfrak{m}_U), \end{aligned}$$

using that e is in $\mathfrak{m}_U$. Since the function $g_i w^{b_i}$ is G-invariant on W, we have $0 = I(g_i w^{b_i}) = \sigma(g_i) I(w^{b_i}) + I(g_i) w^{b_i} = (g_i + I(g_i)) I(w^{b_i}) + I(g_i) w^{b_i}$. When we consider this equality modulo $e\,\mathfrak{m}_U^2\, O(W)$, the term $I(g_i) I(w^{b_i})$ can be omitted.

Namely, we have

$$\begin{aligned} 0 &\equiv e(-b_i g_i w^{g_i}(1+\mathfrak{m}) + \varphi(g_i) w^{g_i}) \pmod{e\,\mathfrak{m}_U^2\, O(W)} \\ &\equiv e w^{g_i}(-b_i g_i + \varphi(g_i)) \pmod{e\,\mathfrak{m}_U^2\, O(W)}. \end{aligned}$$

Since w is a unit, it follows that $0 \equiv e(-b_i g_i + \varphi(g_i)) \pmod{e\,\mathfrak{m}_U^2\, O(W)}$. Since $W \to U$ is faithfully flat, $e\,\mathfrak{m}_U / e\,\mathfrak{m}_U^2 \to e\,\mathfrak{m}_U\, O(W)/e\,\mathfrak{m}_U^2\, O(W)$ is injective, and so $0 \equiv e(-b_i g_i + \varphi(g_i)) \pmod{e\,\mathfrak{m}_U^2}$. So $\varphi(g_i) = b_i g_i$ in $\mathfrak{m}_U / \mathfrak{m}_U^2$ for each $i = 1, \ldots, n$. Also, $g_1, \ldots, g_n$ form a basis for $\mathfrak{m}_U / \mathfrak{m}_U^2$. So φ is diagonalizable, its eigenvalues $b_1, \ldots, b_n$ are in $\mathbf{F}_p$, and U/G is a μ_p-quotient singularity of the form $\frac{1}{p}(b_1, \ldots, b_n)$, as we want. □

3 Review of ramification theory

We recall here how to compute the ramification behavior of a $\mathbf{Z}/p$-covering in characteristic p or mixed characteristic, following Xiao and Zhukov [30].

Let $G = \mathbf{Z}/p = \langle \sigma : \sigma^p = 1 \rangle$ act nontrivially on a normal noetherian integral scheme Y. Assume that Y is of finite type over a field or over $\mathbf{Z}_p$, so that we can talk about the canonical class K_Y. Write f for the quotient map $Y \to Y/G$. For each irreducible divisor E in Y that is mapped into itself by G, let F be its image (as an irreducible divisor) in Y/G. Assume that $p = 0$ on E. There are several invariants we want to compute in this situation: the *ramification index* of the divisor E in Y (the positive integer e such that $f^*F = eE$), and the coefficient c of E in the *ramification divisor* (meaning that $K_Y = f^*K_{Y/G} + cE$ near the generic point of E). Another name for c is the *valuation of the different* $v_E(\mathcal{D}_{k(Y)/k(Y/G)})$, where the valuation v_E on the function field $k(Y)$ is the order of vanishing along E. Here $ef = p$, where f is the degree of the field extension $k(E)$ over $k(F)$.

An easy case is where G does not fix E (in other words, where G acts nontrivially on E). We say that f is *unramified* along E. In this case, $f^*F = E$, and $K_Y = f^*K_{Y/G}$ near the generic point of E.

Define the *Artin ramification number* $i(E)$ of Y over Y/G along E to be the coefficient of E in the fixed point scheme Y^G. Equivalently, in terms of $I(a) = \sigma(a) - a$:

$$i(E) = \min_{a \in O_{Y,E}} v_E(I(a)).$$

In the ramified case, the field $k(E)$ is purely inseparable over $k(F)$, with degree f equal to 1 or p. We can distinguish the two cases as follows. Since $ef = p$, either $e = p$ and $f = 1$ (called *wild* ramification) or $e = 1$ and $f = p$ (called *fierce* ramification). Let t be a defining function of E, that is, a rational function on Y with valuation $v_E(t) = 1$. It is clear that $v_E(I(t)) \geq i(E)$. A very convenient criterion is: $Y \to Y/G$ is wildly ramified along E if and only if equality holds [30, section 2.1]. Otherwise, it is fiercely ramified.

Furthermore, we can compute the relative canonical class (that is, the valuation of the different) as follows, correcting a typo in [30, section 2.1].

Lemma 3.1. *The valuation of the different is $(p-1)i(E)$. When Y is regular, so that Y/G is $\mathbf{Q}$-factorial, we can equivalently say that*

$$K_Y = f^*K_{Y/G} + (p-1)[Y^G],$$

where $[Y^G]$ denotes the Weil divisor associated to the fixed point scheme.

Proof. If G acts nontrivially on E (the unramified case), then $i(E) = 0$ and the statement is clear. So assume that G fixes E. The local ring $O_{Y,E}$ is a discrete valuation ring. The algebra $O_{Y,E}$ is generated by one element y as an algebra over $O_{Y/G,F}$; one can take y to be a uniformizer in $O_{Y,E}$ if E is wildly ramified, and an element of $O_{Y,E}$ whose restriction to $k(E)$ is not in $k(F)$ if E is fiercely ramified [30, section 2.1]. By Lemma 2.3, we have $i(E) = v_E(I(y))$.

Let $u(X)$ be the minimal polynomial of y over $k(Y/G)$. In this situation of a monogenic algebra extension, the different $\mathcal{D}_{k(Y)/k(Y/G)}$ is generated by $u'(y)$ [26, III, Corollary 2 to Proposition 11]. But $u(X) = \prod_{j=0}^{p-1}(X - \sigma^j(y))$. So

$$u'(y) = \prod_{j=1}^{p-1}(y - \sigma^j(y)),$$

and hence

$$v_E(\mathcal{D}_{k(Y)/k(Y/G)}) = v_E(u'(y)) = (p-1)i(E),$$

using that $i(E)$ is unchanged if we replace σ by another generator of G. □

4 Toric divisors

In addition to recognizing when a quotient by $G = \mathbf{Z}/p$ has a μ_p-quotient singularity (as in Theorem 2.2), Theorem 4.1 analyzes when a G-invariant divisor is pulled back from a divisor on the quotient. Using this, we can view certain G-invariant divisors as toric divisors on the quotient scheme, which will be convenient for applications.

Theorem 4.1.

(1) Let $G = \mathbf{Z}/p$ act on a regular scheme U, with the assumptions of Theorem 2.2. Assume moreover that the function e is the greatest common divisor of the functions $I(x_i)$ for $i = 1, \ldots, n$ in the local ring $O_{U,P}$. Then, for each $y \in \mathfrak{m}_U$ such that $\{y = 0\}$ is an irreducible divisor $E \subset U$ and $I(y) \in (ey)$, E is the pullback of a Weil divisor F in U/G.

(2) If in addition $I(x_i) \in e\,\mathfrak{m}_U$ for $i = 1, \ldots, n$, so that U/G is the quotient of a regular scheme Q by μ_p (by Theorem 2.2), then the pullback of F to Q is the divisor $\{h = 0\}$ for some μ_p-eigenfunction h on Q.

(3) Continue to assume that $I(x_i) \in e\,\mathfrak{m}_U$ for $i = 1, \ldots, n$, so that U/G is the quotient of a regular scheme Q by μ_p. Let $y_1, \ldots, y_r$ be functions on U, vanishing at P, that are linearly independent in $\mathfrak{m}_U / \mathfrak{m}_U^2$. Suppose that $I(y_j) \in (ey_j)$ for $j = 1, \ldots, r$. Then the corresponding μ_p-eigenfunctions $h_1, \ldots, h_r$ on Q (from (2)) are linearly independent in $\mathfrak{m}_Q / \mathfrak{m}_Q^2$. That is, these functions are part of a toric coordinate system on Q. Finally, if $I(s) = es$ in Theorem 2.2 (as we can assume), then the μ_p-weight of h_j is equal to the eigenvalue of $\varphi(y) := I(y)/e$ on $y_j \in \mathfrak{m}_U / \mathfrak{m}_U^2$, which is in $\mathbf{F}_p$.

Proof. (1) Let E be the irreducible divisor $\{y = 0\}$ in U. The assumptions imply that E is mapped into itself by G, and that there is an $i \in \{1, \dots, n\}$ such that $v_E(I(y)) > v_E(I(x_i))$. By section 3, $f\colon U \to U/G$ is either unramified (if E is not fixed by G) or fiercely ramified along E. In either case, there is an irreducible divisor F on U/G such that $E = f^*F$, as we want. (The divisor F need not be Cartier, but the pullback of a Weil divisor is still a Weil divisor (with integer coefficients). Indeed, F is a Cartier divisor outside a codimension-2 subset of U/G, by normality of U/G.)

(2) Since Q is regular, the pullback of F to Q is a Cartier divisor, hence (after shrinking U and Q around P) of the form $\{t = 0\}$ for some function $t \in \mathfrak{m}_Q - \{0\}$. Clearly the divisor $\{t = 0\}$ is μ_p-invariant. I claim that t times some unit is a μ_p-eigenfunction on Q. Indeed, in algebraic terms, the action of μ_p on Q makes $O(Q)$ a comodule over $O(\mu_p)$, and the ideal (t) is an sub-$O(\mu_p)$-comodule. Every $O(\mu_p)$-comodule (with no finiteness assumption needed) is the direct sum of its weight spaces, indexed by $\mathbf{Z}/p$. So we can write $t = t_0 + \cdots + t_{p-1}$ with $t_i \in (t)$ and t_i of weight i. Since $t_i \in (t)$, we can write $t_i = a_i t$ for some $a_i \in O_{Q,P}$. Since $O_{Q,P}$ is regular, it is a domain, and hence $1 = a_0 + \cdots + a_{p-1}$. So at least one a_i is not in $\mathfrak{m}_Q$, hence is a unit. Then $h := t_i = a_i t$ is a unit times t and also a μ_p-eigenfunction (of weight i), as we want.

(3) After multiplying e by a unit, we can assume that $I(s) = es$, in the terminology of Theorem 2.2. The assumption that $I(y_j) \in (ey_j)$ for $j = 1, \dots, r$ implies that $y_1, \dots, y_r$ in $\mathfrak{m}_U / \mathfrak{m}_U^2$ are eigenvectors of the map φ. By the proof of Theorem 2.2, the corresponding eigenvalues are in $\mathbf{F}_p \subset k_U$.

For $j = 1, \dots, r$, we know from (1) that the divisor $\{y_j = 0\}$ on U is pulled back from a divisor F_j on U/G. (Here F_j is a Weil divisor, but it is a Cartier divisor outside a codimension-2 subset of U/G, since U/G is normal.) By (2), F_j pulls back to a divisor $\{h_j = 0\}$ on Q with h_j a μ_p-eigenfunction. It follows that the Cartier divisors $\{y_j = 0\}$ on U and $\{h_j = 0\}$ on Q have the same pullback to W; that is, $h_j = y_j(\text{unit})$ on W. Since h_j is a μ_p-eigenfunction of some weight $b_j \in \{0, \dots, p-1\}$, we have $h_j = g_j w^{b_j}$ for some function g_j on U, in the notation of the proof of Theorem 2.2. Therefore, $g_j = y_j(\text{unit})$ on U. Since $y_1, \dots, y_r$ are linearly independent in $\mathfrak{m}_U / \mathfrak{m}_U^2$, the same is true for $g_1, \dots, g_r$. By the proof of Theorem 2.2, the μ_p-weight of h_j is equal to the eigenvalue of φ on the eigenvector $g_j \in \mathfrak{m}_U / \mathfrak{m}_U^2$. Since $g_j \in \mathfrak{m}_U / \mathfrak{m}_U^2$ is a nonzero multiple of y_j, this is the same as the eigenvalue of φ on y_j.

The ring $O_{W,P}$ is faithfully flat over $O_{U,P}$ and over $O_{Q,P}$. By Lemma 2.7, the maximal ideals $\mathfrak{m}_U$ and $\mathfrak{m}_Q$ generate the same ideal in $O_{W,P}$. Since $g_1, \dots, g_r$ are linearly independent in $\mathfrak{m}_U / \mathfrak{m}_U^2$, it follows that $h_1, \dots, h_r$ are linearly independent in $\mathfrak{m}_Q / \mathfrak{m}_Q^2$, as we want. ☐

5 The example over the 2-adic integers

Theorem 5.1. *Let $Y = \{(x, y, i) \in A^3_{\mathbf{Z}_2} : x \neq 0, y \neq 0, i^2 = -1\}$. Let the group $G = \mathbf{Z}/2 = \{1, \sigma\}$ act on Y by*

$$\sigma(x, y, i) = (1/x, 1/y, -i).$$

Then the scheme Y/G is terminal, not Cohen-Macaulay, of dimension 3, and flat over $\mathbf{Z}_2$. Also, the canonical class of Y/G over $\mathbf{Z}_2$ is Cartier.

Proof. The scheme Y is regular, being an open subset of the affine plane over the discrete valuation ring $\mathbf{Z}_2[\zeta_4] = \mathbf{Z}_2[i]/(i^2+1)$. Since Y is a normal integral affine scheme of dimension 3, so is Y/G. The ring $O(Y/G)$ of regular functions is a torsion-free $\mathbf{Z}_2$-module, since it is a subring of the torsion-free $\mathbf{Z}_2$-module $O(Y)$; so Y/G is flat over $\mathbf{Z}_2$. The fixed point scheme of G on Y is defined by: $I(x) = (1-x^2)/x$, $I(y) = (1-y^2)/y$, and $I(i) = \sigma(i) - i = -2i$, hence by $\{x^2 = 1, y^2 = 1, 2i = 0\}$. Together with the equation $i^2 = -1$ on Y, these equations imply set-theoretically that $2 = 0$, $x = 1$, $y = 1$, and $i = 1$; so the fixed point set of G is a single closed point P in Y, with residue field $\mathbf{F}_2$. Since Y is regular, it follows that Y/G is regular outside the image of P, which we also call P.

For Y/G to be Cohen-Macaulay at P would mean that the local cohomology $H^i_P(Y/G, O)$ was zero for $i < \dim(Y/G) = 3$. Consider the exact sequence $H^1(Y/G, O) \to H^1(Y/G - P, O) \to H^2_P(Y/G, O)$. Since Y/G is affine, we have $H^1(Y/G, O) = 0$. So Cohen-Macaulayness of Y/G would imply that $H^1(Y/G - P, O) = 0$.

Fogarty showed that for $G = \mathbf{Z}/p$ acting with an isolated fixed point on a normal scheme W over $\mathbf{F}_p$ of dimension at least 3, W/G is not Cohen-Macaulay [13, Proposition 4]. When W has mixed characteristic $(0, p)$, he needed $\dim(W) \geq 4$ to get the same conclusion. Nonetheless, we can build on his ideas to study the 3-dimensional scheme Y in mixed characteristic.

We first show that $H^1(G, O(Y))$ is not zero. This cohomology group is $\ker(\mathrm{tr})/\operatorname{im}(1-\sigma)$ on $O(Y)$, where the trace is $1+\sigma$. Since $\mathrm{tr}(i) = 0$, i defines an element of $H^1(G, O(Y))$. Note that i restricts to $1 \in O(P) = \mathbf{F}_2$ on the fixed point P. Therefore, i has nonzero image under the restriction map $H^1(G, O(Y)) \to H^1(G, O(P)) \cong \mathbf{F}_2$. So i is nonzero in $H^1(G, O(Y))$, as we want.

Since G acts freely on Y outside P, we have a spectral sequence (as discussed in [13]):

$$E_2^{pq} = H^p(G, H^q(Y - P, O)) \Rightarrow H^{p+q}(Y/G - P, O).$$

Here $H^0(Y-P, O) = O(Y-P)$ is equal to $O(Y)$, since Y is normal and P has codimension 3 in Y (at least 2 would suffice). So $H^1(G, H^0(Y-P, O)) = H^1(G, O(Y)) \neq 0$. The spectral sequence shows that this group injects into $H^1(Y/G - P, O)$, and so $H^1(Y/G - P, O) \neq 0$. As discussed above, it follows that Y/G is not Cohen-Macaulay.

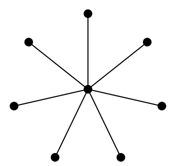
It remains to show that Y/G is terminal. Let us recall the definition. For a normal quasi-projective scheme X over a regular base scheme S, Hartshorne defined the *canonical sheaf* $\omega_{X/S}$ [20, Definition 1.6]. It is a reflexive sheaf of rank 1, or equivalently the sheaf associated to a Weil divisor. In this paper, S will be Spec of the p-adic integers or of a field, and we write K_X for $\omega_{X/S}$. Toward the end of the proof of Theorem 5.1, we compute K_X directly from the definition in our example.

A normal scheme X is *terminal* if K_X is $\mathbf{Q}$-Cartier and, for every normal scheme Z with a proper birational morphism $\pi\colon Z \to X$, we have

$$K_Z = \pi^*(K_X) + \sum_j a_j E_j$$

with $a_j > 0$ for every exceptional divisor E_j of π. If X has a resolution of singularities, terminality of X is equivalent to positivity of the discrepancies a_j on this one resolution [20, Corollary 2.12].

Let $Y_0 = Y$. To prove that our example Y_0/G is terminal, one approach would be to construct an explicit resolution of singularities. As with the analogous example in characteristic 2 [29, Theorem 5.1], this can be done by making G-equivariant blow-ups of Y_0 along regular closed subschemes. Namely, we can make G-equivariant blow-ups $Y_2 \to Y_1 \to Y_0$ such that Y_2/G is regular. That resolution of Y_0/G has dual complex a star, with one edge from a vertex F_0 to each of seven other vertices $F_1, \ldots, F_7$.



However, we can simplify the proof that Y_0/G is terminal by stopping with a partial resolution with toric singularities. Namely, after only one blow-up $Y_1 \to Y_0$, we can recognize the seven singularities of Y_1/G as μ_2-quotient singularities of the form $\frac{1}{2}(1,1,1)$, thanks to Theorem 2.2. So Y_1/G is terminal, and it is then straighforward to compute that Y_0/G is terminal. This method would also simplify the proof of terminality for the example in characteristic 2 that we are imitating [29]. The simplification is more striking for our more complicated examples in characteristic 3 or mixed characteristic $(0,3)$ (Theorems 6.1 and 7.1), and even more significant for our even more complicated examples in characteristic 5 or mixed characteristic $(0,5)$ (Theorems 8.1 and 9.1).

We now begin to blow up. To simplify the equations, change coordinates by $x_0 := x - 1$, $x_1 := y - 1$, and $e_2 := 1 + i$, so that the G-fixed point is defined by $0 = x_0 = x_1 = e_2$. Then

$$Y_0 = \{(x_0, x_1, e_2) \in A^3_{\mathbf{Z}_2} : 0 = e_2^2 - 2e_2 + 2,\ 1 + x_0 \neq 0,\ 1 + x_1 \neq 0\},$$

and G acts by

$$\sigma(x_0, x_1, e_2) = \left(\frac{-x_0}{1+x_0},\ \frac{-x_1}{1+x_1},\ 2 - e_2\right).$$

The blow-up at the G-fixed point is:

$$Y_1 = \{((x_0, x_1, e_2), [y_0, y_1, y_2]) \in A^3_{\mathbf{Z}_2} \times_{\mathbf{Z}_2} \mathbf{P}^2_{\mathbf{Z}_2} : e_2^2 - 2e_2 + 2 = 0,$$
$$x_0 y_1 = x_1 y_0,\ x_0 y_2 = e_2 y_0,\ x_1 y_2 = e_2 y_1,\ 1 + x_0 \neq 0,\ 1 + x_1 \neq 0\}.$$

The exceptional divisor $E_0 \subset Y_1$ is isomorphic to $\mathbf{P}^2_{\mathbf{F}_2}$. Here G acts on Y_1 by

$$\sigma((x_0, x_1, e_2), [y_0, y_1, y_2]) = \left(\left(\frac{-x_0}{1+x_0}, \frac{-x_1}{1+x_1}, 2 - e_2\right), \left[\frac{-y_0}{1+x_0}, \frac{-y_1}{1+x_1}, y_2(1-e_2)\right]\right).$$

First consider the open subset $U_0 = \{y_0 = 1\}$ in Y_1. Then $x_1 = x_0 y_1$ and $e_2 = x_0 y_2$, so

$$U_0 = \{(x_0, y_1, y_2) \in A^3_{\mathbf{Z}_2} : 0 = (x_0 y_2)^2 - 2(x_0 y_2) + 2, 1 + x_0 \neq 0, 1 + x_0 y_1 \neq 0\}.$$

Here $E_0 = \{x_0 = 0\}$. The group $G = \mathbf{Z}/2$ acts by

$$\sigma(x_0, y_1, y_2) = \left(\frac{-x_0}{1+x_0},\ \frac{y_1(1+x_0)}{1+x_0 y_1},\ -y_2(1 - x_0 y_2)(1 + x_0).\right).$$

The fixed point scheme Y_1^G is defined by: $I(x_0) = \sigma(x_0) - x_0 = x_0^2(-1 - x_0 y_2^2 + x_0^2 y_2^3)/(1 + x_0)$, $I(y_1) = x_0 y_1(1 - y_1)$, and $I(y_2) = x_0 y_2(-1 - y_2 + x_0 y_2 + x_0 y_2^2)$. We know that Y_1^G (as a set) is contained in E_0. To focus on the fixed point scheme near E_0, we can say (more simply): $I(x_0) = x_0^2(1+O(x_0))$, $I(y_1) = x_0 y_1(1+y_1+O(x_0))$, and $I(y_2) = x_0 y_2(1 + y_2 + O(x_0))$. We see that the fixed point scheme Y_1^G is generically the Cartier divisor $E_0 = \{x_0 = 0\}$. The bad locus in E_0 (where that fails) is given by removing a factor of x_0 from the equations and setting $x_0 = 0$, so we have: $0 = x_0$, $0 = y_1(1 + y_1)$, and $0 = y_2(1 + y_2)$. (Note that $x_0 = 0$ implies $2 = 0$, by the equation for U_0.) So the fixed point scheme Y_1^G (in this chart) is E_0 as a Cartier divisor except at the 4 points (x_0, y_1, y_2) equal to $(0, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, or $(0, 1, 1)$.

The open set $U_1 = \{y_1 = 1\}$ works the same way, by the symmetry switching x_0 and x_1, hence also switching y_0 and y_1. Together, that gives 6 bad points in $E_0 \cong \mathbf{P}^2_{\mathbf{F}_2}$ so far, namely $[y_0, y_1, y_2]$ equal to $[1, 0, 0]$, $[1, 1, 0]$, $[1, 0, 1]$, $[1, 1, 1]$, $[0, 1, 0]$, and $[0, 1, 1]$.

Finally, look at the open set $U_2 = \{y_2 = 1\}$ in Y_1. Then $x_0 = e_2 y_0$, $x_1 = e_2 y_1$, and so

$$U_2 = \{(y_0, y_1, e_2) \in A^3_{\mathbf{Z}_2} : e_2^2 - 2e_2 + 2 = 0, 1 + y_0 e_2 \neq 0, 1 + y_1 e_2 \neq 0\}.$$

Here $E_0 = \{e_2 = 0\}$. On U_2, G acts by

$$\sigma(y_0, y_1, e_2) = \left(\frac{y_0(1 - e_2)}{1 + y_0 e_2},\ \frac{y_1(1 - e_2)}{1 + y_1 e_2},\ 2 - e_2\right).$$

We know that Y_1^G is contained as a set in E_0. The fixed point scheme Y_1^G (near E_0) is defined by: $I(y_0) = y_0 e_2(1 + y_0 + O(e_2))$, $I(y_1) = y_1 e_2(1 + y_1 + O(e_2))$, and $I(e_2) = e_2^2(1 + O(e_2))$. We see that the fixed point scheme Y_1^G is generically $E_0 = \{e_2 = 0\}$ as a Cartier divisor. The bad locus on E_0 (where this fails) is given by removing a factor of e_2 from the equations and setting $e_2 = 0$, so we have: $0 = e_2$, $0 = y_1(1 + y_1)$, and $0 = y_2(1 + y_2)$. So there are 4 bad points on $E_0 = \mathbf{P}^2_{\mathbf{F}_2}$ in this open set: (y_0, y_1, e_2) equal to $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, or $(1, 1, 0)$.

We conclude that the fixed point scheme in Y_1 is $E_0 \cong \mathbf{P}^2_{\mathbf{F}_2}$ with multiplicity 1 except at the 7 points:

$$[y_0, y_1, y_2] = [1, 0, 0], [0, 1, 0], [0, 0, 1], [1, 1, 0], [1, 0, 1], [0, 1, 1], [1, 1, 1].$$

(The same thing happens for the first blow-up of the analogous example in characteristic 2 [29].) By Theorem 2.1, Y_1/G is regular outside the images of these 7 points.

One further G-equivariant blow-up at each of these 7 points suffices to resolve Y_1/G, but the equations for these blow-ups are a bit messy. Instead, we will use Theorem 2.2 to show that the 7 singular points of Y_1/G are all mixed-characteristic analogs of the singularity A^3/μ_2, the simplest terminal singularity whose canonical class is not Cartier. More precisely, each of these 7 singular points is of the form $\frac{1}{2}(1, 1, 1)$, meaning that it can be written as Q/μ_2 for some regular scheme Q of dimension 3 (at an isolated fixed point of μ_2). As a result, Y_1/G is terminal. With one last calculation, we will deduce that Y_0/G is terminal.

We first consider the singularities of Y_1/G in the chart $U_0 = \{y_0 = 1\}$, as above. Here U_0 has coordinates (x_0, y_1, y_2) with $0 = (x_0 y_2)^2 + 2(x_0 y_2) - 2$, and

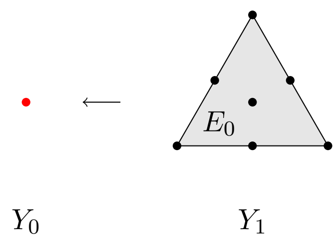


Figure 1: The fixed point schemes in Y_0 and in the blow-up Y_1. Here $E_0 \cong \mathbf{P}^2$, which we view as a toric variety; so the three edges of the triangle denote the coordinate lines in $\mathbf{P}^2$. We consider three coordinate charts on Y_1, each containing one vertex of the triangle.

$E_0 = \{x_0 = 0\}$. We saw that the fixed point scheme U_0^G is the Cartier divisor $E_0 = \{x_0 = 0\}$ except at the 4 points (x_0, y_1, y_2) equal to $(0,0,0), (0,0,1), (0,1,0)$, or $(0,1,1)$. At each of these points P, the G-action has the form required for Theorem 2.2 with $e = s := x_0$, namely that $I(x_0) = x_0^2(1+\mathfrak{m})$ and $I(z_j) = x_0(z_j + \mathfrak{m}^2)$ for $j = 1,2$, for some coordinates x_0, z_1, z_2 at P, with $\mathfrak{m}$ the maximal ideal at P. Also, since $x_0^2 y_2^2 = 2(\text{unit})$, 2 is in the ideal (x_0^2), hence in $x_0\,\mathfrak{m}$, which is another assumption in Theorem 2.2. So the theorem gives that these 4 singular points of Y_1/G are of the form $\frac{1}{2}(1,1,1)$.

The calculations are identical in the chart $\{y_1 = 1\}$ in Y_1. They are slightly different in the chart $\{y_2 = 1\}$, but the conclusion is the same: the singularities of Y_1/G in this chart are again of the form $\frac{1}{2}(1,1,1)$. Namely, this chart has coordinates (y_0, y_1, e_2) with $0 = e_2^2 - 2e_2 + 2$, and $E_0 = \{e_2 = 0\}$. The fixed point scheme Y_1^G is the Cartier divisor E_0 except at the 4 points (y_0, y_1, e_2) equal to $(0,0,0), (1,0,0), (0,1,0)$, or $(1,1,0)$. At each of these points P, the G-action has the form required for Theorem 2.2 with $e = s := e_2$, namely that $I(e_2) = e_2^2(1+\mathfrak{m})$ and $I(z_j) = e_2(z_j + \mathfrak{m}^2)$ for $j = 0,1$, for some coordinates (z_0, z_1, e_2) at P, where $\mathfrak{m}$ is the maximal ideal at P. Also, the equation for e_2 implies that 2 is in the ideal (e_2^2), hence in $e_2\,\mathfrak{m}$, which is another assumption in Theorem 2.2. So the theorem gives that these 4 singular points of Y_1/G are again of the form $\frac{1}{2}(1,1,1)$.

Thus all 7 singular points of Y_1/G are of the form $\frac{1}{2}(1,1,1)$. By the Reid-Tai criterion (Theorem 1.1), they are terminal. (To check that by hand: each singular point has a resolution $Z \to U_1/G$ whose exceptional divisor is $E_j \cong \mathbf{P}^2_{\mathbf{F}_2}$ with normal bundle $O(-2)$. As a result, the singularities of Y_1/G are terminal, with $K_Z = \pi^*(K_{Y_1/G}) + \frac{1}{2}E_j$ near each E_j.)

Recall that $Y_0 = Y$ is the regular scheme of dimension 3 that we started with. (Thus Y_1 is the blow-up of Y_0 at the G-fixed point.) We now go on to show that $X = Y_0/G$ is terminal. Write F_0 for the image in Y_1/G of the exceptional divisor E_0. Note that although G fixes E_0 in Y_1, the morphism $E_0 \to F_0$ is a finite purely inseparable morphism, not necessarily an isomorphism. (Indeed, $G = \mathbf{Z}/2$ is not linearly reductive over $\mathbf{Z}_2$. So if G acts on an affine scheme T preserving a closed subscheme S, the morphism $S/G \to T/G$ need not be a closed immersion. Equivalently, the G-equivariant surjection $O(T) \to O(S)$ need not yield a surjection $O(T)^G \to O(S)^G$.)

Write K_{Y_0} for the canonical sheaf $\omega_{Y_0/\mathbf{Z}_2}$. Since Y_0 is regular, K_{Y_0} is a line bundle, described as follows [20, Definition 1.6]. First, let $R = \mathbf{Z}_2[i]/(i^2+1)$. Then we have an embedding $D = \operatorname{Spec} R \subset A^1_{\mathbf{Z}_2}$; write I for the ideal $(i^2+1) \subset \mathbf{Z}_2[i]$

defining this subscheme. Then the adjunction formula $K_D = (K_{A^1} + D)|_D$ is made into a definition:

$$\omega_{R/\mathbf{Z}_2} = \Omega^1_{\mathbf{Z}_2[i]/\mathbf{Z}_2} \otimes_{\mathbf{Z}_2[i]} R \otimes_R (I/I^2)^*.$$

In these terms, one trivializing section of $\omega_{R/\mathbf{Z}_2}$ is $\alpha := \frac{di}{i} \cdot f$, where $f\colon I/I^2 \to R = \mathbf{Z}_2[i]/I$ is the map sending i^2+1 to 1. (Formally, one could think of this section of $\omega_{R/\mathbf{Z}_2}$ as $\frac{1}{i}\frac{di}{d(i^2+1)}$.) Next, since $\pi\colon Y_0 \to \operatorname{Spec} R$ is smooth of relative dimension 2, we have $K_{Y_0} = \omega_{Y_0/\mathbf{Z}_2} = \Omega^2_{Y_0/R} \otimes \pi^*\omega_{R/\mathbf{Z}_2}$. So one trivializing section of K_{Y_0} is $\beta := \frac{dx}{x} \wedge \frac{dy}{y} \wedge \frac{di}{i} \cdot f$. I claim that this section is fixed by G. Indeed, if we extend the action of G on R to $A^1_{\mathbf{Z}_2}$ by $\sigma(i) = -i$, then $\sigma(f) = f$ and $\sigma(di) = -di$, so $\sigma(\frac{di}{i}f) = \frac{di}{i}f$. Also, $\sigma(dx/x) = -dx/x$ and $\sigma(dy/y) = -dy/y$, from which we see that $\sigma(\beta) = \beta$ as claimed. It follows that the divisor class $K_{Y_0/G}$ is linearly equivalent to zero, in particular Cartier. Here $K_{Y_0/G}$ is the canonical sheaf in the sense of [20, Definition 1.6]; Y_0/G is not Gorenstein, since (as we have shown) it is not Cohen-Macaulay.

Since K_X is Cartier, we can write

$$K_{Y_1/G} = \pi^* K_X + a_0 F_0$$

for an integer a_0. Since Y_1/G is terminal, X is terminal if and only if the discrepancy a_0 is positive. Here and below, we write π for all the relevant contractions, which in the formula above means $\pi\colon Y_1/G \to Y_0/G = X$.

The analogous formula for Y_1 is easy. Since Y_1 is the blow-up of the regular 3-dimensional scheme Y_0 at a closed point,

$$K_{Y_1} = \pi^* K_{Y_0} + 2E_0.$$

Write f for the quotient map $Y_0 \to Y_0/G$ or $Y_1 \to Y_1/G$. The ramification of f along E_0 can be computed as follows.

Corollary 5.2. *Let U and G be as in Theorem 2.2. So U is a regular scheme with an action of the group $G = \mathbf{Z}/p = \langle \sigma : \sigma^p = 1\rangle$, for a prime number p, and assume that U^G is generically a regular divisor $E_1 = \{x_1 = 0\}$ and that $I(x_1) = x_1^2(unit)$. Assume that U is of finite type over a regular base scheme S and that G acts on U over S. Write K_U and $K_{U/G}$ for the canonical classes over S. Then $U \to U/G$ is fiercely ramified along E_1. In particular, the image F_1 of E_1 in U/G is $\mathbf{Q}$-Cartier, and $f\colon U \to U/G$ satisfies $f^*F_1 = E_1$ and $K_U = f^*K_{U/G} + (p-1)E_1$.*

Proof. The norm $N(x_1)$ is a function on U/G that defines a positive multiple of the divisor F_1, and so F_1 is $\mathbf{Q}$-Cartier. Since the fixed point scheme U^G is generically the Cartier divisor E_1, the ramification divisor of f is $(p-1)E_1$ by Lemma 3.1. That is, $K_U = f^*K_{U/G} + (p-1)E_1$. Also, the fixed point scheme is generically $E_1 = \{x_1 = 0\}$ with coefficient 1, whereas $I(x_1)$ vanishes to order 2 along E_1; so section 3 gives that the ramification of f along E_1 is fierce. In particular, the ramification index e is 1, meaning that $f^*F_1 = E_1$. □

In particular, returning to our example with $p = 2$, we have seen that the divisor E_0 has multiplicity 1 in the fixed point scheme $(Y_1)^G$. Also, Corollary 5.2 gives that $f\colon Y_1 \to Y_1/G$ is fiercely ramified along E_0. So we have

$$K_{Y_1} = f^* K_{Y_1/G} + E_0$$

and $f^*F_0 = E_0$. (The same is true for the example in characteristic 2 that we are imitating [29].)

Since $f\colon Y_0 \to Y_0/G$ is étale in codimension 1, we have $K_{Y_0} = f^*K_{Y_0/G}$. It follows that

$$\begin{aligned} f^*K_{Y_1/G} &= K_{Y_1} - E_0 \\ &= (\pi^*K_{Y_0} + 2E_0) - E_0 \\ &= \pi^*f^*K_{Y_0/G} + E_0 \\ &= f^*(\pi^*K_{Y_0/G} + F_0). \end{aligned}$$

Therefore,

$$K_{Y_1/G} = \pi^*K_{Y_0/G} + F_0.$$

Because the coefficient of the exceptional divisor F_0 is positive, and Y_1/G is terminal as shown above, $X = Y_0/G$ is terminal. □

6 Characteristic 3

Theorem 6.1. *Let the group $G = \mathbf{Z}/3$ with generator τ act on $\mathbf{P}^2$ over $\mathbf{F}_3$ by*

$$\tau([u_0, u_1, u_2]) = [u_1, u_2, u_0]$$

and on $\mathbf{P}^1$ by

$$\tau([y_0, y_1]) = [y_0, y_0 + y_1].$$

Then $(\mathbf{P}^2 \times \mathbf{P}^1)/G$ is terminal, not Cohen-Macaulay, and of dimension 3 over $\mathbf{F}_3$.

Proof. We work throughout over $k = \mathbf{F}_3$. Write $G = \mathbf{Z}/3 = \langle \sigma : \sigma^3 = 1\rangle$, with $\tau := \sigma^{-1}$. Let $Y_0 = \mathbf{P}^2 \times \mathbf{P}^1$ and $X = Y_0/G$. The only fixed point of G on Y_0 is $P = ([1, 1, 1], [0, 1])$. So X is normal of dimension 3, and X is smooth over k outside the image of P, which we also call P. Also, $3K_X$ is Cartier. By Fogarty, since P is an isolated fixed point of $G = \mathbf{Z}/p$ on a smooth 3-fold in characteristic p, X is not Cohen-Macaulay at P [13, Proposition 4].

It remains to show that X is terminal. One can resolve the singularities of X by performing G-equivariant blow-ups of Y_0. However, as in section 5, we will shorten the proof by recognizing that, after two G-equivariant blow-ups $Y_2 \to Y_1 \to Y_0$, the singularities of Y_2/G become toric, namely quotients of a regular scheme by μ_3. That makes it easy to check that Y_0/G is terminal, without having to continue making G-equivariant blow-ups.

Before this approach, I found a G-equivariant blow-up $Y_{18} \to \cdots \to Y_0$ with Y_{18}/G regular, but the construction involved 18 blow-ups along points or curves. The approach here, looking for toric singularities instead of regularity, saves a lot of work.

To put the fixed point at the origin, we change coordinates on Y_0 by: $x_0 = (u_0 + u_1 + u_2)/u_1$, $x_1 = (-u_1 + u_2)/u_1$, and $x_2 = y_0/y_1$. Then G acts on the open subset U of Y_0 given by

$$U = \{(x_0, x_1, x_2) \in A^3 : 1 + x_2 \neq 0,\ 1 - x_2 \neq 0,\ 1 + x_1 \neq 0,\ 1 + x_0 - x_1 \neq 0\}.$$

The G-action on U is given by

$$\tau(x_0, x_1, x_2) = \left(\frac{x_0}{1+x_1}, \frac{x_1+x_0}{1+x_1}, \frac{x_2}{1+x_2}\right).$$

As we blow up, we will not need to keep track of the precise affine open set on which G acts, since we are only concerned with the action near the fixed point set.

Let Y_1 be the blow-up of Y_0 at the G-fixed point, which is the origin in these coordinates. Then the open subset of Y_1 over $U \subset Y_0$ is

$$\{((x_0, x_1, x_2), [y_0, y_1, y_2]) \in U \times \mathbf{P}^2 : x_0y_1 = x_1y_0,\ x_0y_2 = x_2y_0, x_1y_2 = x_2y_1\}.$$

Clearly the fixed point set Y_1^G is contained in the exceptional divisor $E_0 \cong \mathbf{P}^2$. It turns out to be a curve isomorphic to $\mathbf{P}^1$. We need three coordinate charts to cover E_0. First consider the open subset $\{y_0 = 1\}$ in Y_1. Here $(x_0, x_1, x_2) = (x_0, x_0y_1, x_0y_2)$, and G acts by

$$\tau(x_0, y_1, y_2) = \left(\frac{x_0}{1+x_0y_1},\ 1+y_1,\ \frac{y_2(1+x_0y_1)}{1+x_0y_2}\right).$$

By the action on the y_1 coordinate, there are no fixed points in this open set.

Next, work in the open set $\{y_1 = 1\} \subset Y_1$. Here $(x_0, x_1, x_2) = (y_0x_1, x_1, y_2x_1)$, $E_0 = \{x_1 = 0\}$, and G acts by

$$\tau(y_0, x_1, y_2) = \left(\frac{y_0}{1+y_0}, \frac{x_1(1+y_0)}{1+x_1}, \frac{y_2(1+x_1)}{(1+y_0)(1+x_1y_2)}\right).$$

The fixed point scheme Y_1^G is defined by: $I(y_0) = -y_0^2/(1+y_0)$, $I(x_1) = x_1(y_0 + O(x_1))$, and $I(y_2) = y_2(-y_0 + O(x_1))/(1 + y_0 + O(x_1))$. Since we know that the fixed point set Y_1^G is contained in $E_0 = \{x_1 = 0\}$, we read off that the fixed point set is the line $\{0 = y_0 = x_1\}$ in E_0.

Finally, consider the open set $\{y_2 = 1\}$. Then $(x_0, x_1, x_2) = (x_2y_0, x_2y_1, x_2)$, and G acts by

$$\tau(y_0, y_1, x_2) = \left(\frac{y_0(1+x_2)}{1+y_1x_2}, \frac{(y_0+y_1)(1+x_2)}{1+y_1x_2}, \frac{x_2}{1+x_2}\right).$$

In these coordinates, the exceptional divisor $E_0 \cong \mathbf{P}^2$ in Y_1 is $\{x_2 = 0\}$. The fixed point scheme of G is defined by: $I(y_0) = y_0x_2(1 - y_1 + O(x_2))$, $I(y_1) = y_0 + O(x_2)$, and $I(x_2) = x_2^2(-1 + O(x_2))$. We know that the fixed point set is contained in E_0, and these equations imply that the fixed point set is the line $\{0 = y_0 = x_2\}$ in E_0, the same line as in the previous chart. Thus we have shown that the fixed point set in all of Y_1 is a curve isomorphic to $\mathbf{P}^1$.

Let Y_2 be the blow-up of Y_1 along the (reduced) G-fixed curve, with exceptional divisor $E_1 \subset Y_2$. It is clear that Y_2^G is contained in E_1 as a set. Write E_0 for the strict transform of $E_0 \subset Y_1$ in Y_2. It turns out that the fixed point scheme Y_2^G is equal to the Cartier divisor E_1 except at six points in E_1, three over the point $[y_0, y_1, y_2] = [0, 1, 0]$ in $E_0 \subset Y_1$ and one each over $[y_0, y_1, y_2]$ equal to $[0, 1, 1]$, $[0, 1, -1]$, or $[0, 0, 1]$. Since E_1 is a $\mathbf{P}^1$-bundle over $\mathbf{P}^1$, we will need to look at four affine charts to see all of it. (See Figure 2. Each affine chart we consider in Y_2 contains exactly one of the four vertices of the square.)

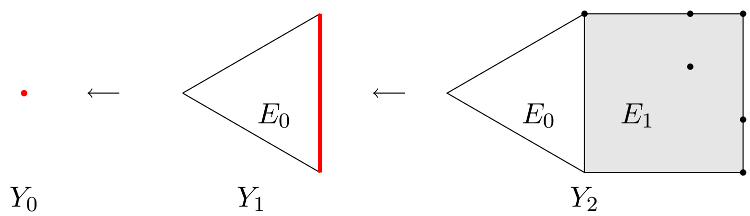


Figure 2: For $j = 0, 1, 2$, G acts freely on Y_j outside the shaded or marked loci; Y_j/G is regular outside the marked loci; and Y_j/G has toric singularities outside the red loci.

First work over the chart $\{y_1 = 1\}$ in Y_1. We wrote out the G-action on this chart above, with coordinates (y_0, x_1, y_2). Here $E_0 = \{x_0 = 0\}$. We defined Y_2 as the blow-up of Y_1 along the G-fixed curve $\{0 = y_0 = x_1\}$ in E_0. So, over this open subset of Y_1, Y_2 is an open subset of

$$\{((y_0, x_1, y_2), [w_0, w_1]) \in A^3 \times \mathbf{P}^1 : y_0 w_1 = x_1 w_0\}.$$

First look at the chart $\{w_0 = 1\}$ in Y_2 over $\{y_1 = 1\}$ in Y_1. (This chart contains the upper left vertex of E_1, in Figure 2.) Then $(y_0, x_1, y_2) = (y_0, y_0 w_1, y_2)$, and $G = \mathbf{Z}/3$ acts by

$$\tau(y_0, w_1, y_2) = \left(\frac{y_0}{1 + y_0}, \frac{w_1(1 + y_0)^2}{1 + y_0 w_1}, \frac{y_2(1 + y_0 w_1)}{(1 + y_0 w_1 y_2)(1 + y_0)} \right).$$

Here $E_0 = \{w_1 = 0\}$ and $E_1 = \{y_0 = 0\}$. The fixed point scheme is defined by: $I(y_0) = y_0^2(-1 + O(y_0))$, $I(w_1) = y_0 w_1(-1 - w_1 + O(y_0))$, and $I(y_2) = y_0 y_2(-1 + w_1 - w_1 y_2 + O(y_0))$. So the fixed point scheme (near E_1) is E_1 with multiplicity 1 except at points with $0 = y_0$, $0 = w_1(w_1 + 1)$, and $0 = y_2(1 - w_1 + w_1 y_2)$. Thus the bad points are $(y_0, w_1, y_2) = (0, 0, 0)$ in $E_0 \cap E_1$, $(0, -1, 0)$, and $(0, -1, -1)$. We have found three bad points in E_1, with the first one in $E_0 \cap E_1$.

The other chart over $\{y_1 = 1\}$ in Y_1 is $\{w_1 = 1\}$ in Y_2 (the chart containing the upper right vertex of E_1 in Figure 2). Then $(y_0, x_1, y_2) = (w_0 x_1, x_1, y_2)$, and G acts by

$$\tau(w_0, x_1, y_2) = \left(\frac{w_0(1 + x_1)}{(1 + w_0 x_1)^2}, \frac{x_1(1 + w_0 x_1)}{1 + x_1}, \frac{y_2(1 + x_1)}{(1 + x_1 y_2)(1 + w_0 x_1)} \right).$$

Here E_0 does not appear, and $E_1 = \{x_1 = 0\}$. The fixed point scheme is defined (near E_1) by: $I(w_0) = w_0 x_1(1 + w_0 + O(x_1))$, $I(x_1) = x_1^2(-1 + w_0 + O(x_1))$, and $I(y_2) = x_1 y_2(1 - w_0 - y_2 + O(x_1))$. So Y_2^G is E_1 with multiplicity 1 except at points where $0 = x_1$, $0 = w_0(1 + w_0)$, and $0 = y_2(1 - w_0 - y_2)$. Thus the bad points in this chart are (w_0, x_1, y_2) equal to $(0, 0, 0)$, $(0, 0, 1)$, $(-1, 0, 0)$, and $(-1, 0, -1)$. The points with $w_0 \neq 0$ appeared in the previous chart, $\{w_0 = 1\}$. So we have two new bad points, (w_0, x_1, y_2) equal to $(0, 0, 0)$ or $(0, 0, 1)$, for a total of five so far.

To see all of E_1 in Y_2, we also need to work over $\{y_2 = 1\} \subset Y_1$. The coordinates are (y_0, y_1, x_2), and $E_0 = \{x_2 = 0\}$. The corresponding open subset of Y_2 is an open subset of

$$\{((y_0, y_1, x_2), [z_0, z_2]) \in A^3 \times \mathbf{P}^1 : y_0 z_2 = x_2 z_0\}.$$

First consider $\{z_0 = 1\}$ in Y_2, the chart containing the lower left vertex of E_1 in Figure 2. Then $(y_0, y_1, x_2) = (y_0, y_1, y_0 z_2)$, and G acts by

$$\tau(y_0, y_1, z_2) = \left(\frac{y_0(1+y_0 z_2)}{1+y_0 y_1 z_2},\ \frac{(y_0+y_1)(1+y_0 z_2)}{1+y_0 y_1 z_2},\ \frac{z_2(1+y_0 y_1 z_2)}{(1+y_0 z_2)^2} \right).$$

Here $E_0 = \{z_2 = 0\}$ and $E_1 = \{y_0 = 0\}$. The fixed point scheme Y_2^G is defined (near E_1) by: $I(y_0) = y_0^2 z_2(1 - y_1 + O(y_0))$, $I(y_1) = y_0(1 + y_1 z_2 - y_1^2 z_2 + O(y_0))$, and $I(z_2) = y_0 z_2^2(1 + y_1 + O(y_0))$. This is equal to E_1 with multiplicity 1 except at the point $(y_0, y_1, z_2) = (0, -1, -1)$. This appeared in the chart $\{w_0 = 1\}$ over $\{y_1 = 1\}$ (as the point $(y_0, w_1, y_2) = (0, -1, -1)$).

The other chart is $\{z_2 = 1\}$ in Y_2, which contains the lower right vertex of E_1 in Figure 2. Then $(y_0, y_1, x_2) = (z_0 x_2, y_1, x_2)$, and G acts by

$$\tau(z_0, y_1, x_2) = \left(\frac{z_0(1+x_2)^2}{1+y_1 x_2},\ \frac{(y_1+z_0 x_2)(1+x_2)}{1+y_1 x_2},\ \frac{x_2}{1+x_2} \right).$$

In this open set, E_0 does not appear, and $E_1 = \{x_2 = 0\}$. The fixed point scheme Y_2^G is defined by: $I(z_0) = z_0 x_2(-1 - y_1 + O(x_2))$, $I(y_1) = x_2(z_0 + y_1 - y_1^2 + O(x_2))$, and $I(x_2) = x_2^2(-1 + O(x_2))$. We know that Y_2^G is contained in $E_1 = \{x_2 = 0\}$ as a set. We read off that the fixed point scheme Y_2^G is generically the Cartier divisor E_1. The bad locus (where that fails) is given, in E_1, by: $0 = x_2$, $0 = z_0(1 + y_1)$, and $0 = z_0 + y_1 - y_1^2$. So we see three bad points in E_1: $(z_0, y_1, x_2) = (0, 1, 0)$, $(-1, -1, 0)$, or $(0, 0, 0)$. The first two points appeared in the chart $\{w_1 = 1\}$ in Y_2 over $\{y_1 = 1\}$ in Y_1 (as $(w_0, x_1, y_2) = (0, 0, 1)$ or $(-1, 0, -1)$, respectively). On the other hand, the origin in this chart is new; so we have a total of six bad points.

Thus the fixed point scheme Y_2^G is E_1 with multiplicity 1 except at six points on E_1. We will use Theorem 2.2 to recognize the resulting six singularities of Y_2/G as toric; so we have no further need for G-equivariant blow-ups.

First consider three of the six bad points of $E_1 \subset Y_2$, the ones with $[y_0, y_1, y_2]$ equal to $[0, 0, 1]$, $[0, 1, 1]$, or $[0, 1, -1]$. In these cases, our calculation of the action of G shows (by Theorem 2.2) that the singularity of Y_2/G at these points is of the form $\frac{1}{3}(1, 1, 2)$. (For example, for the point $[y_0, y_1, y_2] = [0, 0, 1]$, work in the chart $\{z_2 = 1\}$ in Y_2 over $\{y_2 = 1\}$ in Y_1. Here we have coordinates (z_0, y_1, x_2), $(Y_2)^G_{\text{red}}$ is $\{x_2 = 0\}$, and the point we are considering is the origin. As above, we have $I(z_0) = z_0 x_2(-1 - y_1 + O(x_2))$, $I(y_1) = x_2(z_0 + y_1 - y_1^2 + O(x_2))$, and $I(x_2) = x_2^2(-1 + O(x_2))$. So Theorem 2.2 applies, with $e = s := x_2$. The linear map φ in the theorem is given by: $\varphi(z_0) = -z_0$, $\varphi(y_1) = z_0 + y_1$, and $\varphi(x_2) = -x_2$. This map has eigenvalues $-1, 1, -1$ in $\mathbf{F}_3$, and so Theorem 2.2 gives that the singularity of Y_2/G at this point is of the form $\frac{1}{3}(-1, 1, -1) \cong \frac{1}{3}(1, 1, 2)$.) By the Reid-Tai criterion (Theorem 1.1), this singularity is terminal.

Next, consider the chart $\{w_1 = 1\}$ in Y_2 over $\{y_1 = 1\}$ in Y_1. As shown above, in coordinates (w_0, x_1, y_2), the points $(0, 0, 0)$ and $(-1, 0, 0)$ are bad for the action of G on Y_2, and $(Y_2)_{\text{red}} = \{x_1 = 0\}$. Here again, the singularity of Y_2/G at these points is of the form $\frac{1}{3}(1, 1, 2)$. (For example, for the point $(w_0, x_1, y_2) = (0, 0, 0)$, we saw that $I(w_0) = w_0 x_1(1 + w_0 + O(x_1))$, $I(x_1) = x_1^2(-1 + w_0 + O(x_1))$, and $I(y_2) = x_1 y_2(1 - w_0 - y_2 + O(x_1))$. So Theorem 2.2 applies with $e = s := x_1$. The linear map φ in the theorem is given by $\varphi(w_0) = w_0$, $\varphi(x_1) = -x_1$, and $\varphi(y_2) = y_2$. This has eigenvalues $1, -1, 1$, and so the theorem gives that the singularity of Y_2/G

at this point is of the form $\frac{1}{3}(1,-1,1) \cong \frac{1}{3}(1,1,2)$.) In particular, these two points are terminal. Thus Y_2/G is terminal at five of its six singular points.

Finally, we consider the last singular point of Y_2/G. In the chart $\{w_0 = 1\}$ in Y_2 over $\{y_1 = 1\}$ in Y_1, the point P is $(y_0, w_1, y_2) = (0, 0, 0)$, and $(Y_2)^G_{\text{red}} = \{y_0 = 0\}$. In this case, our calculation of the action of G shows (by Theorem 2.2) that the singularity of Y_2/G is of the form $\frac{1}{3}(1,1,1)$. (As above, we have $I(y_0) = y_0^2(-1 + O(y_0))$, $I(w_1) = y_0 w_1(-1 - w_1 + O(y_0))$, and $I(y_2) = y_0 y_2(-1 + w_1 - w_1 y_2 + O(y_0))$. So Theorem 2.2 applies with $e = s := y_0$. The linear map φ in the theorem is given by $\varphi(y_0) = -y_0$, $\varphi(w_1) = -w_1$, and $\varphi(y_2) = -y_2$. So Theorem 2.2 gives that the singularity of Y_2/G at this point is of the form $\frac{1}{3}(-1,-1,-1) \cong \frac{1}{3}(1,1,1)$.) By the Reid-Tai criterion (Theorem 1.1), Y_2/G is canonical at this point, but not terminal.

We can now begin the proof that Y_0/G is terminal. Write f for any of the quotient maps $Y_j \to Y_j/G$. The fixed point scheme Y_2^G is the Cartier divisor E_1 except at six points on E_1. Clearly $3K_{Y_2/G}$ is Cartier. Write $E_0 \subset Y_2$ for the strict transform of $E_0 \subset Y_1$. For $j = 0, 1$ in Y_2, let F_j be the image (as an irreducible divisor) of E_j in Y_2/G. Since G acts nontrivially on E_0, we have $f^*F_0 = E_0$. The divisor E_1 is fixed by G. By Corollary 5.2, f is fiercely ramified along E_1. So the ramification divisor is $p - 1$ times E_1, meaning that $K_{Y_2} = f^*(K_{Y_2/G}) + 2E_1$, and we have $f^*F_1 = E_1$.

Write π_{ij} for the birational morphism $Y_i \to Y_j$ or $Y_i/G \to Y_j/G$ (with $i > j$). Since $\pi_{20}\colon Y_2 \to Y_0$ is defined by blowing up points and smooth curves on a smooth 3-fold, we have:

$$K_{Y_1} = \pi_{10}^*(K_{Y_0}) + 2E_0$$

and

$$\begin{aligned} K_{Y_2} &= \pi_{21}^*(K_{Y_1}) + E_1 \\ &= \pi_{20}^*(K_{Y_0}) + 2(E_0 + E_1) + E_1 \\ &= \pi_{20}^*(K_{Y_0}) + 2E_0 + 3E_1. \end{aligned}$$

Therefore,

$$\begin{aligned} f^*\pi_{20}^* K_{Y_0/G} &= \pi_{20}^* f^* K_{Y_0/G} \\ &= \pi_{20}^* K_{Y_0} \\ &= K_{Y_2} - 2E_0 - 3E_1 \\ &= f^*(K_{Y_2/G}) + 2E_1 - 2E_0 - 3E_1 \\ &= f^*(K_{Y_2/G}) - 2E_0 - E_1 \\ &= f^*(K_{Y_2/G} - 2F_0 - F_1). \end{aligned}$$

So

$$K_{Y_2/G} = \pi_{20}^*(K_{Y_0/G}) + 2F_0 + F_1.$$

Here every exceptional divisor of the birational morphism $Y_2/G \to Y_0/G$ has positive coefficient. Also, we have shown that Y_2/G is terminal outside one point which is canonical, and that point lies on F_1. Therefore, Y_0/G is terminal. We showed earlier that it is not Cohen-Macaulay. Theorem 6.1 is proved. □

Remark 6.2. The divisor class $\pi^* K_{Y_0/G} = K_{Y_2/G} - 2F_0 - F_1$ has some non-integer discrepancies, for example over the origin in the chart $\{z_2 = 1\}$ in Y_2 over $\{y_2 = 1\}$

in Y_1. As a result, $K_{Y_0/G}$ is not Cartier (as one can also check directly), in contrast to our examples in residue characteristic 2 ([29, Theorem 5.1] and Theorem 5.1). I expect that there is also a 3-fold X over $\mathbf{F}_3$ that is terminal and non-Cohen-Macaulay with K_X Cartier. Namely, one should replace $\mathbf{P}^1$ in Theorem 6.1 by the "Harbater-Katz-Gabber" curve $C = \{0 = y^p - yz^{p-1} - x^{p-1}z\}$ in $\mathbf{P}^2_{\mathbf{F}_p}$ [9], here with $p = 3$, which has a $\mathbf{Z}/p$-action by $\tau([x, y, z]) = [x, y + z, z]$ that preserves a nonzero 1-form near the fixed point $[1, 0, 0]$. For $p = 3$, C is a supersingular elliptic curve.

7 The example over the 3-adic integers

Theorem 7.1. *Let G be the group $G = \mathbf{Z}/3$ with generator τ. Let $R = \mathbf{Z}_3[e]/(e^3 - 3e^2 + 3)$, which is the ring of integers in a Galois extension of $\mathbf{Q}_3$ with group $G = \mathbf{Z}/3$. Let G act on the scheme $\mathbf{P}^2_R$ by*

$$\tau([u_0, u_1, u_2], e) = ([u_1, u_2, u_0], 3 + e - e^2).$$

Then the scheme $\mathbf{P}^2_R/G$ is terminal, not Cohen-Macaulay, of dimension 3, and flat over $\mathbf{Z}_3$.

This example behaves much like the example over $\mathbf{F}_3$, Theorem 6.1. Theorem 8.1. *In particular, Figure 2 accurately depicts the blow-ups we make in mixed characteristic* $(0, 3)$*, just as in characteristic 3.* We can view R as the subring $\mathbf{Z}_3[\zeta_9]^+$ of the cyclotomic ring $\mathbf{Z}_3[\zeta_9]$ fixed by $\zeta_9 \mapsto \zeta_9^{-1}$, with $e = \zeta_9 + 1 + \zeta_9^{-1}$. Informally, R is the simplest ramified $\mathbf{Z}/3$-extension of $\mathbf{Z}_3$. More broadly, this action of G on $\mathbf{P}^2_R$ was chosen as possibly the simplest action of $\mathbf{Z}/3$ on a 3-fold in mixed characteristic $(0, 3)$ with an isolated fixed point. The simplicity helps to ensure that the quotient scheme is terminal.

Proof. We write $G = \mathbf{Z}/3 = \langle \sigma : \sigma^3 = 1 \rangle$, with $\tau := \sigma^{-1}$. Let $Y_0 = \mathbf{P}^2_R$ with G acting diagonally on $\mathbf{P}^2$ and on R, and let $X = Y_0/G$. Write e_2 for the generator e of R, to fit better with our numbering of coordinates on Y_0; so we have

$$0 = e_2^3 - 3e_2^2 + 3.$$

The only fixed point of G on Y_0 is the closed point $P \cong \operatorname{Spec} \mathbf{F}_3$ given by $([u_0, u_1, u_2], e_2) = ([1, 1, 1], 0)$. So X is normal of dimension 3, and X is regular outside the image of P, which we also call P. Clearly $3K_X$ is Cartier.

It is not automatic from Fogarty's results [13], but we can use his methods to show that X is not Cohen-Macaulay at P. As in the proof of Theorem 5, using that G has an isolated fixed point on the 3-fold Y_0, it suffices to show that $H^1(G, O(Y_0))$ is not zero. This cohomology group is $\ker(\mathrm{tr})/\operatorname{im}(1 - \sigma)$ on $O(Y_0)$, where the trace is $1 + \sigma + \sigma^2$. The equation $0 = e_2^3 - 3e_2^2 + 3$ (specifically, the coefficient of e_2^2) implies that e_2 has trace 3. So $\mathrm{tr}(1 - e_2) = 0$, and hence $1 - e_2$ defines an element of $H^1(G, O(Y_0))$. Note that $1 - e_2$ restricts to $1 \in O(P) = \mathbf{F}_3$ on the fixed point P. Therefore, $1 - e_2$ has nonzero image under the restriction map $H^1(G, O(Y_0)) \to H^1(G, O(P)) \cong \mathbf{F}_3$. So $H^1(G, O(Y_0))$ is not zero, and hence Y_0/G is not Cohen-Macaulay.

It remains to show that X is terminal. One can resolve the singularities of X by performing G-equivariant blow-ups of Y_0. However, as in sections 5 and 6, we will shorten the proof by recognizing that, after two G-equivariant blow-ups $Y_2 \to Y_1 \to Y_0$, the singularities of Y_2/G become toric (in Kato's mixed-characteristic sense), namely quotients of a regular scheme by μ_3. That makes it easy to check that Y_0/G is terminal, without having to continue making G-equivariant blow-ups.

To put the fixed point at the origin, we change coordinates on Y_0 by: $x_0 = (u_0 + u_1 + u_2 - 3)/u_1$ and $x_1 = (-u_1 + u_2)/u_1$. Then G acts on

$$U := \{(x_0, x_1, e_2) \in A^3_{\mathbf{Z}_3} : 0 = 3 - 3e_2^2 + e_2^3, 1 + x_1 \neq 0, 1 + x_0 - x_1 \neq 0\}$$

by

$$\tau(x_0, x_1, e_2) = \left(\frac{x_0 - 3x_1}{1 + x_1}, \frac{x_0 - 2x_1}{1 + x_1}, 3 + e_2 - e_2^2\right).$$

In what follows, we will often not need to keep track of the precise open set on which G acts, because we are only concerned with the G-fixed point scheme.

The blow-up $Y_1 \to Y_0$ at the G-fixed point is, over the open set $U \subset Y_0$:

$$\{((x_0, x_1, e_2), [y_0, y_1, y_2]) \in U \times_{\mathbf{Z}_3} \mathbf{P}^2_{\mathbf{Z}_3} : x_0 y_1 = x_1 y_0,\ x_0 y_2 = e_2 y_0,\ x_1 y_2 = e_2 y_1\}.$$

Clearly the fixed point set Y_1^G is contained in the exceptional divisor $E_0 \cong \mathbf{P}^2_{\mathbf{F}_3}$. It turns out to be a curve isomorphic to $\mathbf{P}^1_{\mathbf{F}_3}$. We need three coordinate charts to cover E_0. First look at the open set $U_0 = \{y_0 = 1\}$ in Y_1. Then $(x_0, x_1, e_2) = (x_0, x_0 y_1, x_0 y_2)$, and so

$$U_0 = \{(x_0, y_1, y_2) \in A^3_{\mathbf{Z}_3} : 0 = x_0^3 y_2^3 - 3x_0^2 y_2^2 + 3,\ 1 + x_0 y_1 \neq 0,\ 1 + x_0 - x_0 y_1 \neq 0\}.$$

The exceptional divisor E_0 is $\{x_0 = 0\}$. Here G acts by

$$\tau(x_0, y_1, y_2) = \left(\frac{x_0(1 - 3y_1)}{1 + x_0 y_1}, \frac{1 - 2y_1}{1 - 3y_1}, \frac{y_2(1 + 2x_0 y_2 - x_0^2 y_2^2)(1 + x_0 y_1)}{1 - 3y_1}\right).$$

We know that the fixed point scheme Y_1^G is contained in $E_0 = \{x_0 = 0\}$ as a set, and that $3 = 0$ on E_0. The fixed point scheme is defined by: $I(x_0) = x_0^2 y_1(-1 + O(x_0))$, $I(y_1) = 1 + O(x_0)$, and $I(y_2) = x_0 y_2(y_1 - y_2 + O(x_0))$. By the second equation, Y_1^G is empty in this open set.

Next, consider the open set $U_1 = \{y_1 = 1\}$ in Y_1. Then $(x_0, x_1, e_2) = (x_1 y_0, x_1, x_1 y_2)$, and G acts on

$$U_1 = \{(y_0, x_1, y_2) \in A^3_{\mathbf{Z}_3} : 0 = 3 - 3x_1^2 y_2^2 + x_1^3 y_2^3,\ 1 + x_1 \neq 0,\ 1 - x_1 + x_1 y_2 \neq 0\}.$$

Namely, G acts by

$$\tau(y_0, x_1, y_2) = \left(\frac{y_0 - 3}{y_0 - 2}, \frac{x_1(y_0 - 2)}{1 + x_1}, \frac{y_2(1 + x_1)(1 + 2x_1 y_2 - x_1^2 y_2^2)}{y_0 - 2}\right).$$

Here $E_0 = \{x_1 = 0\}$. We know that Y_1^G is contained in $E_0 = \{x_1 = 0\}$, as a set (and hence $3 = 0$ on $(Y_1)^G_{\text{red}}$). More precisely, the fixed point scheme Y_1^G is defined by: $I(y_0) = (-y_0^2 + O(x_1))/(1 + y_0 + O(x_1))$, $I(x_1) = x_1(y_0 + O(x_1))$, and $I(y_2) = y_2(-y_0 + O(x_1))/(1 + y_0 + O(x_1))$. So Y_1^G is the line $\{y_0 = x_1 = 0\}$, as a set.

Finally, consider the chart $\{y_2 = 1\}$ in Y_1. We have coordinates (y_0, y_1, e_2), with $(x_0, x_1, e_2) = (e_2y_0, e_2y_1, e_2)$, and $E_0 = \{e_2 = 0\}$. Here G acts on the open set

$$U_2 = \{(y_0, y_1, e_2) \in A^3_{\mathbf{Z}_3} : 0 = 3 - 3e_2^2 + e_2^3,\ 1 + e_2y_0 \neq 0,\ 1 + e_2y_0 - e_2y_1 \neq 0\}.$$

Namely, G acts by

$$\tau(y_0, y_1, e_2) = \left(\frac{(y_0 - 3y_1)(1 + e_2 - e_2^2)}{1 + e_2y_1},\ \frac{(y_0 - 2y_1)(1 + e_2 - e_2^2)}{1 + e_2y_1},\ 3 + e_2 - e_2^2 \right).$$

The fixed point scheme Y_1^G is defined by: $I(y_0) = e_2(y_0 - y_0y_1 + O(e_2))$, $I(y_1) = y_0 + O(e_2)$, and $I(e_2) = e_2^2(-1 + O(e_2))$. We know that the fixed point set is contained in $E_0 = \{e_2 = 0\}$. We read off that the fixed point set is the line $0 = y_0 = e_2$, which is the same line seen in the previous chart. Thus we have shown that the fixed point set in all of Y_1 is a curve isomorphic to $\mathbf{P}^1_{\mathbf{F}_3}$ in E_0.

Seeking to make the fixed point set a divisor, we let Y_2 be the blow-up of Y_1 along the (reduced) G-fixed curve, with exceptional divisor $E_1 \subset Y_2$. It is clear that Y_2^G is contained in E_1 as a set. We will see that the fixed point scheme Y_2^G is equal to the Cartier divisor E_1 except at six points on E_1. These correspond exactly to the six bad points that occur in the example over $\mathbf{F}_3$ (Figure 2). Since E_1 is a $\mathbf{P}^1$-bundle over $\mathbf{P}^1_{\mathbf{F}_3}$, we will need to look at four affine charts to see all of it.

First work over the open subset $\{y_1 = 1\}$ in Y_1, with coordinates (y_0, x_1, y_2), where $E_0 = \{x_1 = 0\}$. Since Y_2 is the blow-up of Y_1 along the G-fixed curve $\{0 = y_0 = x_1\}$, this part of Y_2 is given by

$$\{((y_0, x_1, y_2), [w_0, w_1]) \in A^3_{\mathbf{Z}_3} \times \mathbf{P}^1_{\mathbf{Z}_3} : 0 = 3 - 3x_1^2y_2^2 + x_1^3y_2^3,\ y_0w_1 = x_1w_0\}.$$

First consider the open set $\{w_0 = 1\} \subset Y_2$ over $\{y_1 = 1\} \subset Y_1$. Then $(y_0, x_1, y_2) = (y_0, y_0w_1, y_2)$, $E_0 = \{w_1 = 0\}$, and $E_1 = \{y_0 = 0\}$. Also, $e_2 = y_0w_1y_2$, and so $3 = 3y_0^2w_1^2y_2^2 - y_0^3w_1^3y_2^3$. Here G acts by

$$\tau(y_0, w_1, y_2) = \left(\frac{y_0 - 3}{y_0 - 2},\ \frac{w_1(y_0 - 2)^2}{(1 + y_0w_1)(1 - 3y_0w_1^2y_2^2 + y_0^2w_1^3y_2^3)}, \right.$$
$$\left. \frac{y_2(1 + y_0w_1)(1 + 2y_0w_1y_2 - y_0^2w_1^2y_2^2)}{y_0 - 2} \right).$$

The fixed point scheme Y_2^G (near E_1) is defined by: $I(y_0) = y_0^2(-1 + O(y_0))$, $I(w_1) = y_0w_1(-1 - w_1 + O(y_0))$, and $I(y_2) = y_0y_2(-1 + w_1 - w_1y_2 + O(y_0))$. So Y_2^G is the Cartier divisor E_1 except where $0 = y_2(1 - w_1 + w_1y_2)$ and $0 = w_1(1 + w_1)$. So we have found three bad points, $(y_0, w_1, y_2) = (0, 0, 0)$ in $E_0 \cap E_1$ and $(0, -1, 0)$, and $(0, -1, -1)$ in E_1.

The other chart over $\{y_1 = 1\}$ in Y_1 is $\{w_1 = 1\}$ in Y_2. Then $(y_0, x_1, y_2) = (w_0x_1, x_1, y_2)$, E_0 does not appear, and $E_1 = \{x_1 = 0\}$. Also, $e_2 = x_1y_2$, and so $3 = 3x_1^2y_2^2 - x_1^3y_2^3$. Here G acts by

$$\tau(w_0, x_1, y_2) = \left(\frac{(w_0 - 3x_1y_2^2 + x_1^2y_2^3)(1 + x_1)}{(w_0x_1 - 2)^2},\ \frac{x_1(w_0x_1 - 2)}{1 + x_1}, \right.$$
$$\left. \frac{y_2(1 + x_1)(1 + 2x_1y_2 - x_1^2y_2^2)}{w_0x_1 - 2} \right).$$

So the fixed point scheme Y_2^G (near E_1) is defined by: $I(w_0) = x_1(w_0 + w_0^2 + O(x_1))$, $I(x_1) = x_1^2(-1 + w_0 + O(x_1))$, and $I(y_2) = x_1 y_2(1 - w_0 - y_2 + O(x_1))$. So Y_2^G is the Cartier divisor E_1 except where $x_1 = 0$ (so $3 = 0$), $0 = w_0(1 + w_0)$, and $0 = y_2(1 - w_0 - y_2)$. So the bad points are $(w_0, x_1, y_2) = (0, 0, 0)$, $(0, 0, 1)$, $(-1, 0, 0)$, and $(-1, 0, -1)$. The points with $w_0 \neq 0$ appeared in the previous chart, $\{w_0 = 1\}$. So we have two new bad points, (w_0, x_1, y_2) equal to $(0, 0, 0)$ or $(0, 0, 1)$, for a total of five so far.

To see all of E_1 in Y_2, we also have to work over the open set $\{y_2 = 1\} \subset Y_1$, with coordinates (y_0, y_1, x_2), where $E_0 = \{x_2 = 0\}$. The corresponding open subset of Y_2 is an open subset of

$$\{((y_0, y_1, e_2), [z_0, z_2]) \in A^3_{\mathbf{Z}_3} \times_{\mathbf{Z}_3} \mathbf{P}^1_{\mathbf{Z}_3} : 0 = 3 - 3e_2^2 + e_2^3,\ y_0 z_2 = e_2 z_0\}.$$

First consider the chart $\{z_0 = 1\} \subset Y_2$. Then $(y_0, y_1, e_2) = (y_0, y_1, y_0 z_2)$, $E_0 = \{z_2 = 0\}$, and $E_1 = \{y_1 = 0\}$. Also, $e_2 = y_0 z_2$, and so $0 = 3 - 3y_0^2 z_2^2 + y_0^3 z_2^3$. Here G acts by

$$\tau(y_0, y_1, z_2) = \left(\frac{(y_0 - 3y_1)(1 + y_0 z_2 - y_0^2 z_2^2)}{1 + y_0 y_1 z_2}, \right.$$
$$\left. \frac{(y_0 - 2y_1)(1 + y_0 z_2 - y_0^2 z_2^2)}{1 + y_0 y_1 z_2}, \frac{z_2(4 + y_0 z_2 - y_0^2 z_2^2)(1 + y_0 y_1 z_2)}{1 - 3y_0 y_1 z_2^2 + y_0^2 y_1 z_2^3} \right).$$

The fixed point scheme (near E_1) is defined by: $I(y_0) = y_0^2 z_2(1 - y_1 + O(y_0))$, $I(y_1) = y_0(1 + y_1 z_2 - y_1^2 z_2 + O(y_0))$, and $I(z_2) = y_0 z_2^2(1 + y_1 + O(y_0))$. So Y_2^G is the Cartier divisor E_1 except where $y_0 = 0$ (so $3 = 0$), $0 = 1 + y_1 z_2 - y_1^2 z_2$, and $0 = z_2^2(1 + y_1)$. So we have one bad point in this open set, $(y_0, y_1, z_2) = (0, -1, -1)$. This already appeared in the chart $\{w_0 = 1\}$ over $\{y_1 = 1\}$ (as the point $(y_0, w_1, y_2) = (0, -1, -1)$).

The other chart is $\{z_2 = 1\} \subset Y_2$ over $\{y_2 = 1\}$ in Y_1, with coordinates (z_0, y_1, e_2). Here $y_0 = e_2 z_0$, E_0 does not appear, and $E_1 = \{e_2 = 0\}$. Here G acts by

$$\tau(z_0, y_1, e_2) = \left(\frac{(z_0 - 3e_2 y_1 + e_2^2 y_1)(-2 - e_2 + 2e_2^2)}{1 + e_2 y_1}, \right.$$
$$\left. \frac{(e_2 z_0 - 2y_1)(1 + e_2 - e_2^2)}{1 + e_2 y_1}, 3 + e_2 - e_2^2 \right).$$

We know that the fixed point scheme Y_2^G is contained in E_1 as a set. (Also, $3 = O(e_2)$ by the equation for Y_2.) Explicitly, the fixed point scheme (near E_2) is defined by: $I(z_0) = e_2(-z_0 - z_0 y_1 + O(e_2))$, $I(y_1) = e_2(z_0 + y_1 - y_1^2 + O(e_2))$, and $I(e_2) = e_2^2(-1 + O(e_2))$. So Y_2^G is the Cartier divisor E_1 except where $e_2 = 0$ (so $3 = 0$), $0 = z_0(1 + y_1)$, and $0 = z_0 + y_1 - y_1^2$. So we see three bad points in E_1: $(z_0, y_1, x_2) = (0, 1, 0)$, $(-1, -1, 0)$, or $(0, 0, 0)$. The first two points appeared in the chart $\{w_1 = 1\}$ in Y_2 over $\{y_1 = 1\}$ in Y_1 (as $(w_0, x_1, y_2) = (0, 0, 1)$ or $(-1, 0, -1)$, respectively). On the other hand, the origin in this chart is new; so we have a total of six bad points.

Thus the fixed point scheme Y_2^G is E_1 with multiplicity 1 except at six points on E_1. As in Theorem 6.1, Theorem 2.2 shows that five of the singular points of

Y_2/G are toric singularities of the form $\frac{1}{3}(1,1,2)$ (hence terminal), while the sixth is a toric singularity of the form $\frac{1}{3}(1,1,1)$ (hence canonical). In fact, our calculations of $I = \sigma - 1$ on the coordinates in this section are identical to those in section 6, to the accuracy we state. We also need to check the assumption in Theorem 2.2 that $p \in e^{p-1}\mathfrak{m}$, that is, that $3 \in e^2\mathfrak{m}$. This is true because $3 = e_2^3(\text{unit})$ on Y_2, and e_2 is a multiple of the function e defining E_1 in each coordinate chart; so 3 is in the ideal (e^3), hence in $e^2\mathfrak{m}$ at each of the bad points. As a result, Theorem 2.2 gives the conclusions above about the 6 singular points of Y_2/G.

The calculation of the discrepancies of Y_0/G is likewise identical to the calculation in section 6. Therefore, Y_0/G is terminal. We showed earlier that it is not Cohen-Macaulay. Theorem 7.1 is proved. ☐

Remark 7.2. In Theorem 7.1, the canonical class of Y_0/G is not Cartier. I expect that there is also a 3-dimensional scheme X, flat over $\mathbf{Z}_3$, that is terminal and non-Cohen-Macaulay with K_X Cartier. Namely, one should replace the p-adic integer ring $R = \mathbf{Z}_3[\zeta_9]^{\mathbf{Z}/2}$ in Theorem 7.1 by $S = \mathbf{Z}_3[\zeta_9]$, with the action of $G = \mathbf{Z}/3 \subset (\mathbf{Z}/9)^*$. The point is that the canonical sheaf of S over $\mathbf{Z}_3$ has a G-equivariant trivialization.

8 Characteristic 5

Theorem 8.1. *Let the group $G = \mathbf{Z}/5$ with generator τ act on the quintic del Pezzo surface S_5 over $\mathbf{F}_5$ by an embedding of G into the symmetric group $\Sigma_5 = \mathrm{Aut}(S_5)$, and let G act on $\mathbf{P}^1$ by*

$$\tau([y_0, y_1]) = [y_0, y_0 + y_1].$$

Then $(S_5 \times \mathbf{P}^1)/G$ is terminal, not Cohen-Macaulay, and of dimension 3 over $\mathbf{F}_5$.

We define the quintic del Pezzo surface (over any field) as the moduli space $\overline{M_{0,5}}$ of 5-pointed stable curves of genus 0. That makes it clear that the symmetric group Σ_5 acts on this surface.

Proof. We work throughout over $k = \mathbf{F}_5$. Write $G = \mathbf{Z}/5 = \langle \sigma : \sigma^5 = 1 \rangle$, with $\tau := \sigma^{-1}$. Let $Y_0 = S_5 \times \mathbf{P}^1$ and $X = Y_0/G$. In characteristic 5, G has only one fixed point in S_5, and so G has only one fixed point P in Y_0. So X is normal of dimension 3, and X is smooth over k outside the image of P (which we also call P). Also, $5K_X$ is Cartier. By Fogarty, since P is an isolated fixed point of $G = \mathbf{Z}/p$ on a smooth 3-fold in characteristic p, X is not Cohen-Macaulay at P [13, Proposition 4].

It remains to show that X is terminal. This example is more complicated than those in characteristics 2 and 3, and it may be impossible to resolve the singularities of X by performing G-equivariant blow-ups of Y_0. (Indeed, in the simpler situation of actions of $G = \mathbf{Z}/p$ in characteristic zero, one cannot always resolve the singularities of a quotient Y/G via G-equivariant blow-ups of Y when $p \geq 5$ [19, Claim 2.29.2].) Fortunately, as in earlier sections, we can reach toric singularities after some G-equivariant blow-ups. It will then be easy to check that Y_0/G is terminal.

The G-action on S_5 over k is given on an open subset isomorphic to an open subset of A^2 by:

$$\tau(s_0, s_1) = \left(\frac{s_0 - s_1 + s_0^2 + s_0 s_1}{(1-2s_0)(1-s_0-s_1)}, \frac{s_1 - 2s_0^2 - 2s_0 s_1}{(1-2s_0)(1-s_0-s_1)} \right).$$

Here the fixed point is at the origin. (Section 9 explains where this formula comes from.) So the G-action on an open subset $U \subset Y_0$ is given (on an open neighborhood of the origin in A^3) by:

$$\tau(s_0, s_1, s_2) = \left(\frac{s_0 - s_1 + s_0^2 + s_0 s_1}{(1-2s_0)(1-s_0-s_1)}, \frac{s_1 - 2s_0^2 - 2s_0 s_1}{(1-2s_0)(1-s_0-s_1)}, \frac{s_2}{1+s_2} \right).$$

As we blow up, we will not need to keep track of the precise affine open set on which G acts, since we are only concerned with the action near the fixed point set.

Let Y_1 be the blow-up of Y_0 at the G-fixed point, which is the origin in these coordinates. Then the open subset of Y_1 over $U \subset Y_0$ is

$$\{((s_0, s_1, s_2), [y_0, y_1, y_2]) \in U \times \mathbf{P}^2 : s_0 y_1 = s_1 y_0,\ s_0 y_2 = s_2 y_0, s_1 y_2 = s_2 y_1\}.$$

The exceptional divisor E_0 is isomorphic to $\mathbf{P}^2$. It turns out that the fixed point set of G on Y_1 is a curve isomorphic to $\mathbf{P}^1$ in E_0. To check that, first work in the open subset $\{y_0 = 1\}$ in Y_1. Here $(s_0, s_1, s_2) = (s_0, s_0 y_1, s_0 y_2)$, and G acts by

$$\tau(s_0, y_1, y_2) = \left(\frac{s_0(1 + s_0 - y_1 + s_0 y_1)}{(1-2s_0)(1-s_0-s_0 y_1)}, \frac{-2s_0 + y_1 - 2s_0 y_1}{1 + s_0 - y_1 + s_0 y_1}, \right.$$
$$\left. \frac{y_2(1-2s_0)(1-s_0-s_0 y_1)}{(1+s_0 y_2)(1+s_0-y_1+s_0 y_1)} \right).$$

Here $E_0 = \{s_0 = 0\}$. The fixed point scheme Y_1^G is defined by the vanishing of: $I(s_0) = s_0(-y_1 + O(s_0))$, $I(y_1) = (y_1^2 + O(s_0))/(1 - y_1 + O(s_0))$, and $I(y_2) = y_2(y_1 + O(s_0))/(1 - y_1 + O(s_0))$. We know that Y_1^G is contained (as a set) in E_0 (since Y_0^G is only the origin). So the fixed point set is the line $\{0 = s_0 = y_1\}$, in this chart.

In the chart $\{y_1 = 1\}$ in Y_1, we have $s_0 = s_1 y_0$ and $s_2 = s_1 y_2$, so we have coordinates (y_0, s_1, y_2). Here $E_0 = \{s_1 = 0\}$. We can write the action of G in these coordinates (for example using Magma). We find that the fixed point scheme Y_1^G is defined by: $I(y_0) = -1 + O(s_1)$, $I(s_1) = s_1^2(1 + y_0 - 2y_0^2)$, and $I(y_2) = s_1 y_2(-1 - y_0 - y_2 + 2y_0^2)$. Since Y_1^G is contained (as a set) in E_0, the first equation shows that Y_1^G is empty, in this chart. In the last chart $\{y_2 = 1\}$ in Y_1, we have coordinates (y_0, y_1, s_2), and $E_0 = \{s_2 = 0\}$. The fixed point scheme is defined by: $I(y_0) = -y_1 + O(s_2)$, $I(y_1) = s_2(y_1 + y_1^2 + y_0 y_1 - 2y_0^2 + O(s_2))$, and $I(s_2) = s_2^2(-1 + O(s_2))$. Since Y_1^G is contained (as a set) in E_0, the fixed point set is the line $\{0 = y_1 = s_2\}$, the same line seen in an earlier chart.

Thus $(Y_1^G)_{\text{red}}$ is isomorphic to $\mathbf{P}^1$. Our criterion for a quotient by G to have toric singularities (Theorem 2.2) requires the G-fixed locus to have codimension 1; so let Y_2 be the blow-up of Y_1 along this $\mathbf{P}^1$. Clearly G continues to act on Y_2. The exceptional divisor E_1 in Y_2 is a $\mathbf{P}^1$-bundle over $\mathbf{P}^1$, and so the natural way to cover E_1 by affine charts involves 4 charts, as follows.

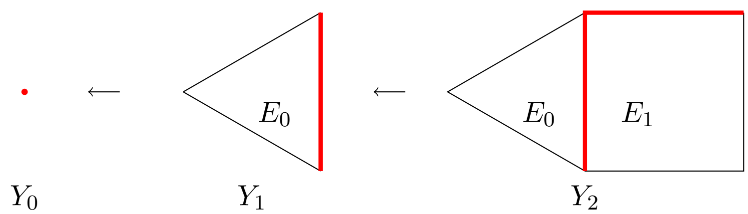


Figure 3: For $j = 0, 1, 2$, G acts freely on Y_j outside the red loci.

Over the open set $\{y_0 = 1\}$ in Y_1, Y_2 is the blow-up along the G-fixed curve $\{0 = s_0 = y_1\}$, so Y_2 has coordinates $((s_0, y_1, y_2), [w_0, w_1])$. First take $\{w_0 = 1\}$, so $y_1 = s_0 w_1$, and we have coordinates (s_0, w_1, y_2). Here E_0 does not appear, and $E_1 = \{s_0 = 0\}$. The fixed point scheme Y_2^G is defined by: $I(s_0) = s_0^2(-1 - w_1 + O(s_0))$, $I(w_1) = -2 + O(s_0)$, and $I(y_2) = s_0 y_2(1 + w_1 - y_2 + O(s_0))$. We know that the fixed point set is contained in E_1, and so the formula for $I(w_1)$ implies that Y_2^G is empty, in this chart. In the other chart $\{w_1 = 1\}$ in Y_2 over the same open set in Y_1, we have $s_0 = y_1 w_0$, and so Y_2 has coordinates (w_0, y_1, y_2). Here $E_0 = \{w_0 = 0\}$ and $E_1 = \{y_1 = 0\}$. The fixed point scheme is defined by $I(w_0) = w_0(2w_0 + O(y_1))/(1 - 2w_0 + O(y_1))$, $I(y_1) = y_1(-2w_0 + O(y_1))$, and $I(y_2) = y_1 y_2(1 + w_0 - w_0 y_2 + O(y_1))$. So Y_2^G is the line $\{0 = w_0 = y_1\} = E_0 \cap E_1$, in this chart.

To see the rest of $E_1 \subset Y_2$, work over the open set $\{y_2 = 1\}$ in Y_1. Here Y_2 is the blow-up along the G-fixed curve $\{0 = y_1 = s_2\}$, so Y_2 has coordinates $((y_0, y_1, s_2), [r_1, r_2])$. First take $\{r_1 = 1\}$ in Y_2, so $s_2 = y_1 r_2$, and we have coordinates (y_0, y_1, r_2). Here $E_0 = \{r_2 = 0\}$ and $E_1 = \{y_1 = 0\}$. Here Y_2^G is given by $I(y_0) = y_1(-1 + y_0 r_2 - y_0^2 r_2 + O(y_1))$, $I(y_1) = y_1 r_2(-2y_0^2 + O(y_1))$, and $I(r_2) = r_2^2(2y_0^2 + O(y_1))/(1 - 2y_0^2 r_2 + O(y_1))$. We know that the fixed point set is contained in E_1, and we read off that it is the union of the two lines $\{0 = y_1 = r_2\} = E_0 \cap E_1$ and $\{0 = y_0 = y_1\}$ in E_1. The first curve appeared in an earlier chart, and the second is new. Finally, the other open set is $\{r_2 = 1\}$ in Y_2, so $y_1 = s_2 r_1$, and we have coordinates (y_0, r_1, s_2). Here E_0 does not appear, and $E_1 = \{s_2 = 0\}$. Here Y_2^G is given by $I(y_0) = s_2(y_0 - r_1 - y_0^2 + O(s_2))$, $I(r_1) = -2y_0^2 + O(s_2)$, and $I(s_2) = s_2^2(-1 + O(s_2))$. We read off that the fixed point set is the curve $\{0 = y_0 = s_2\}$, which is the second curve in the previous chart.

Thus $(Y_2)^G$ as a set is the union of two $\mathbf{P}^1$'s meeting at a point. We are trying to make the fixed locus have codimension 1, and so our next step is to blow up one of those curves. Namely, let Y_3 be the blow-up of Y_2 along the G-fixed curve $E_0 \cap E_1$. The exceptional divisor E_2 in Y_3 is a $\mathbf{P}^1$-bundle over $\mathbf{P}^1$, and so we need to look at four affine charts to see all of it.

First, work over the open set $\{r_1 = 1\}$ in Y_2 over $\{y_2 = 1\}$ in Y_1. Then Y_3 is the blow-up along the curve $\{0 = y_1 = r_2\} = E_0 \cap E_1$, and so Y_3 has coordinates $(y_0, y_1, r_2), [z_1, z_2]$. First take $\{z_1 = 1\}$, so $r_2 = y_1 z_2$, and we have coordinates (y_0, y_1, z_2). Here $E_0 = \{z_2 = 0\}$, E_1 does not appear, and $E_2 = \{y_1 = 0\}$. The fixed point scheme Y_3^G is defined by: $I(y_0) = y_1(-1 + O(y_1))$, $I(y_1) = y_1^2 z_2(-2y_0^2 + O(y_1))$, and $I(z_2) = y_1 z_2^2(-y_0^2 + O(y_1))$. These equations are equivalent to $y_1 = 0$, near E_2; so the fixed point scheme Y_2^G is the Cartier divisor E_2, in this chart. (Thus, by Theorem 2.1, Y_2/G is smooth over $k = \mathbf{F}_5$, in this open set.)

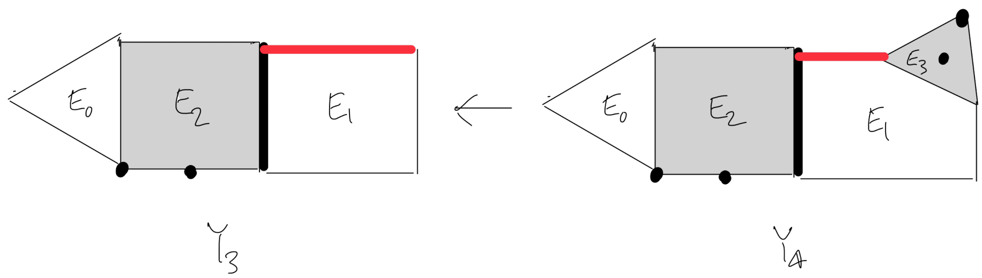


Figure 4: For $j = 3, 4$, G acts freely on Y_j outside the shaded or marked loci; Y_j/G is regular outside the marked loci; and Y_j/G has toric singularities outside the red loci.

The other chart is $\{z_2 = 1\}$ in Y_3, so $y_1 = r_2z_1$, and we have coordinates (y_0, z_1, r_2). Here E_0 does not appear, $E_1 = \{z_1 = 0\}$, and $E_2 = \{r_2 = 0\}$. The fixed point scheme Y_3^G is given by $I(y_0) = z_1r_2(-1+O(r_2))$, $I(z_1) = z_1r_2(y_0^2+O(r_2))$, and $I(r_2) = r_2^2(2y_0^2 + O(r_2))$. The fixed point scheme is generically E_2 with multiplicity 1, together with the other fixed curve we knew from Y_2, here given by $\{0 = y_0 = z_1\} \subset E_1$. In more detail, the "bad locus" where the scheme Y_3^G is not just E_2 as a Cartier divisor is given by removing a factor of r_2 from these equations, yielding: $0 = z_1(-1 + O(r_2))$, $0 = z_1(y_0^2 + O(r_2))$, and $0 = r_2(2y_0^2 + O(r_2))$. We know the fixed locus away from E_2, so assume that $r_2 = 0$; then these equations show that the bad locus inside E_2 is the curve $\{0 = z_1 = r_2\} = E_1 \cap E_2$.

Fortunately, Theorem 2.2 implies that Y_3/G has toric singularities at points of $E_1 \cap E_2$ outside the origin. Namely, let $e = r_2$ and $s = z_1$; then $I(s) = es(\text{unit})$ near $E_1 \cap E_2 = \{0 = z_1 = r_2\}$ outside the origin. The theorem gives that Y_3/G has singularity $\frac{1}{5}(0, 1, 2)$ at points of $E_1 \cap E_2$ outside the origin.

To see all of E_2, we also have to work over $\{w_1 = 1\}$ in Y_2, with coordinates (w_0, y_1, y_2), over $\{y_0 = 1\}$ in Y_1. Here Y_3 is the blow-up along the G-fixed curve $\{0 = w_0 = y_1\} = E_0 \cap E_1$, so Y_3 has coordinates $(w_0, y_1, y_2), [v_0, v_1]$. First take $\{v_0 = 1\}$, so $y_1 = w_0v_1$, and we have coordinates (w_0, v_1, y_2) on Y_3. Here E_0 does not appear, $E_1 = \{v_1 = 0\}$, and $E_2 = \{w_0 = 0\}$. The fixed point scheme is defined by: $I(w_0) = w_0^2(2 - 2v_1 + O(w_0))$, $I(v_1) = w_0v_1(1 - 2v_1 + O(w_0))$, and $I(y_2) = w_0v_1y_2(1+O(w_0))$. In the chart we are working over in Y_2, the fixed set Y_2^G is only the curve $E_0 \cap E_1$ we are blowing up, and so Y_3^G (in this chart) is contained in E_2 as a set. By the equations, Y_3^G is generically the Cartier divisor E_2, and the bad locus (where that fails) is given by $0 = w_0$, $0 = v_1(1 - 2v_1)$, and $0 = v_1y_2$. So the bad locus is the union of the curve $\{0 = w_0 = v_1\} = E_1 \cap E_2$ and the point $(w_0, v_1, y_2) = (0, -2, 0)$ in E_2. By Theorem 2.2 (using $e = s = w_0$), Y_3/G has singularity $\frac{1}{5}(2, 1, 0)$ everywhere on the curve $E_1 \cap E_2$ (in this chart), in agreement with an earlier calculation.

To analyze the bad point above, change coordinates temporarily by $t_1 = v_1 + 2$; then the bad point becomes the origin in coordinates (w_0, t_1, y_2). In these coordinates, we have $I(w_0) = w_0^2(1 - 2t_1 + O(w_0))$, $I(t_1) = I(v_1) = (-t_1 - 2t_1^2 + O(w_0))$, and $I(y_2) = w_0y_2(-2+O(w_0))$. Theorem 2.2 applies, with $s = e = w_0$, and we read off that Y_3/G has singularity $\frac{1}{5}(1, -1, -2)$ at this point. That is terminal, by the Reid-Tai criterion (Theorem 1.1).

The last chart we need to consider in Y_3 is the other open set $\{v_1 = 1\}$ over the open set above in Y_2, $\{w_1 = 1\} \subset Y_2$ over $\{y_0 = 1\} \subset Y_1$. So $w_0 = y_1 v_0$, and we have coordinates (v_0, y_1, y_2). Here $E_0 = \{v_0 = 0\}$, E_1 does not appear, and $E_2 = \{y_1 = 0\}$. Here Y_3^G is defined by: $I(v_0) = v_0 y_1(2 - v_0 + O(y_1))$, $I(y_1) = y_1^2(1 - 2v_0 + O(y_1))$, and $I(y_2) = y_1 y_2(1 + O(y_1))$. As in the previous chart, we know that Y_3^G is contained in E_2 as a set. By the equations, Y_3^G is generically the Cartier divisor E_2, and the bad locus (where that fails) is given by $0 = y_1$, $0 = v_0(2 - v_0)$, and $0 = y_2$. Thus there are two bad points in this chart, (v_0, y_1, y_2) equal to $(2, 0, 0) \in E_2$ or $(0, 0, 0) \in E_0 \cap E_2$. The first is the bad point from the previous chart, but the second one is new. Theorem 2.2 works to analyze the second point (the origin), with $e = s = y_1$. We read off that Y_3/G has singularity $\frac{1}{5}(2, 1, 1)$ at this point.

That finishes the analysis of Y_3. In particular, as a set, Y_3^G is the union of the divisor E_2 and a curve in E_1. It is tempting to blow up the G-fixed curve next, but that leads to a large number of blow-ups over one point of the curve, where the fixed point scheme is especially complicated. We therefore define Y_4 as the blow-up at that point, and only later blow up the whole curve. This leads more efficiently to toric singularities.

Namely, let Y_4 be the blow-up of Y_3 at the origin in the chart $\{r_2 = 1\}$ in Y_2 (unchanged in Y_3), with coordinates (y_0, r_1, s_2). So Y_4 has coordinates $(y_0, r_1, s_2), [q_0, q_1, q_2]$. The exceptional divisor E_3 is isomorphic to $\mathbf{P}^2$, and so it is covered by 3 affine charts. First take $\{q_0 = 1\}$ in Y_4, so $r_1 = y_0 q_1$ and $s_2 = y_0 q_2$, and we have coordinates (y_0, q_1, q_2). Here $E_1 = \{q_2 = 0\}$ and $E_3 = \{y_0 = 0\}$. The fixed point scheme Y_4^G is defined by: $I(y_0) = y_0^2 q_2(1 - q_1 + O(y_0))$, $I(q_1) = y_0(-2 + q_1 q_2 + q_1^2 q_2 + O(y_0))$, and $I(q_2) = y_0 q_2^2(-2 + q_1 + O(y_0))$. So Y_4^G is generically the Cartier divisor E_3; the G-fixed curve in E_1 does not appear in this chart. The bad locus (where the scheme Y_4^G is not just E_3) is given by $0 = y_0$, $0 = -2 + q_1 q_2 + q_1^2 q_2$, and $0 = q_2^2(-2 + q_1)$. By the second equation, $q_2 \neq 0$, and so the third equation gives that $q_1 = 2$. Then the second equation gives that $0 = -2 + 2q_2 - q_2 = -2 + q_2$, so $q_2 = 2$. That is, there is only one bad point in this chart, $(y_0, q_1, q_2) = (0, 2, 2) \in E_3$. To analyze that point, change coordinates temporarily by $s_1 = q_1 - 2$ and $s_2 = q_2 - 2$. In these coordinates, $I(y_0) = y_0^2(-2 - s_1 - s_2 - s_1 s_2 + O(y_0))$, $I(s_1) = I(q_1) = y_0(s_2 + 2s_1^2 + s_1^2 s_2 + O(y_0))$, and $I(s_2) = I(q_2) = y_0(-s_1 - s_1 s_2 + s_1 s_2^2 + O(y_0))$. By Theorem 2.2, with $e = s = y_0$, Y_4/G has a μ_5-quotient singularity. Explicitly, the linear map φ over k in the theorem is $\varphi(y_0) = -2y_0$, $\varphi(s_1) = s_2$, and $\varphi(s_2) = -s_1$, which has eigenvalues $-2, 2, -2$. So Y_4/G has singularity $\frac{1}{5}(-2, 2, -2)$ at this point. This is terminal, by the Reid-Tai criterion.

Next, take the open set $\{q_1 = 1\}$ in Y_4, so $y_0 = r_1 q_0$ and $s_2 = r_1 q_2$, and Y_4 has coordinates (q_0, r_1, q_2). Here $E_1 = \{q_2 = 0\}$ and $E_3 = \{r_1 = 0\}$. The fixed point scheme Y_4^G is defined by: $I(q_0) = r_1(-q_2 - q_0 q_2 + 2q_0^3 + O(r_1))$, $I(r_1) = r_1^2(2q_2 - 2q_0^2 + O(r_1))$, and $I(q_2) = r_1 q_2(2q_2 + 2q_0^2 + O(r_1))$. So Y_4^G is generically the Cartier divisor E_3, together with the G-fixed curve $\{0 = q_0 = q_2\}$ in E_1. The bad locus in E_3 is given by $0 = r_1$, $0 = -q_2 - q_0 q_2 + 2q_0^3$, and $0 = q_2(2q_2 + 2q_0^2)$. This yields two bad points, (q_0, r_1, q_2) equal to $(-2, 0, 1)$ or $(0, 0, 0)$. The first one is the bad point from the previous chart, and the second is not surprising, as it is the intersection point of E_3 with the G-fixed curve.

Finally, take the open set $\{q_2 = 1\}$ in Y_4, so $y_0 = s_2 q_0$ and $r_1 = s_2 q_1$, and we have coordinates (q_0, q_1, s_2). Here E_1 does not appear, and $E_3 = \{s_2 = 0\}$.

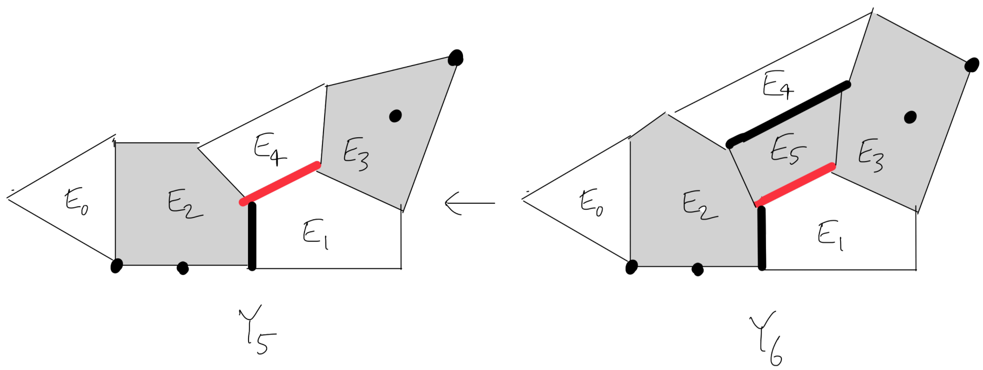


Figure 5: For $j = 5, 6$, G acts freely on Y_j outside the shaded or marked loci; Y_j/G is regular outside the marked loci; and Y_j/G has toric singularities outside the red loci.

The fixed point scheme Y_4^G is defined by: $I(q_0) = s_2(2q_0 - q_1 + O(s_2))$, $I(q_1) = s_2(-2q_1 - 2q_0^2 + O(s_2))$, and $I(s_2) = s_2^2(-1 + O(s_2))$. So Y_4^G is generically E_3. The bad locus in E_3 is given by: $0 = s_2$, $0 = 2q_0 - q_1$, and $0 = -2q_1 - 2q_0^2$. This yields two bad points, (q_0, q_1, s_2) equal to $(-2, 1, 0)$ (seen in the previous two charts) or $(0, 0, 0)$, which is new. Theorem 2.2 applies at this new point, with $e = s = s_2$. The k-linear map φ is given by $\varphi(q_0) = 2q_0 - q_1$, $\varphi(q_1) = -2q_1$, and $\varphi(s_2) = -s_2$. So φ has eigenvalues $(2, -2, -1)$, and hence Y_4/G has singularity $\frac{1}{5}(2, -2, -1)$ at this point. This is terminal, by the Reid-Tai criterion.

That completes our description of Y_4. Next, let Y_5 be the blow-up of Y_4 along the G-fixed curve in E_1. The exceptional divisor E_4 in Y_5 is a $\mathbf{P}^1$-bundle over $\mathbf{P}^1$, and so it is covered by four affine charts. First work over the open set $\{z_2 = 1\}$ in Y_3 (unchanged in Y_4), with coordinates (y_0, z_1, r_2); this contains the point where the G-fixed curve in $E_1 = \{z_1 = 0\}$ meets $E_2 = \{r_2 = 0\}$. Here Y_5 is the blow-up along the G-fixed curve $\{0 = y_0 = z_1\}$, and so Y_5 has coordinates $(y_0, z_1, r_2), [n_0, n_1]$. First take the open set $\{n_0 = 1\}$ in Y_5, so $z_1 = y_0 n_1$, and we have coordinates (y_0, n_1, r_2). Here $E_1 = \{n_1 = 0\}$, $E_2 = \{r_2 = 0\}$, and $E_4 = \{y_0 = 0\}$. The fixed point scheme Y_5^G is defined by: $I(y_0) = y_0 n_1 r_2(-1 + O(y_0))$, $I(n_1) = n_1 r_2(n_1 + O(y_0))/(1 - n_1 r_2 + O(y_0))$, and $I(r_2) = y_0 r_2^2(-2n_1 r_2 + O(y_0))$. So Y_5^G, as a set, is the union of the divisor E_2 and the curve $\{0 = y_0 = n_1\} = E_1 \cap E_4$. (In particular, the fixed point set is still not all of codimension 1.) We have analyzed the bad locus of E_2 in previous steps, but we have to add here that the bad locus of E_2 is disjoint from $E_2 \cap E_4$ except for the point where E_2 meets the G-fixed curve, by the formula for $I(n_1)$.

The other chart is $\{n_1 = 1\}$ in Y_5, so $y_0 = z_1 n_0$, and we have coordinates (n_0, z_1, r_2). Here E_1 does not appear, $E_2 = \{r_2 = 0\}$, and $E_4 = \{z_1 = 0\}$. The fixed point scheme Y_5^G is defined by: $I(n_0) = r_2(-1 + O(z_1))$, $I(z_1) = z_1^2 r_2(-2r_2 + O(z_1))$, and $I(r_2) = z_1 r_2^2(-2r_2 + O(z_1))$. These equations reduce to $r_2 = 0$ near E_4, and so Y_5^G is the Cartier divisor E_2, in this chart.

To finish our description of E_4 in Y_5, we work over the open set where the G-fixed curve in Y_4 meets E_3, namely $\{q_1 = 1\}$ in Y_4. Here Y_4 has coordinates (q_0, r_1, q_2), $E_1 = \{q_2 = 0\}$, $E_3 = \{r_1 = 0\}$, and the G-fixed curve is $\{0 = q_0 = q_2\}$ in

E_1. So the blow-up Y_5 along the G-fixed curve has coordinates $(q_0, r_1, q_2), [u_0, u_2]$. First take $\{u_0 = 1\}$ in Y_5, so $q_2 = q_0 u_2$, and we have coordinates (q_0, r_1, u_2). Here $E_1 = \{u_2 = 0\}$, $E_3 = \{r_1 = 0\}$, and $E_4 = \{q_0 = 0\}$. The fixed point scheme Y_5^G is defined by: $I(q_0) = q_0 r_1(-u_2 + O(q_0))$, $I(r_1) = q_0 r_1^2(2u_2 + O(q_0))$, and $I(u_2) = r_1 u_2^2(1 + O(q_0))/(1 - r_1 u_2 + O(q_0))$. We know the fixed set outside E_4, and so we read off that the fixed set is the divisor E_3 together with the G-fixed curve $E_1 \cap E_4$ found earlier. We have analyzed the bad set of E_3 away from E_4 in earlier blow-ups, and we see from the formula for $I(u_2)$ that the bad set of E_3 near $E_3 \cap E_4$ is only the point $E_1 \cap E_3 \cap E_4$ where the G-fixed curve meets E_3.

The other chart is $\{u_2 = 1\}$ in Y_5. Here $q_0 = q_2 u_0$, and so we have coordinates (u_0, r_1, q_2). Here E_1 does not appear, $E_3 = \{r_1 = 0\}$, and $E_4 = \{q_2 = 0\}$. The fixed point scheme Y_5^G is defined by: $I(u_0) = r_1(-1 + O(q_2))$, $I(r_1) = r_1^2 q_2(2 + O(q_2))$, and $I(q_2) = r_1 q_2^2(2 + O(q_2))$. These equations reduce to $r_1 = 0$ near E_4, and so the fixed point scheme Y_5^G is the Cartier divisor E_3, in this chart.

That completes our description of Y_5. Let Y_6 be the blow-up of Y_5 along the G-fixed curve $E_1 \cap E_4$. The exceptional divisor E_5 in Y_6 is a $\mathbf{P}^1$-bundle over $\mathbf{P}^1$, covered by four affine charts. First take the open set $\{n_0 = 1\}$ in Y_5, which contains the point where the G-fixed curve meets E_2. Here Y_5 has coordinates (y_0, n_1, r_2), with $E_1 = \{n_1 = 0\}$, $E_2 = \{r_2 = 0\}$, and $E_4 = \{y_0 = 0\}$. Since Y_6 is the blow-up along the G-fixed curve $\{0 = y_0 = n_1\} = E_1 \cap E_4$, Y_6 has coordinates $(y_0, n_1, r_2), [m_0, m_1]$. First take $\{m_0 = 1\}$ in Y_6, so $n_1 = y_0 m_1$, and we have coordinates (y_0, m_1, r_2). Here $E_1 = \{m_1 = 0\}$, $E_2 = \{r_2 = 0\}$, E_4 does not appear, and $E_5 = \{y_0 = 0\}$. The fixed point scheme Y_6^G is defined by: $I(y_0) = y_0^2 m_1 r_2(-1 + O(y_0))$, $I(m_1) = y_0 m_1 r_2(2m_1 + O(y_0))$, and $I(r_2) = y_0^2 r_2^2(2 - 2m_1 r_2 + O(y_0))$. We know the fixed point set away from E_5, and so we read off that the fixed point scheme is generically the Cartier divisor $E_2 + E_5$. (Since E_5 is fixed by G, we have finally made the fixed point set of codimension 1.) Let $e = y_0 r_2$. The bad locus (where the scheme Y_6^G is more than the Cartier divisor $E_2 + E_5$), on E_5, is given by factoring out e from the equations and setting $y_0 = 0$, so we get: $0 = y_0$ and $0 = 2m_1^2$. So, as a set, the bad locus is the curve $\{0 = y_0 = m_1\} = E_1 \cap E_5$. Theorem 2.2 does not seem to apply to this curve, and so Y_6/G might not have toric singularities there; we will have to blow up one more time.

For now, look at the other open set, $\{m_1 = 1\}$ in Y_6. So $s_0 = n_1 m_0$, and we have coordinates (m_0, n_1, r_2). Here E_1 does not appear, $E_2 = \{r_2 = 0\}$, $E_4 = \{m_0 = 0\}$, and $E_5 = \{n_1 = 0\}$. The fixed point scheme Y_6^G is defined by: $I(m_0) = m_0 n_1 r_2(-2 + O(n_1))$, $I(n_1) = n_1^2 r_2(1 + O(n_1))$, and $I(r_2) = m_0 n_1^2 r_2^2(2m_0 - 2r_2 + O(n_1))$. Since we know the fixed point set outside E_5, we read off that the fixed point scheme is generically the Cartier divisor $E_2 + E_5$. Let $e = n_1 r_2$. The bad locus (where the scheme Y_6^G is more than the Cartier divisor $E_2 + E_5$) is the curve $\{0 = m_0 = n_1\} = E_4 \cap E_5$. Fortunately, Theorem 2.2 applies, with $s = m_0$. We read off that Y_6/G has singularity $\frac{1}{5}(-2, 1, 0)$ along the whole curve $E_4 \cap E_5$, in this chart.

To finish describing $E_5 \subset Y_6$, we have to work over the open set $\{u_0 = 1\}$ in Y_5, where the G-fixed curve $E_1 \cap E_4$ in Y_5 meets E_3. Here Y_5 has coordinates (q_0, r_1, u_2), with $E_1 = \{u_2 = 0\}$, $E_3 = \{r_1 = 0\}$, and $E_4 = \{q_0 = 0\}$. Then Y_6 is the blow-up along the G-fixed curve $\{0 = q_0 = u_2\} = E_1 \cap E_4$, so Y_6 has coordinates $(q_0, r_1, u_2), [t_0, t_2]$. First take $\{t_0 = 1\}$ in Y_6, so $u_2 = q_0 u_2$ and we have coordinates (q_0, r_1, t_2). Here $E_1 = \{t_2 = 0\}$, $E_3 = \{r_1 = 0\}$, E_4 does not appear,

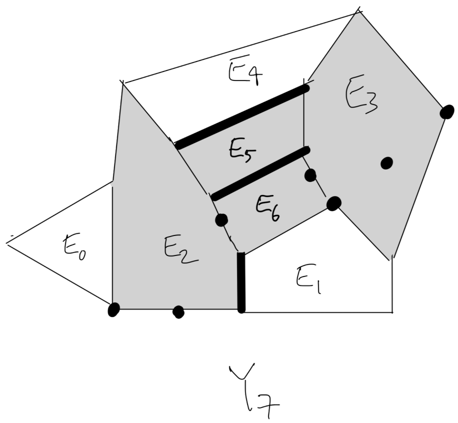


Figure 6: G acts freely on Y_7 outside the shaded or marked loci, and Y_7/G is regular outside the marked loci. Here Y_7/G has toric singularities.

and $E_5 = \{q_0 = 0\}$. The fixed point scheme is defined by: $I(q_0) = q_0^2 r_1(-t_2 + O(q_0))$, $I(r_1) = q_0^2 r_1^2(-2 + 2t_2 + O(q_0))$, and $I(t_2) = q_0 r_1 t_2(2t_2 + O(q_0))$. So the fixed point scheme is generically $E_3 + E_5$. Let $e = q_0 r_1$. The bad locus (where the scheme Y_6^G is more than the Cartier divisor $E_3 + E_5$), in E_5, is given by $0 = q_0$ and $0 = 2t_2^2$, so (as a set) it is the curve $\{0 = q_0 = t_2\} = E_1 \cap E_5$, which we met in an earlier chart.

The other chart is $\{t_2 = 1\}$ in Y_6, so $q_0 = u_2 t_0$, and we have coordinates (t_0, r_1, u_2). Here E_1 does not appear, $E_3 = \{r_1 = 0\}$, $E_4 = \{t_0 = 0\}$, and $E_5 = \{u_2 = 0\}$. The fixed point scheme Y_6^G is defined by: $I(t_0) = t_0 r_1 u_2(-2 + O(u_2))$, $I(r_1) = t_0 r_1^2 u_2^2(2 - 2t_0 + O(u_2))$, and $I(u_2) = r_1 u_2^2(1 + O(u_2))$. So Y_6^G is generically $E_3 + E_5$. Let $e = r_1 u_2$. The bad locus (where the scheme Y_6^G is more than the Cartier divisor $E_3 + E_5$), in E_5, is the curve $\{0 = t_0 = u_2\} = E_4 \cap E_5$, which we met in an earlier chart. Theorem 2.2 applies, with $s = t_0$. Namely, Y_6/G has singularity $\frac{1}{5}(-2, 0, 1)$ everywhere on the curve $E_4 \cap E_5$ in this chart (including the origin, which did not appear in the earlier chart).

That completes our description of Y_6. In particular, the G-fixed locus has codimension 1 in Y_6, and Y_6/G has toric singularities outside the image of the curve $E_1 \cap E_5$. Let Y_7 be the blow-up of Y_6 along that curve. The exceptional divisor E_6 in Y_7 is a $\mathbf{P}^1$-bundle over $\mathbf{P}^1$, and so we will cover E_6 with four affine charts. First, work over the open set $\{m_0 = 1\}$ in Y_6, where the bad curve $E_1 \cap E_5$ meets E_2. Here Y_6 has coordinates (y_0, m_1, r_2), with $E_1 = \{m_1 = 0\}$, $E_2 = \{r_2 = 0\}$, and $E_5 = \{y_0 = 0\}$. Since Y_7 is the blow-up along the curve $\{0 = y_0 = m_1\} = E_1 \cap E_5$, Y_7 has coordinates $(y_0, m_1, r_2), [j_0, j_1]$.

First take $\{j_0 = 1\}$ in Y_7, so $m_1 = y_0 j_1$, and we have coordinates (y_0, j_1, r_2). Here $E_1 = \{j_1 = 0\}$, $E_2 = \{r_2 = 0\}$, E_5 does not appear, and $E_6 = \{y_0 = 0\}$. The fixed point scheme Y_7^G is defined by: $I(y_0) = y_0^3 j_1 r_2(-1 + O(y_0))$, $I(j_1) = y_0^2 j_1 r_2(1 - 2j_1 + O(y_0))$, and $I(r_2) = y_0^2 r_2^2(2 + O(y_0))$. So Y_7^G is generically the Cartier divisor $E_2 + 2E_6$. Let $e = y_0^2 r_2$. The bad locus (where the scheme Y_7^G is more than the Cartier divisor $E_2 + 2E_6$), in E_6, is given by $0 = y_0$, $0 = j_1(1 - 2j_1)$, and $0 = r_2$, so it consists of the two points (y_0, j_1, r_2) equal to $(0, 0, 0) = E_1 \cap E_2 \cap E_6$ or $(0, -2, 0) \in E_2 \cap E_6$. At the first point, Theorem 2.2 applies, with $s = r_2$. We read off that Y_7/G has singularity $(0,1,2)$ everywhere on the curve $E_1 \cap E_2$ (including the origin, which did not appear when we saw $E_1 \cap E_2$ in an earlier chart). To analyze the second point, change coordinates temporarily by $s_1 = j_1 + 2$; then that point becomes the origin in coordinates (y_0, s_1, r_2). We have $I(y_0) = y_0^3 r_2(2 - s_1 + O(y_0))$, $I(s_1) = I(j_1) = y_0^2 r_2(-s_1 - 2s_1^2 + O(y_0))$, and $I(r_2) = y_0^2 r_2^2(2 + O(y_0))$. Theorem 2.2 applies, with $s = r_2$. We read off that Y_7/G has singularity $\frac{1}{5}(2,-1,2)$ at this point.

The other open set is $\{j_1 = 1\}$ in Y_7, so $y_0 = m_1 j_0$, and we have coordinates (j_0, m_1, r_2). Here E_1 does not appear, $E_2 = \{r_2 = 0\}$, $E_5 = \{j_0 = 0\}$, and $E_6 = \{m_1 = 0\}$. The fixed point scheme Y_7^G is defined by: $I(j_0) = j_0^2 m_1^2 r_2(2 - j_0 + O(m_1))$, $I(m_1) = j_0 m_1^3 r_2(2 + j_0 + O(m_1))$, and $I(r_2) = j_0^2 m_1^2 r_2^2(2 + O(m_1))$. So Y_7^G is generically $E_2 + E_5 + 2E_6$. Let $e = j_0 m_1^2 r_2$. The bad locus (where the scheme Y_7^G is more than the Cartier divisor $E_2 + E_5 + 2E_6$), on E_6, is given by: $0 = m_1$, $0 = j_0(2 - j_0)$, and $0 = j_0 r_2$. So the bad locus is the union of the curve $\{0 = j_0 = m_1\} = E_5 \cap E_6$ and the point $(j_0, m_1, r_2) = (2,0,0)$ in $E_2 \cap E_6$. That point is the one we analyzed in the previous chart. For the curve, Theorem 2.2 applies, using $s = j_0$. We read off that Y_7/G has singularity $\frac{1}{5}(2,2,0)$ everywhere on the curve $E_5 \cap E_6$, in this chart.

Last, work over the open set $\{t_0 = 1\}$ in Y_6, where the bad curve $E_1 \cap E_5$ meets E_3. Here Y_6 has coordinates (q_0, r_1, t_2), with $E_1 = \{t_2 = 0\}$, $E_3 = \{r_1 = 0\}$, and $E_5 = \{q_0 = 0\}$. We obtain Y_7 by blowing up along the curve $\{0 = q_0 = t_2\} = E_1 \cap E_5$, so Y_7 has coordinates $(q_0, r_1, t_2), [x_0, x_2]$. First take $\{x_0 = 1\}$, so $t_2 = q_0 x_2$, and we have coordinates (q_0, r_1, x_2). Here $E_1 = \{x_2 = 0\}$, $E_3 = \{r_1 = 0\}$, E_5 does not appear, and $E_6 = \{q_0 = 0\}$. The fixed point scheme Y_7^G is defined by: $I(q_0) = q_0^3 r_1(2 - x_2 + O(q_0))$, $I(r_1) = q_0^2 r_1^2(-2 + O(q_0))$, and $I(x_2) = q_0^2 r_1 x_2(1 - 2x_2 + O(q_0))$. So Y_7^G is generically $E_3 + 2E_6$. Let $e = q_0^2 r_1$. The bad locus (where the scheme Y_7^G is more than the Cartier divisor $E_3 + 2E_6$), in E_6, is given by: $0 = q_0$, $0 = r_1$, and $0 = x_2(1 - 2x_2)$, so it consists of the two points (q_0, r_1, x_2) equal to $(0,0,0) = E_1 \cap E_3 \cap E_6$ or $(0,0,-2)$ in $E_3 \cap E_6$. Since $I(r_1) = e r_1(\text{unit})$, Theorem 2.2 applies at both points. At the origin, the theorem gives that Y_7/G has singularity $\frac{1}{5}(2,-2,1)$, which is terminal. For the other point, change coordinates temporarily by $y_2 = x_2 + 2$, so that the point becomes the origin in coordinates (q_0, r_1, y_2). We have $I(q_0) = q_0^3 r_1(-1 - y_2 + O(q_0))$, $I(r_1) = q_0^2 r_1^2(-2 + O(q_0))$, and $I(y_2) = I(x_2) = q_0^2 r_1(-y_2 - 2y_2^2 + O(q_0))$. So Theorem 2.2 gives that Y_7/G has singularity $\frac{1}{5}(-1,-2,-1)$ at this point.

The other chart is $\{x_2 = 1\}$ in Y_7, so $q_0 = t_2 x_0$, and we have coordinates (x_0, r_1, t_2). Here E_1 does not appear, $E_3 = \{r_1 = 0\}$, $E_5 = \{x_0 = 0\}$, and $E_6 = \{t_2 = 0\}$. The fixed point scheme Y_7^G is defined by: $I(x_0) = x_0^2 r_1 t_2^2(2 - x_0 + O(t_2))$, $I(r_1) = x_0^2 r_1^2 t_2^2(-2 + O(t_2))$, and $I(t_2) = x_0 r_1 t_2^3(2 - 2x_0 + O(t_2))$. So Y_7^G is generically $E_3 + E_5 + 2E_6$. Let $e = x_0 r_1 t_2^2$. The bad locus (where Y_7^G is more than the Cartier divisor $E_3 + E_5 + 2E_6$), in E_6, is given by: $0 = t_2$, $0 = x_0(2 - x_0)$, and $0 = x_0 r_1$, which is the union of the curve $\{0 = x_0 = t_2\} = E_5 \cap E_6$ and the point $(x_0, r_1, t_2) = (2,0,0)$ in $E_3 \cap E_6$. We analyzed that point in the previous chart. Theorem 2.2 applies to the curve, using $s = x_0$. We read off that Y_7/G has singularity $\frac{1}{5}(2,0,2)$ everywhere on the curve $E_5 \cap E_6$ (including the origin, which did not appear in the earlier chart where we met this curve).

That completes our analysis of Y_7; we have shown that Y_7/G has toric singularities. It will now be straightforward to show that Y_0/G is terminal.

First, we can compute the canonical class of Y_7, since Y_7 is obtained from Y_0 by a sequence of blow-ups along points and smooth curves. Write E_j for the strict transform of the exceptional divisor in Y_{j+1} to any higher model. Write π_{ij} for the morphism $Y_i \to Y_j$ (with $i > j$), and also for the resulting morphism $Y_i/G \to Y_j/G$. First, $K_{Y_1} = \pi_{10}^* K_{Y_0} + 2E_0$, since $Y_1 \to Y_0$ is the blow-up of a smooth 3-fold at a point. Next, $K_{Y_2} = \pi_{21}^* K_{Y_1} + E_1$, since $Y_2 \to Y_1$ is the blow-up along a smooth curve, and we have $\pi_{21}^* E_0 = E_0 + E_1$ because the curve being blown up is contained in E_0. Likewise, we have:

$$\begin{aligned} K_{Y_3} &= \pi_{32}^* K_{Y_2} + E_2,\ \pi_{32}^* E_0 = E_0 + E_2, \pi_{32}^* E_1 = E_1 + E_2, \\ K_{Y_4} &= \pi_{43}^* K_{Y_3} + 2E_3,\ \pi_{43}^* E_0 = E_0, \pi_{43}^* E_1 = E_1 + E_3,\ \pi_{43}^* E_2 = E_2, \\ K_{Y_5} &= \pi_{54}^* K_{Y_4} + E_4,\ \pi_{54}^* E_0 = E_0, \pi_{54}^* E_1 = E_1 + E_4,\ \pi_{54}^* E_2 = E_2,\ \pi_{54}^* E_3 = E_3, \\ K_{Y_6} &= \pi_{65}^* K_{Y_5} + E_5,\ \pi_{65}^* E_j = E_j \text{ for } j \in \{0, \ldots, 4\} - \{1, 4\}, \\ &\pi_{65}^* E_1 = E_1 + E_5, \pi_{65}^* E_4 = E_4 + E_5, \\ K_{Y_7} &= \pi_{76}^* K_{Y_6} + E_6, \pi_{76}^* E_j = E_j \text{ for } j \in \{0, \ldots, 5\} - \{1, 5\}, \\ &\pi_{76}^* E_1 = E_1 + E_6, \pi_{76}^* E_5 = E_5 + E_6. \end{aligned}$$

Combining these equations gives that

$$K_{Y_7} = \pi_{70}^* K_{Y_0} + 2E_0 + 3E_1 + 6E_2 + 5E_3 + 4E_4 + 8E_5 + 12E_6.$$

Write f for any of the quotient maps $Y_j \to Y_j/G$. First, $K_{Y_0/G}$ is $\mathbf{Q}$-Cartier since Y_0 is smooth, and $K_{Y_0} = f^* K_{Y_0/G}$ because $f\colon Y_0 \to Y_0/G$ is étale in codimension 1. Next, we computed that the fixed point scheme Y_7^G is the Cartier divisor $E_2 + E_3 + E_5 + 2E_6$ outside a codimension-2 subset of Y_7. By section 3, it follows that

$$\begin{aligned} K_{Y_7} &= f^* K_{Y_7/G} + (p-1)(E_2 + E_3 + E_5 + 2E_6) \\ &= f^* K_{Y_7/G} + 4E_2 + 4E_3 + 4E_5 + 8E_6. \end{aligned}$$

For each $j \in \{0, \ldots, 7\}$, let F_j be the image of E_j in Y_7/G, as an irreducible divisor. For $j \in \{0, 1, 4\}$, G acts nontrivially on E_j (so f is unramified along E_j), and hence $E_j = f^* F_j$. For the other j's, in $\{2, 3, 5, 6\}$, section 3 and our calculations imply that f is fiercely ramified along E_j, and so again we have $E_j = f^* F_j$. (For example, for E_2, use the first chart where E_2 appeared, $\{z_1 = 1\}$ in Y_3. There E_2 is the divisor $\{y_1 = 0\}$, and Y_3^G has multiplicity 1 along E_2, but $I(y_1) = y_1^2 z_2(-2y_0^2 + O(y_1))$ vanishes to order $2 > 1$ along E_2; so section 3 gives that f is fiercely ramified along E_2.)

We can combine these results to compute the discrepancies of the morphism $Y_7/G \to Y_0/G$. Namely, we have

$$\begin{aligned} f^*(K_{Y_7/G} - \pi_{70}^* K_{Y_0/G}) &= f^* K_{Y_7/G} - \pi_{70}^* f^* K_{Y_0/G} \\ &= K_{Y_7} - 4E_2 - 4E_3 - 4E_5 - 8E_6 - \pi_{70}^* K_{Y_0} \\ &= 2E_0 + 3E_1 + 2E_2 + E_3 + 4E_4 + 4E_5 + 4E_6 \\ &= f^*(2F_0 + 3F_1 + 2F_2 + F_3 + 4F_4 + 4F_5 + 4F_6). \end{aligned}$$

Therefore, $K_{Y_7/G} = \pi_{70}^*(K_{Y_0/G}) + 2F_0 + 3F_1 + 2F_2 + F_3 + 4F_4 + 4F_5 + 4F_6$. In particular, these coefficients are all positive, which is part of showing that Y_0/G is terminal. (That would be all we need if Y_7/G were smooth.)

To show that Y_0/G is terminal, it now suffices to show that the pair $(Y_7/G, D)$ is terminal, where $D := -2F_0 - 3F_1 - 2F_2 - F_3 - 4F_4 - 4F_5 - 4F_6$. Because the coefficients of D are negative (which works to our advantage), this is clear at points where Y_7/G is terminal. There are 6 subvarieties (points or curves) where Y_7/G is not terminal, as we now address.

(1) Along the curve $E_1 \cap E_2$, Y_7/G has singularity $\frac{1}{5}(0, 1, 2)$, with F_1 a toric divisor of weight 1 and F_2 a toric divisor of weight 2, using Theorem 4.1. To show that $(Y_7/G, D) = (Y_7/G, -3F_1 - 2F_2 - \cdots)$ is terminal along this curve, we need to show that $4(i \bmod 5) + 3(2i \bmod 5) > 5$ for $i = 1, \ldots 4$, by Theorem 1.2. This is clear, since the left side is $\geq 4 + 3 = 7 > 5$.

(2) At a point in $E_0 \cap E_2$, Y_7/G has singularity $\frac{1}{5}(2, 1, 1)$, with F_0 of weight 2 and F_2 of weight 1. To show that $(Y_7/G, D) = (Y_7/G, -2F_0 - 2F_2 - \cdots)$ is terminal, we need that $3(2i \bmod 5) + 3(i \bmod 5) + (i \bmod 5) > 5$ for $i = 1, \ldots 4$. Indeed, the left side is $\geq 3 + 3 + 1 = 7 > 5$.

(3) Along the curve $E_4 \cap E_5$, Y_7/G has singularity $\frac{1}{5}(-2, 1, 0)$, with E_4 of weight -2 and E_5 of weight 1. To show that $(Y_7/G, D) = (Y_7/G, -4F_4 - 4F_5 - \cdots)$ is terminal, we need that $5(-2i \bmod 5) + 5(i \bmod 5) > 5$ for $i = 1, \ldots 4$. Indeed, the left side is $\geq 5 + 5 = 10 > 5$.

(4) At a point in $E_2 \cap E_6$, Y_7/G has singularity $\frac{1}{5}(2, -1, 2)$, with E_2 and E_6 both of weight 2. To show that $(Y_7/G, D) = (Y_7/G, -2F_2 - 4F_6 - \cdots)$ is terminal, we need that $3(2i \bmod 5) + 5(2i \bmod 5) + (-i \bmod 5) > 5$ for $i = 1, \ldots 4$. Indeed, the left side is $\geq 3 + 5 + 1 = 9 > 5$.

(5) Along the curve $E_5 \cap E_6$, Y_7/G has singularity $\frac{1}{5}(2, 2, 0)$, with E_5 and E_6 both of weight 2. To show that $(Y_7/G, D) = (Y_7/G, -4F_5 - 4F_6 - \cdots)$ is terminal, we need that $5(2i \bmod 5) + 5(2i \bmod 5) > 5$ for $i = 1, \ldots 4$. Indeed, the left side is $\geq 5 + 5 = 10 > 5$.

(6) At a point in $E_3 \cap E_6$, Y_7/G has singularity $\frac{1}{5}(-1, -2, -1)$, with E_3 of weight -2 and E_6 of weight -1. To show that $(Y_7/G, D) = (Y_7/G, -F_3 - 4F_6 - \cdots)$ is terminal, we need that $2(-2i \bmod 5) + 5(-i \bmod 5) + (-i \bmod 5) > 5$ for $i = 1, \ldots 4$. Indeed, the left side is $\geq 2 + 5 + 1 = 8 > 5$.

That completes the proof that Y_0/G is terminal. Theorem 8 is proved. □

Remark 8.2. The divisor class $\pi^* K_{Y_0/G} = K_{Y_7/G} + D$ happens to be Cartier on the loci (1)–(6), above. However, it is not Cartier at the terminal singularity $\frac{1}{5}(2, -2, -1)$ in E_3; one can compute that some discrepancies at divisors over that point are not integers. As a result, $K_{Y_0/G}$ is not Cartier (as one can also check directly). I expect that there is also a 3-fold X over $\mathbf{F}_5$ that is terminal and non-Cohen-Macaulay with K_X Cartier. Namely, one should replace $\mathbf{P}^1$ in Theorem 8.1 by the Harbater-Katz-Gabber curve of Remark 6.2, now with $p = 5$.

9 The example over the 5-adic integers

Theorem 9.1. *Let the group $G = \mathbf{Z}/5$ act on the quintic del Pezzo surface S_5 over $\mathbf{Z}_5$ by an embedding of G into the symmetric group $\Sigma_5 = \mathrm{Aut}(S_5)$. Let $R = \mathbf{Z}_5[e]/(e^5 - 5e^4 + 25e^2 - 25e + 5)$, which is the ring of integers in a Galois extension of $\mathbf{Q}_5$ with group $G = \mathbf{Z}/5$. Let G act on the scheme $(S_5)_R$ by the diagonal action on S_5 and on R. Then the scheme $(S_5)_R/G$ is terminal, not Cohen-Macaulay, of dimension 3, and flat over $\mathbf{Z}_5$.*

We define the quintic del Pezzo surface S_5 (over any commutative ring) as the moduli space $\overline{M_{0,5}}$ of 5-pointed stable curves of genus 0. That makes it clear that the symmetric group Σ_5 acts on S_5.

This example behaves much like the example over $\mathbf{F}_5$, Theorem 8.1. *In particular, the figures in section 8 accurately depict the blow-ups we make in mixed characteristic* $(0,5)$*, just as in characteristic 5.* We can view R as the subring of the cyclotomic ring $\mathbf{Z}_5[\zeta_{25}]$ fixed by the automorphism $\zeta_{25} \mapsto \zeta_{25}^7$ of order 4, with $e = 1 + \zeta_{25} + \zeta_{25}^{-1} + \zeta_{25}^7 + \zeta_{25}^{-7}$. Informally, R is the simplest ramified $\mathbf{Z}/5$-extension of $\mathbf{Z}_5$. More broadly, this action of G on $(S_5)_R$ was chosen as possibly the simplest action of $\mathbf{Z}/5$ on a 3-fold in mixed characteristic $(0,5)$ with an isolated fixed point. The simplicity helps to ensure that the quotient scheme is terminal.

Proof. We work throughout over $\mathbf{Z}_5$. Write $G = \mathbf{Z}/5 = \langle \sigma : \sigma^5 = 1 \rangle$, with $\tau := \sigma^{-1}$. By de Fernex [11], the action of G on S_5 is conjugate to the birational action of G on $\mathbf{P}^2$ by

$$\tau([x,y,z]) = [x(z-y), z(x-y), xz].$$

The fixed point over $\mathbf{F}_5$ is $[-2,1,-1]$. Let us change variables over $\mathbf{Z}_5$ to move that point to $[0,0,1]$ (although it is only fixed over $\mathbf{F}_5$). Namely, let $v_0 = x + 2y$, $v_1 = z - x - y$, and $v_2 = y$. In these coordinates, the action of G becomes

$$\begin{aligned}\tau[v_0, v_1, v_2] = [&3v_0^2 + 3v_0v_1 - 12v_0v_2 - 8v_1v_2 + 10v_2^2,\\ &- v_0^2 - v_0v_1 + 5v_0v_2 + 3v_1v_2 - 5v_2^2, (v_0 + v_1 - v_2)(v_0 - 3v_2)].\end{aligned}$$

Therefore, in affine coordinates $(s_0, s_1) := (v_0/v_2, v_1/v_2)$, G acts by

$$\tau(s_0, s_1) = \left(\frac{10 - 12s_0 - 8s_1 + 3s_0^2 + 3s_0s_1}{(3 - s_0)(1 - s_0 - s_1)}, \frac{-5 + 5s_0 + 3s_1 - s_0^2 - s_0s_1}{(3 - s_0)(1 - s_0 - s_1)} \right).$$

This reduces modulo 5 to the formula for the action of G on S_5 over $\mathbf{F}_5$ in section 8.

Let $Y_0 = (S_5)_R$, with the diagonal action of G on S_5 and on R. Write e_2 for the generator e of R, to fit with our numbering of coordinates on Y_0; so we have

$$0 = e_2^5 - 5e_2^4 + 25e_2^2 - 25e_2 + 5.$$

Then G acts on an affine neighborhood U of the origin by:

$$\begin{aligned}\tau(s_0, s_1, e_2) = \Bigg(&\frac{10 - 12s_0 - 8s_1 + 3s_0^2 + 3s_0s_1}{(3 - s_0)(1 - s_0 - s_1)}, \frac{-5 + 5s_0 + 3s_1 - s_0^2 - s_0s_1}{(3 - s_0)(1 - s_0 - s_1)},\\ &\frac{1}{7}(20 - 53e_2 + 8e_2^2 + 9e_2^3 - 2e_2^4) \Bigg).\end{aligned}$$

(The last expression is a generator of the Galois group of R over $\mathbf{Z}_5$, as one can check via Magma. The denominator 7 occurs because the ring of integers of $\mathbf{Q}(\zeta_{25})^{\mathbf{Z}/4}$ is not monogenic, hence not generated over $\mathbf{Z}$ by e_2. This causes no difficulties, because we are working over $\mathbf{Z}_5$.) Note that U is written with three variables over $\mathbf{Z}_5$, but this is a regular scheme of dimension 3 because of the equation satisfied by e_2:

$$0 = e_2^5 - 5e_2^4 + 25e_2^2 - 25e_2 + 5.$$

We will apply Theorem 2.2 repeatedly to recognize the singularities of Y_j/G, for various blow-ups Y_j of Y_0. We remark now that the assumption in Theorem 2.2 that $p \in e^{p-1}\,\mathfrak{m}$ will be valid in each case, that is, that $5 \in e^4\,\mathfrak{m}$. Indeed, we have $5 = e_2^5(\text{unit})$ on Y_0, hence on each blow-up Y_j, and e_2 is a multiple of the function e defining the Weil divisor $[Y_j^G]$ in each case, that being the function e we will use for Theorem 2.2. So 5 is in the ideal (e^5), hence in $e^4\,\mathfrak{m}$ at each of the bad points.

Let $X = Y_0/G$. Since G acts freely on $\operatorname{Spec} R$ outside its closed point, the only fixed point of G on Y_0 is the closed point $P \cong \operatorname{Spec} \mathbf{F}_5$ given by $(s_0, s_1, e_2) = (0, 0, 0)$. So X is normal of dimension 3, and X is regular outside the image of P, which we also call P. Also, $5K_X$ is Cartier.

It is not automatic from Fogarty's results [13], but we can use his methods to show that X is not Cohen-Macaulay at P. As in the proof of Theorem 5, using that G has an isolated fixed point on the 3-fold Y_0, it suffices to show that $H^1(G, O(Y_0))$ is not zero. This cohomology group is $\ker(\text{tr})/\operatorname{im}(1-\sigma)$ on $O(Y_0)$, where the trace is $1 + \sigma + \cdots + \sigma^4$. The equation $0 = e_2^5 - 5e_2^4 + 25e_2^2 - 25e_2 + 5$ (specifically, the coefficient of e_2^4) implies that e_2 has trace 5. So $\text{tr}(1 - e_2) = 0$, and hence $1 - e_2$ defines an element of $H^1(G, O(Y_0))$. Note that $1 - e_2$ restricts to $1 \in O(P) = \mathbf{F}_5$ on the fixed point P. Therefore, $1 - e_2$ has nonzero image under the restriction map $H^1(G, O(Y_0)) \to H^1(G, O(P) \cong \mathbf{F}_5$. So $H^1(G, O(Y_0))$ is not zero, and hence Y_0/G is not Cohen-Macaulay.

It remains to show that Y_0/G is terminal. This example is complicated, and it may be impossible to resolve the singularities of X by performing G-equivariant blow-ups of Y_0. Fortunately, as in earlier sections, we can make Y_7/G have toric singularities after some G-equivariant blow-ups $Y_7 \to \cdots \to Y_0$, exactly parallel to those in the characteristic 5 example (section 8). In fact, all the formulas we write for the fixed point loci will look *identical* to those in the characteristic 5 example, because we only need to write those formulas modulo suitable error terms. It will then be easy to check that Y_0/G is terminal.

The blow-up $Y_1 \to Y_0$ at the G-fixed point is, over the open set $U \subset Y_0$:

$$\{((x_0, x_1, e_2), [y_0, y_1, y_2]) \in U \times_{\mathbf{Z}_5} \mathbf{P}^2_{\mathbf{Z}_5} : x_0y_1 = x_1y_0,\ x_0y_2 = e_2y_0,\ x_1y_2 = e_2y_1\}.$$

We will see that the fixed point set in Y_1 is a curve isomorphic to $\mathbf{P}^1_{\mathbf{F}_5}$. To check that, first work in the open subset $\{y_0 = 1\}$ in Y_1, with coordinates (s_0, y_1, y_2); here $(s_0, s_1, e_2) = (s_0, s_0y_1, s_0y_2)$. This is an open neighborhood of the origin in

$$\operatorname{Spec} \mathbf{Z}_5[s_0, y_1, y_2]/((s_0y_2)^5 - 5(s_0y_2)^4 + 25(s_0y_2)^2 - 25(s_0y_2) + 5),$$

by the equation for e_2. Since $e_2 = s_0y_2$, e_2 is in the ideal (s_0), and hence 5 is also in (s_0) (which lets us simplify formulas written modulo (s_0)). It is straightforward to compute how G acts in this chart, but we do not write it out, for brevity. The exceptional divisor E_0 is $\{s_0 = 0\}$, in this chart (and so E_0 is isomorphic to $\mathbf{P}^2$ over $\mathbf{F}_5$). The fixed point scheme Y_1^G is defined by the vanishing of: $I(s_0) = s_0(-y_1 + O(s_0))$, $I(y_1) = (y_1^2 + O(s_0))/(1 - y_1 + O(s_0))$, and $I(y_2) = y_2(y_1 + O(s_0))/(1 - y_1 + O(s_0))$. We know that Y_1^G is contained (as a set) in E_0 (since Y_0^G is only the origin in characteristic 5). So the fixed point set is the line $\{0 = s_0 = y_1\}$, in this chart.

In the chart $\{y_1 = 1\}$ in Y_1, we have $s_0 = s_1y_0$ and $e_2 = s_1y_2$, so we have coordinates (y_0, s_1, y_2). Here $E_0 = \{s_1 = 0\}$. We can write the action of G in these coordinates (for example using Magma). We find that the fixed point scheme Y_1^G is defined by the vanishing of: $I(y_0) = -1 + O(s_1)$, $I(s_1) = s_1^2(1 + y_0 - 2y_0^2 + O(s_1))$, and $I(y_2) = s_1 y_2(-1 - y_0 - y_2 + 2y_0^2 + O(s_1))$. Since Y_1^G is contained (as a set) in E_0, the first equation shows that Y_1^G is empty, in this chart. In the last chart $\{y_2 = 1\}$ in Y_1, we have coordinates (y_0, y_1, e_2), and $E_0 = \{e_2 = 0\}$. The fixed point scheme is defined by: $I(y_0) = -y_1 + O(e_2)$, $I(y_1) = e(y_1 + y_1^2 + y_0 y_1 - 2y_0^2 + O(e_2))$, and $I(e_2) = e_2^2(-1 + O(e_2))$. Since Y_1^G is contained (as a set) in E_0, the fixed point set is the line $\{0 = y_1 = e_2\}$, the same line seen in an earlier chart.

Thus $(Y_1^G)_{\text{red}}$ is isomorphic to $\mathbf{P}^1_{\mathbf{F}_5}$. Our criterion for a quotient by G to have toric singularities (Theorem 2.2) requires the G-fixed locus to have codimension 1; so let Y_2 be the blow-up of Y_1 along this $\mathbf{P}^1$. Clearly G continues to act on Y_2. The exceptional divisor E_1 in Y_2 is a $\mathbf{P}^1$-bundle over $\mathbf{P}^1_{\mathbf{F}_5}$, and so the natural way to cover E_1 by affine charts involves 4 charts, as follows. (See Figure 3, which applies to the current example as well.)

Over the open set $\{y_0 = 1\}$ in Y_1, Y_2 is the blow-up along the G-fixed curve $\{0 = s_0 = y_1\}$, so Y_2 has coordinates $((s_0, y_1, y_2), [w_0, w_1])$. First take $\{w_0 = 1\}$, so $y_1 = s_0 w_1$, and we have coordinates (s_0, w_1, y_2). As in every other chart, there are three variables over $\mathbf{Z}_5$, but this is a regular scheme of dimension 3 because of the equation satisfied by e_2. In this case, we have $e_2 = s_0 y_2$, and so

$$0 = (s_0 y_2)^5 - 5(s_0 y_2)^4 + 25(s_0 y_2)^2 - 25(s_0 y_2) + 5.$$

In this chart, E_0 does not appear, and $E_1 = \{s_0 = 0\}$. The fixed point scheme Y_2^G is defined by: $I(s_0) = s_0^2(-1 - w_1 + O(s_0))$, $I(w_1) = -2 + O(s_0)$, and $I(y_2) = s_0 y_2(1 + w_1 - y_2 + O(s_0))$. We know that the fixed point set is contained in E_1, and so the formula for $I(w_1)$ implies that Y_2^G is empty, in this chart.

In the other chart $\{w_1 = 1\}$ in Y_2 over the same open set in Y_1, we have $s_0 = y_1 w_0$, and so Y_2 has coordinates (w_0, y_1, y_2). Here $E_0 = \{w_0 = 0\}$, $E_1 = \{y_1 = 0\}$. Also, $e_2 = w_0 y_1 y_2$. The fixed point scheme is defined by $I(w_0) = w_0(2w_0 + O(y_1))/(1 - 2w_0 + O(y_1))$, $I(y_1) = y_1(-2w_0 + O(y_1))$, and $I(y_2) = y_1 y_2(1 + w_0 - w_0 y_2 + O(y_1))$. So Y_2^G is the line $\{0 = w_0 = y_1\} = E_0 \cap E_1$ over $\mathbf{F}_5$, in this chart.

To see the rest of $E_1 \subset Y_2$, work over the open set $\{y_2 = 1\}$ in Y_1. Here Y_2 is the blow-up along the G-fixed curve $\{0 = y_1 = e_2\}$, so Y_2 has coordinates $((y_0, y_1, e_2), [r_1, r_2])$. First take $\{r_1 = 1\}$ in Y_2, so $e_2 = y_1 r_2$, and we have coordinates (y_0, y_1, r_2). Here $E_0 = \{r_2 = 0\}$ and $E_1 = \{y_1 = 0\}$. Here Y_2^G is given by $I(y_0) = y_1(-1 + y_0 r_2 - y_0^2 r_2 + O(y_1))$, $I(y_1) = y_1 r_2(-2y_0^2 + O(y_1))$, and $I(r_2) = r_2^2(2y_0^2 + O(y_1))/(1 - 2y_0^2 r_2 + O(y_1))$. We know that the fixed point set is contained in E_1, and we read off that it is the union of the two lines $\{0 = y_1 = r_2\} = E_0 \cap E_1$ and $\{0 = y_0 = y_1\}$ in E_1. The first curve appeared in an earlier chart, and the second is new. Finally, the other open set is $\{r_2 = 1\}$ in Y_2, so $y_1 = e_2 r_1$, and we have coordinates (y_0, r_1, e_2). Here E_0 does not appear, and $E_1 = \{e_2 = 0\}$. Here Y_2^G is given by $I(y_0) = e_2(y_0 - r_1 - y_0^2 + O(e_2))$, $I(r_1) = -2y_0^2 + O(e_2)$, and $I(e_2) = e_2^2(-1 + O(e_2))$. We read off that the fixed point set is the curve $\{0 = y_0 = e_2\}$, which is the second curve in the previous chart.

Thus $(Y_2)^G$ as a set is the union of two $\mathbf{P}^1$'s over $\mathbf{F}_5$ meeting at a point. We are trying to make the fixed locus have codimension 1, and so our next step is to blow up one of those curves. Namely, let Y_3 be the blow-up of Y_2 along the G-fixed curve $E_0 \cap E_1$. The exceptional divisor E_2 in Y_3 is a $\mathbf{P}^1$-bundle over $\mathbf{P}^1_{\mathbf{F}_5}$, and so we need to look at four affine charts to see all of it. (See Figure 4, which applies to the current example as well.)

First, work over the open set $\{r_1 = 1\}$ in Y_2 over $\{y_2 = 1\}$ in Y_1. Then Y_3 is the blow-up along the curve $\{0 = y_1 = r_2\} = E_0 \cap E_1$, and so Y_3 has coordinates $(y_0, y_1, r_2), [z_1, z_2]$. First take $\{z_1 = 1\}$, so $r_2 = y_1 z_2$, and we have coordinates (y_0, y_1, z_2). As in every other chart, there are three variables over $\mathbf{Z}_5$, but this is a regular scheme of dimension 3 because of the equation satisfied by e_2. In this case, we have $e_2 = y_1^2 z_2$, and so

$$0 = (y_1^2 z_2)^5 - 5(y_1^2 z_2)^4 + 25(y_1^2 z_2)^2 - 25(y_1^2 z_2) + 5.$$

In this chart, $E_0 = \{z_2 = 0\}$, E_1 does not appear, and $E_2 = \{y_1 = 0\}$. The fixed point scheme Y_3^G is defined by: $I(y_0) = y_1(-1+O(y_1))$, $I(y_1) = y_1^2 z_2(-2y_0^2+O(y_1))$, and $I(z_2) = y_1 z_2^2(-y_0^2 + O(y_1))$. These equations are equivalent to $y_1 = 0$, near E_2; so the fixed point scheme Y_2^G is the Cartier divisor E_2, in this chart. (Thus, by Theorem 2.1, Y_2/G is regular, in this open set.)

The other chart is $\{z_2 = 1\}$ in Y_3, so $y_1 = r_2 z_1$, and we have coordinates (y_0, z_1, r_2). Here E_0 does not appear, $E_1 = \{z_1 = 0\}$, and $E_2 = \{r_2 = 0\}$. Also, $e_2 = z_1 r_2^2$. The fixed point scheme Y_3^G is given by $I(y_0) = z_1 r_2(-1 + O(r_2))$, $I(z_1) = z_1 r_2(y_0^2 + O(r_2))$, and $I(r_2) = r_2^2(2y_0^2 + O(r_2))$. The fixed point scheme is generically E_2 with multiplicity 1, together with the other fixed curve we knew from Y_2, here given by $\{0 = y_0 = z_1\} \subset E_1$. In more detail, the "bad locus" where the scheme Y_3^G is not just E_2 as a Cartier divisor is given by removing a factor of r_2 from these equations, yielding: $0 = z_1(-1 + O(r_2))$, $0 = z_1(y_0^2 + O(r_2))$, and $0 = r_2(2y_0^2 + O(r_2))$. We know the fixed locus away from E_2, so assume that $r_2 = 0$; then these equations show that the bad locus inside E_2 is the curve $\{0 = z_1 = r_2\} = E_1 \cap E_2$.

Fortunately, Theorem 2.2 implies that Y_3/G has toric singularities at points of $E_1 \cap E_2$ outside the origin. Namely, let $e = r_2$ and $s = z_1$; then $I(s) = es(\text{unit})$ near $E_1 \cap E_2 = \{0 = z_1 = r_2\}$ outside the origin. The theorem gives that Y_3/G has singularity $\frac{1}{5}(0, 1, 2)$ at points of $E_1 \cap E_2$ outside the origin.

To see all of $E_2 \subset Y_3$, we also have to work over $\{w_1 = 1\}$ in Y_2, with coordinates (w_0, y_1, y_2), over $\{y_0 = 1\}$ in Y_1. Here Y_3 is the blow-up along the G-fixed curve $\{0 = w_0 = y_1\} = E_0 \cap E_1$, so Y_3 has coordinates $(w_0, y_1, y_2), [v_0, v_1]$. First take $\{v_0 = 1\}$, so $y_1 = w_0 v_1$, and we have coordinates (w_0, v_1, y_2) on Y_3. Here E_0 does not appear, $E_1 = \{v_1 = 0\}$, and $E_2 = \{w_0 = 0\}$. Also, $e_2 = w_0^2 v_1 y_2$. The fixed point scheme is defined by: $I(w_0) = w_0^2(2 - 2v_1 + O(w_0))$, $I(v_1) = w_0 v_1(1 - 2v_1 + O(w_0))$, and $I(y_2) = w_0 v_1 y_2(1 + O(w_0))$. In the chart we are working over in Y_2, the fixed set Y_2^G is only the curve $E_0 \cap E_1$ we are blowing up, and so Y_3^G (in this chart) is contained in E_2 as a set. By the equations, Y_3^G is generically the Cartier divisor E_2, and the bad locus (where that fails) is given by $0 = w_0$, $0 = v_1(1 - 2v_1)$, and $0 = v_1 y_2$. So the bad locus is the union of the curve $\{0 = w_0 = v_1\} = E_1 \cap E_2$ and the point $(w_0, v_1, y_2) = (0, -2, 0)$ in E_2. By Theorem 2.2 (using $e = s = w_0$), Y_3/G has singularity $\frac{1}{5}(2, 1, 0)$ everywhere on the curve $E_1 \cap E_2$ (in this chart), in agreement with an earlier calculation.

To analyze the bad point above, change coordinates temporarily by $t_1 = v_1 + 2$; then the bad point becomes the origin in coordinates (w_0, t_1, y_2). In these coordinates, we have $I(w_0) = w_0^2(1 - 2t_1 + O(w_0))$, $I(t_1) = I(v_1) = (-t_1 - 2t_1^2 + O(w_0))$, and $I(y_2) = w_0 y_2(-2+O(w_0))$. Theorem 2.2 applies, with $s = e = w_0$, and we read off that Y_3/G has singularity $\frac{1}{5}(1,-1,-2)$ at this point. That is terminal, by the Reid-Tai criterion (Theorem 1.1).

The last chart we need to consider in Y_3 is the other open set $\{v_1 = 1\}$ over the open set above in Y_2, $\{w_1 = 1\} \subset Y_2$ over $\{y_0 = 1\} \subset Y_1$. So $w_0 = y_1 v_0$, and we have coordinates (v_0, y_1, y_2). Here $E_0 = \{v_0 = 0\}$, E_1 does not appear, and $E_2 = \{y_1 = 0\}$. Also, $e_2 = v_0 y_1^2 y_2$. Here Y_3^G is defined by: $I(v_0) = v_0 y_1(2 - v_0 + O(y_1))$, $I(y_1) = y_1^2(1-2v_0+O(y_1))$, and $I(y_2) = y_1 y_2(1+O(y_1))$. As in the previous chart, we know that Y_3^G is contained in E_2 as a set. By the equations, Y_3^G is generically the Cartier divisor E_2, and the bad locus (where that fails) is given by $0 = y_1$, $0 = v_0(2-v_0)$, and $0 = y_2$. Thus there are two bad points in this chart, (v_0, y_1, y_2) equal to $(2,0,0) \in E_2$ or $(0,0,0) \in E_0 \cap E_2$. The first is the bad point from the previous chart, but the second one is new. Theorem 2.2 works to analyze the second point (the origin), with $e = s = y_1$. We read off that Y_3/G has singularity $\frac{1}{5}(2,1,1)$ at this point.

That finishes the analysis of Y_3. In particular, as a set, Y_3^G is the union of the divisor E_2 and a curve in E_1. It is tempting to blow up the G-fixed curve next, but that leads to a large number of blow-ups over one point of the curve, where the fixed point scheme is especially complicated. We therefore define Y_4 as the blow-up at that point, and only later blow up the whole curve. This leads more efficiently to toric singularities.

Namely, let Y_4 be the blow-up of Y_3 at the origin in the chart $\{r_2 = 1\}$ in Y_2 (unchanged in Y_3), with coordinates (y_0, r_1, e_2). So Y_4 has coordinates $(y_0, r_1, e_2), [q_0, q_1, q_2]$. The exceptional divisor E_3 is isomorphic to $\mathbf{P}^2_{\mathbf{F}_5}$, and so it is covered by 3 affine charts. First take $\{q_0 = 1\}$ in Y_4, so $r_1 = y_0 q_1$ and $e_2 = y_0 q_2$, and we have coordinates (y_0, q_1, q_2). As in every other chart, there are three variables over $\mathbf{Z}_5$, but this is a regular scheme of dimension 3 because of the equation satisfied by e_2. In this case, we have $e_2 = y_0 q_2$, and so

$$0 = (y_0 q_2)^5 - 5(y_0 q_2)^4 + 25(y_0 q_2)^2 - 25(y_0 q_2) + 5.$$

Here $E_1 = \{q_2 = 0\}$ and $E_3 = \{y_0 = 0\}$. The fixed point scheme Y_4^G is defined by: $I(y_0) = y_0^2 q_2(1 - q_1 + O(y_0))$, $I(q_1) = y_0(-2 + q_1 q_2 + q_1^2 q_2 + O(y_0))$, and $I(q_2) = y_0 q_2^2(-2+q_1+O(y_0))$. So Y_4^G is generically the Cartier divisor E_3; the G-fixed curve in E_1 does not appear in this chart. The bad locus (where the scheme Y_4^G is not just E_3) is given by $0 = y_0$, $0 = -2+q_1 q_2+q_1^2 q_2$, and $0 = q_2^2(-2+q_1)$. By the second equation, $q_2 \neq 0$, and so the third equation gives that $q_1 = 2$. Then the second equation gives that $0 = -2+2q_2-q_2 = -2+q_2$, so $q_2 = 2$. That is, there is only one bad point in this chart, $(y_0, q_1, q_2) = (0,2,2) \in E_3$. To analyze that point, change coordinates temporarily by $s_1 = q_1 - 2$ and $s_2 = q_2 - 2$. In these coordinates, $I(y_0) = y_0^2(-2-s_1-s_2-s_1 s_2+O(y_0))$, $I(s_1) = I(q_1) = y_0(s_2+2s_1^2+s_1^2 s_2+O(y_0))$, and $I(s_2) = I(q_2) = y_0(-s_1 - s_1 s_2 + s_1 s_2^2 + O(y_0))$. By Theorem 2.2, with $e = s = y_0$, Y_4/G has a μ_5-quotient singularity. Explicitly, the linear map φ over $\mathbf{F}_5$ in the theorem is $\varphi(y_0) = -2y_0$, $\varphi(s_1) = s_2$, and $\varphi(s_2) = -s_1$, which has eigenvalues $-2, 2, -2$. So Y_4/G has singularity $\frac{1}{5}(-2,2,-2)$ at this point. This is terminal, by the Reid-Tai criterion.

Next, take the open set $\{q_1 = 1\}$ in Y_4, so $y_0 = r_1 q_0$ and $e_2 = r_1 q_2$, and Y_4 has coordinates (q_0, r_1, q_2). Here $E_1 = \{q_2 = 0\}$ and $E_3 = \{r_1 = 0\}$. The fixed point scheme Y_4^G is defined by: $I(q_0) = r_1(-q_2 - q_0q_2 + 2q_0^3 + O(r_1))$, $I(r_1) = r_1^2(2q_2 - 2q_0^2 + O(r_1))$, and $I(q_2) = r_1q_2(2q_2 + 2q_0^2 + O(r_1))$. So Y_4^G is generically the Cartier divisor E_3, together with the G-fixed curve $\{0 = q_0 = q_2\}$ in E_1. The bad locus in E_3 is given by $0 = r_1$, $0 = -q_2 - q_0q_2 + 2q_0^3$, and $0 = q_2(2q_2 + 2q_0^2)$. This yields two bad points, (q_0, r_1, q_2) equal to $(-2, 0, 1)$ or $(0, 0, 0)$. The first one is the bad point from the previous chart, and the second is not surprising, as it is the intersection point of E_3 with the G-fixed curve.

Finally, take the open set $\{q_2 = 1\}$ in Y_4, so $y_0 = e_2q_0$ and $r_1 = e_2q_1$, and we have coordinates (q_0, q_1, e_2). Here E_1 does not appear, and $E_3 = \{e_2 = 0\}$. The fixed point scheme Y_4^G is defined by: $I(q_0) = e(2q_0 - q_1 + O(e_2))$, $I(q_1) = e(-2q_1 - 2q_0^2 + O(e_2))$, and $I(e_2) = e_2^2(-1 + O(e_2))$. So Y_4^G is generically E_3. The bad locus in E_3 is given by: $0 = e_2$, $0 = 2q_0 - q_1$, and $0 = -2q_1 - 2q_0^2$. This yields two bad points, (q_0, q_1, e_2) equal to $(-2, 1, 0)$ (seen in the previous two charts) or $(0, 0, 0)$, which is new. Theorem 2.2 applies at this new point, with $e = s := e_2$. The $\mathbf{F}_5$-linear map φ is given by $\varphi(q_0) = 2q_0 - q_1$, $\varphi(q_1) = -2q_1$, and $\varphi(e_2) = -e_2$. So φ has eigenvalues $(2, -2, -1)$, and hence Y_4/G has singularity $\frac{1}{5}(2, -2, -1)$ at this point. This is terminal, by the Reid-Tai criterion.

That completes our description of Y_4. Next, let Y_5 be the blow-up of Y_4 along the G-fixed curve in E_1. The exceptional divisor E_4 in Y_5 is a $\mathbf{P}^1$-bundle over $\mathbf{P}^1_{\mathbf{F}_5}$, and so it is covered by four affine charts. First work over the open set $\{z_2 = 1\}$ in Y_3 (unchanged in Y_4), with coordinates (y_0, z_1, r_2); this contains the point where the G-fixed curve in $E_1 = \{z_1 = 0\}$ meets $E_2 = \{r_2 = 0\}$. Here Y_5 is the blow-up along the G-fixed curve $\{0 = y_0 = z_1\}$, and so Y_5 has coordinates $(y_0, z_1, r_2), [n_0, n_1]$. First take the open set $\{n_0 = 1\}$ in Y_5, so $z_1 = y_0n_1$, and we have coordinates (y_0, n_1, r_2). As in every other chart, there are three variables over $\mathbf{Z}_5$, but this is a regular scheme of dimension 3 because of the equation satisfied by e_2. In this case, we have $e_2 = y_0n_1r_2^2$, and so

$$0 = (y_0n_1r_2^2)^5 - 5(y_0n_1r_2^2)^4 + 25(y_0n_1r_2^2)^2 - 25(y_0n_1r_2^2) + 5.$$

In this chart, $E_1 = \{n_1 = 0\}$, $E_2 = \{r_2 = 0\}$, and $E_4 = \{y_0 = 0\}$. The fixed point scheme Y_5^G is defined by: $I(y_0) = y_0n_1r_2(-1 + O(y_0))$, $I(n_1) = n_1r_2(n_1 + O(y_0))/(1 - n_1r_2 + O(y_0))$, and $I(r_2) = y_0r_2^2(-2n_1r_2 + O(y_0))$. So Y_5^G, as a set, is the union of the divisor E_2 and the curve $\{0 = y_0 = n_1\} = E_1 \cap E_4$. (In particular, the fixed point set is still not all of codimension 1.) We have analyzed the bad locus of E_2 in previous steps, but we have to add here that the bad locus of E_2 is disjoint from $E_2 \cap E_4$ except for the point where E_2 meets the G-fixed curve, by the formula for $I(n_1)$. (See Figure 5, which applies to the current example as well.)

The other chart is $\{n_1 = 1\}$ in Y_5, so $y_0 = z_1n_0$, and we have coordinates (n_0, z_1, r_2). Here E_1 does not appear, $E_2 = \{r_2 = 0\}$, and $E_4 = \{z_1 = 0\}$. Also, $e_2 = z_1r_2^2$. The fixed point scheme Y_5^G is defined by: $I(n_0) = r_2(-1 + O(z_1))$, $I(z_1) = z_1^2r_2(-2r_2 + O(z_1))$, and $I(r_2) = z_1r_2^2(-2r_2 + O(z_1))$. These equations reduce to $r_2 = 0$ near E_4, and so Y_5^G is the Cartier divisor E_2, in this chart.

To finish our description of E_4 in Y_5, we work over the open set where the G-fixed curve in Y_4 meets E_3, namely $\{q_1 = 1\}$ in Y_4. Here Y_4 has coordinates (q_0, r_1, q_2), $E_1 = \{q_2 = 0\}$, $E_3 = \{r_1 = 0\}$, and the G-fixed curve is $\{0 = q_0 = q_2\}$ in E_1. So the blow-up Y_5 along the G-fixed curve has coordinates $(q_0, r_1, q_2), [u_0, u_2]$. First take $\{u_0 = 1\}$ in Y_5, so $q_2 = q_0u_2$, and we have coordinates (q_0, r_1, u_2). Here

$E_1 = \{u_2 = 0\}$, $E_3 = \{r_1 = 0\}$, and $E_4 = \{q_0 = 0\}$. Also, $e_2 = q_0 r_1 u_2$. The fixed point scheme Y_5^G is defined by: $I(q_0) = q_0 r_1(-u_2 + O(q_0))$, $I(r_1) = q_0 r_1^2(2u_2 + O(q_0))$, and $I(u_2) = r_1 u_2^2(1 + O(q_0))/(1 - r_1 u_2 + O(q_0))$. We know the fixed set outside E_4, and so we read off that the fixed set is the divisor E_3 together with the G-fixed curve $E_1 \cap E_4$ found earlier. We have analyzed the bad set of E_3 away from E_4 in earlier blow-ups, and we see from the formula for $I(u_2)$ that the bad set of E_3 near $E_3 \cap E_4$ is only the point $E_1 \cap E_3 \cap E_4$ where the G-fixed curve meets E_3.

The other chart is $\{u_2 = 1\}$ in Y_5. Here $q_0 = q_2 u_0$, and so we have coordinates (u_0, r_1, q_2). Here E_1 does not appear, $E_3 = \{r_1 = 0\}$, and $E_4 = \{q_2 = 0\}$. Also, $e_2 = r_1 q_2$. The fixed point scheme Y_5^G is defined by: $I(u_0) = r_1(-1 + O(q_2))$, $I(r_1) = r_1^2 q_2(2 + O(q_2))$, and $I(q_2) = r_1 q_2^2(2 + O(q_2))$. These equations reduce to $r_1 = 0$ near E_4, and so the fixed point scheme Y_5^G is the Cartier divisor E_3, in this chart.

That completes our description of Y_5. Let Y_6 be the blow-up of Y_5 along the G-fixed curve $E_1 \cap E_4$. The exceptional divisor E_5 in Y_6 is a $\mathbf{P}^1$-bundle over $\mathbf{P}^1_{\mathbf{F}_5}$, covered by four affine charts. First take the open set $\{n_0 = 1\}$ in Y_5, which contains the point where the G-fixed curve meets E_2. Here Y_5 has coordinates (y_0, n_1, r_2), with $E_1 = \{n_1 = 0\}$, $E_2 = \{r_2 = 0\}$, and $E_4 = \{y_0 = 0\}$. Since Y_6 is the blow-up along the G-fixed curve $\{0 = y_0 = n_1\} = E_1 \cap E_4$, Y_6 has coordinates $(y_0, n_1, r_2), [m_0, m_1]$. First take $\{m_0 = 1\}$ in Y_6, so $n_1 = y_0 m_1$, and we have coordinates (y_0, m_1, r_2). As in every other chart, there are three variables over $\mathbf{Z}_5$, but this is a regular scheme of dimension 3 because of the equation satisfied by e_2. In this case, we have $e_2 = y_0^2 m_1 r_2^2$, and so

$$0 = (y_0^2 m_1 r_2^2)^5 - 5(y_0^2 m_1 r_2^2)^4 + 25(y_0^2 m_1 r_2^2)^2 - 25(y_0^2 m_1 r_2^2) + 5.$$

In this chart, $E_1 = \{m_1 = 0\}$, $E_2 = \{r_2 = 0\}$, E_4 does not appear, and $E_5 = \{y_0 = 0\}$. The fixed point scheme Y_6^G is defined by: $I(y_0) = y_0^2 m_1 r_2(-1 + O(y_0))$, $I(m_1) = y_0 m_1 r_2(2m_1 + O(y_0))$, and $I(r_2) = y_0^2 r_2^2(2 - 2m_1 r_2 + O(y_0))$. We know the fixed point set away from E_5, and so we read off that the fixed point scheme is generically the Cartier divisor $E_2 + E_5$. (Since E_5 is fixed by G, we have finally made the fixed point set of codimension 1.) Let $e = y_0 r_2$. The bad locus (where the scheme Y_6^G is more than the Cartier divisor $E_2 + E_5$), on E_5, is given by factoring out e from the equations and setting $y_0 = 0$, so we get: $0 = y_0$ and $0 = 2m_1^2$. So, as a set, the bad locus is the curve $\{0 = y_0 = m_1\} = E_1 \cap E_5$. Theorem 2.2 does not seem to apply to this curve, and so Y_6/G might not have toric singularities there; we will have to blow up one more time.

For now, look at the other open set, $\{m_1 = 1\}$ in Y_6. So $s_0 = n_1 m_0$, and we have coordinates (m_0, n_1, r_2). Here E_1 does not appear, $E_2 = \{r_2 = 0\}$, $E_4 = \{m_0 = 0\}$, and $E_5 = \{n_1 = 0\}$. Also, $e_2 = y_0 m_1^2 r_2^2$. The fixed point scheme Y_6^G is defined by: $I(m_0) = m_0 n_1 r_2(-2 + O(n_1))$, $I(n_1) = n_1^2 r_2(1 + O(n_1))$, and $I(r_2) = m_0 n_1^2 r_2^2(2m_0 - 2r_2 + O(n_1))$. Since we know the fixed point set outside E_5, we read off that the fixed point scheme is generically the Cartier divisor $E_2 + E_5$. Let $e = n_1 r_2$. The bad locus (where the scheme Y_6^G is more than the Cartier divisor $E_2 + E_5$) is the curve $\{0 = m_0 = n_1\} = E_4 \cap E_5$. Fortunately, Theorem 2.2 applies, with $s = m_0$. We read off that Y_6/G has singularity $\frac{1}{5}(-2, 1, 0)$ along the whole curve $E_4 \cap E_5$, in this chart.

To finish describing $E_5 \subset Y_6$, we have to work over the open set $\{u_0 = 1\}$ in Y_5, where the G-fixed curve $E_1 \cap E_4$ in Y_5 meets E_3. Here Y_5 has coordinates

(q_0, r_1, u_2), with $E_1 = \{u_2 = 0\}$, $E_3 = \{r_1 = 0\}$, and $E_4 = \{q_0 = 0\}$. Then Y_6 is the blow-up along the G-fixed curve $\{0 = q_0 = u_2\} = E_1 \cap E_4$, so Y_6 has coordinates $(q_0, r_1, u_2), [t_0, t_2]$. First take $\{t_0 = 1\}$ in Y_6, so $u_2 = q_0 u_2$ and we have coordinates (q_0, r_1, t_2). Here $E_1 = \{t_2 = 0\}$, $E_3 = \{r_1 = 0\}$, E_4 does not appear, and $E_5 = \{q_0 = 0\}$. Also, $e_2 = q_0^2 r_1 t_2$. The fixed point scheme is defined by: $I(q_0) = q_0^2 r_1(-t_2 + O(q_0))$, $I(r_1) = q_0^2 r_1^2(-2 + 2t_2 + O(q_0))$, and $I(t_2) = q_0 r_1 t_2(2t_2 + O(q_0))$. So the fixed point scheme is generically $E_3 + E_5$. Let $e = q_0 r_1$. The bad locus (where the scheme Y_6^G is more than the Cartier divisor $E_3 + E_5$), in E_5, is given by $0 = q_0$ and $0 = 2t_2^2$, so (as a set) it is the curve $\{0 = q_0 = t_2\} = E_1 \cap E_5$, which we met in an earlier chart.

The other chart is $\{t_2 = 1\}$ in Y_6, so $q_0 = u_2 t_0$, and we have coordinates (t_0, r_1, u_2). Here E_1 does not appear, $E_3 = \{r_1 = 0\}$, $E_4 = \{t_0 = 0\}$, and $E_5 = \{u_2 = 0\}$. Also, $e_2 = q_0^2 r_1 t_2$. The fixed point scheme Y_6^G is defined by: $I(t_0) = t_0 r_1 u_2(-2 + O(u_2))$, $I(r_1) = t_0 r_1^2 u_2^2(2 - 2t_0 + O(u_2))$, and $I(u_2) = r_1 u_2^2(1 + O(u_2))$. So Y_6^G is generically $E_3 + E_5$. Let $e = r_1 u_2$. The bad locus (where the scheme Y_6^G is more than the Cartier divisor $E_3 + E_5$), in E_5, is the curve $\{0 = t_0 = u_2\} = E_4 \cap E_5$, which we met in an earlier chart. Theorem 2.2 applies, with $s = t_0$. Namely, Y_6/G has singularity $\frac{1}{5}(-2, 0, 1)$ everywhere on the curve $E_4 \cap E_5$ in this chart (including the origin, which did not appear in the earlier chart).

That completes our description of Y_6. In particular, the G-fixed locus has codimension 1 in Y_6, and Y_6/G has toric singularities outside the image of the curve $E_1 \cap E_5$. Let Y_7 be the blow-up of Y_6 along that curve. The exceptional divisor E_6 in Y_7 is a $\mathbf{P}^1$-bundle over $\mathbf{P}^1_{\mathbf{F}_5}$, and so we will cover E_6 with four affine charts. First, work over the open set $\{m_0 = 1\}$ in Y_6, where the bad curve $E_1 \cap E_5$ meets E_2. Here Y_6 has coordinates (y_0, m_1, r_2), with $E_1 = \{m_1 = 0\}$, $E_2 = \{r_2 = 0\}$, and $E_5 = \{y_0 = 0\}$. Since Y_7 is the blow-up along the curve $\{0 = y_0 = m_1\} = E_1 \cap E_5$, Y_7 has coordinates $(y_0, m_1, r_2), [j_0, j_1]$. (See Figure 6, which applies to the current example as well.)

First take $\{j_0 = 1\}$ in Y_7, so $m_1 = y_0 j_1$, and we have coordinates (y_0, j_1, r_2). As in every other chart, there are three variables over $\mathbf{Z}_5$, but this is a regular scheme of dimension 3 because of the equation satisfied by e_2. In this case, we have $e_2 = y_0^3 j_1 r_2^2$, and so

$$0 = (y_0^3 j_1 r_2^2)^5 - 5(y_0^3 j_1 r_2^2)^4 + 25(y_0^3 j_1 r_2^2)^2 - 25(y_0^3 j_1 r_2^2) + 5.$$

In this chart, $E_1 = \{j_1 = 0\}$, $E_2 = \{r_2 = 0\}$, E_5 does not appear, and $E_6 = \{y_0 = 0\}$. The fixed point scheme Y_7^G is defined by: $I(y_0) = y_0^3 j_1 r_2(-1 + O(y_0))$, $I(j_1) = y_0^2 j_1 r_2(1 - 2j_1 + O(y_0))$, and $I(r_2) = y_0^2 r_2^2(2 + O(y_0))$. So Y_7^G is generically the Cartier divisor $E_2 + 2E_6$. Let $e = y_0^2 r_2$. The bad locus (where the scheme Y_7^G is more than the Cartier divisor $E_2 + 2E_6$), in E_6, is given by $0 = y_0$, $0 = j_1(1 - 2j_1)$, and $0 = r_2$, so it consists of the two points (y_0, j_1, r_2) equal to $(0, 0, 0) = E_1 \cap E_2 \cap E_6$ or $(0, -2, 0) \in E_2 \cap E_6$. At the first point, Theorem 2.2 applies, with $s = r_2$. We read off that Y_7/G has singularity $(0, 1, 2)$ everywhere on the curve $E_1 \cap E_2$ (including the origin, which did not appear when we saw $E_1 \cap E_2$ in an earlier chart). To analyze the second point, change coordinates temporarily by $s_1 = j_1 + 2$; then that point becomes the origin in coordinates (y_0, s_1, r_2). We have $I(y_0) = y_0^3 r_2(2 - s_1 + O(y_0))$, $I(s_1) = I(j_1) = y_0^2 r_2(-s_1 - 2s_1^2 + O(y_0))$, and $I(r_2) = y_0^2 r_2^2(2 + O(y_0))$. Theorem 2.2 applies, with $s = r_2$. We read off that Y_7/G has singularity $\frac{1}{5}(2, -1, 2)$ at this point.

The other open set is $\{j_1 = 1\}$ in Y_7, so $y_0 = m_1 j_0$, and we have coordinates (j_0, m_1, r_2). Here E_1 does not appear, $E_2 = \{r_2 = 0\}$, $E_5 = \{j_0 = 0\}$, and $E_6 = \{m_1 = 0\}$. Also, $e_2 = j_0^2 m_1^3 r_2^2$. The fixed point scheme Y_7^G is defined by: $I(j_0) = j_0^2 m_1^2 r_2(2 - j_0 + O(m_1))$, $I(m_1) = j_0 m_1^3 r_2(2 + j_0 + O(m_1))$, and $I(r_2) = j_0^2 m_1^2 r_2^2(2 + O(m_1))$. So Y_7^G is generically $E_2 + E_5 + 2E_6$. Let $e = j_0 m_1^2 r_2$. The bad locus (where the scheme Y_7^G is more than the Cartier divisor $E_2 + E_5 + 2E_6$), on E_6, is given by: $0 = m_1$, $0 = j_0(2 - j_0)$, and $0 = j_0 r_2$. So the bad locus is the union of the curve $\{0 = j_0 = m_1\} = E_5 \cap E_6$ and the point $(j_0, m_1, r_2) = (2, 0, 0)$ in $E_2 \cap E_6$. That point is the one we analyzed in the previous chart. For the curve, Theorem 2.2 applies, using $s = j_0$. We read off that Y_7/G has singularity $\frac{1}{5}(2, 2, 0)$ everywhere on the curve $E_5 \cap E_6$, in this chart.

Last, work over the open set $\{t_0 = 1\}$ in Y_6, where the bad curve $E_1 \cap E_5$ meets E_3. Here Y_6 has coordinates (q_0, r_1, t_2), with $E_1 = \{t_2 = 0\}$, $E_3 = \{r_1 = 0\}$, and $E_5 = \{q_0 = 0\}$. We obtain Y_7 by blowing up along the curve $\{0 = q_0 = t_2\} = E_1 \cap E_5$, so Y_7 has coordinates $(q_0, r_1, t_2), [x_0, x_2]$. First take $\{x_0 = 1\}$, so $t_2 = q_0 x_2$, and we have coordinates (q_0, r_1, x_2). Here $E_1 = \{x_2 = 0\}$, $E_3 = \{r_1 = 0\}$, E_5 does not appear, and $E_6 = \{q_0 = 0\}$. Also, $e_2 = q_0^3 r_1 x_2$. The fixed point scheme Y_7^G is defined by: $I(q_0) = q_0^3 r_1(2 - x_2 + O(q_0))$, $I(r_1) = q_0^2 r_1^2(-2 + O(q_0))$, and $I(x_2) = q_0^2 r_1 x_2(1 - 2x_2 + O(q_0))$. So Y_7^G is generically $E_3 + 2E_6$. Let $e = q_0^2 r_1$. The bad locus (where the scheme Y_7^G is more than the Cartier divisor $E_3 + 2E_6$), in E_6, is given by: $0 = q_0$, $0 = r_1$, and $0 = x_2(1 - 2x_2)$, so it consists of the two points (q_0, r_1, x_2) equal to $(0, 0, 0) = E_1 \cap E_3 \cap E_6$ or $(0, 0, -2)$ in $E_3 \cap E_6$. Since $I(r_1) = e r_1(\text{unit})$, Theorem 2.2 applies at both points. At the origin, the theorem gives that Y_7/G has singularity $\frac{1}{5}(2, -2, 1)$, which is terminal. For the other point, change coordinates temporarily by $y_2 = x_2 + 2$, so that the point becomes the origin in coordinates (q_0, r_1, y_2). We have $I(q_0) = q_0^3 r_1(-1 - y_2 + O(q_0))$, $I(r_1) = q_0^2 r_1^2(-2 + O(q_0))$, and $I(y_2) = I(x_2) = q_0^2 r_1(-y_2 - 2y_2^2 + O(q_0))$. So Theorem 2.2 gives that Y_7/G has singularity $\frac{1}{5}(-1, -2, -1)$ at this point.

The other chart is $\{x_2 = 1\}$ in Y_7, so $q_0 = t_2 x_0$, and we have coordinates (x_0, r_1, t_2). Here E_1 does not appear, $E_3 = \{r_1 = 0\}$, $E_5 = \{x_0 = 0\}$, and $E_6 = \{t_2 = 0\}$. Also, $e_2 = x_0^2 r_1 t_2^3$. The fixed point scheme Y_7^G is defined by: $I(x_0) = x_0^2 r_1 t_2^2(2 - x_0 + O(t_2))$, $I(r_1) = x_0^2 r_1^2 t_2^2(-2 + O(t_2))$, and $I(t_2) = x_0 r_1 t_2^3(2 - 2x_0 + O(t_2))$. So Y_7^G is generically $E_3 + E_5 + 2E_6$. Let $e = x_0 r_1 t_2^2$. The bad locus (where Y_7^G is more than the Cartier divisor $E_3 + E_5 + 2E_6$), in E_6, is given by: $0 = t_2$, $0 = x_0(2 - x_0)$, and $0 = x_0 r_1$, which is the union of the curve $\{0 = x_0 = t_2\} = E_5 \cap E_6$ and the point $(x_0, r_1, t_2) = (2, 0, 0)$ in $E_3 \cap E_6$. We analyzed that point in the previous chart. Theorem 2.2 applies to the curve, using $s = x_0$. We read off that Y_7/G has singularity $\frac{1}{5}(2, 0, 2)$ everywhere on the curve $E_5 \cap E_6$ (including the origin, which did not appear in the earlier chart where we met this curve).

That completes our analysis of Y_7; we have shown that Y_7/G has toric singularities. The sequence of blow-ups and the descriptions of the singularities of Y_7/G are identical to those in the characteristic 5 example, Theorem 8.1. Given that, the proof that Y_0/G is terminal is unchanged from that of the characteristic 5 example. Theorem 9.1 is proved. □

Remark 9.2. As in Remark 7.2, I expect that there is also a 3-dimensional scheme X, flat over $\mathbf{Z}_5$, that is terminal and non-Cohen-Macaulay with K_X Cartier. Namely, one should replace the p-adic integer ring $R = \mathbf{Z}_5[\zeta_{25}]^{\mathbf{Z}/4}$ in Theorem 9.1 by $S =$

$\mathbf{Z}_5[\zeta_{25}]$, with the action of $\mathbf{Z}/5 \subset (\mathbf{Z}/25)^*$.

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