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Milnor K-Theory is the Simplest Part of Algebraic K-Theory

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Abstract. We identify the Milnor K-theory of a field with a certain higher Chow group.

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One of the important consequences of Grothendieck's Riemann–Roch theorem is that

$$K_0(X) \otimes \mathbf{Q} \cong \bigoplus_p \mathrm{CH}^p(X) \otimes \mathbf{Q},$$

for any smooth algebraic variety X. Here $K_0(X)$ is the Grothendieck group of vector bundles on X, and $\mathrm{CH}^p(X)$ is the Chow group of codimension-p algebraic cycles on X.

Recently, Bloch [2] has shown that Quillen's higher algebraic K-theory of X has a similar decomposition. He defines groups $\mathrm{CH}^p(X, n)$ in terms of certain codimension-p algebraic cycles on $X \times \mathbf{A}^n$, and we have

$$K_n(X) \otimes \mathbf{Q} \cong \bigoplus_p \mathrm{CH}^p(X, n) \otimes \mathbf{Q}.$$

Surprisingly, if we don't tensor with $\mathbf{Q}$, Bloch's 'Chow groups' seem to have nicer properties than the usual K-groups. An example is given in this paper: if X is a point, viewed as a variety over a field F, then (writing $\mathrm{CH}^p(F, n)$ for $\mathrm{CH}^p(\operatorname{Spec} F, n)$) we have

$$\mathrm{CH}^p(F, n) = 0, \quad \text{for } p > n,$$
$$\mathrm{CH}^n(F, n) \cong K_n^{\mathrm{M}}(F).$$

Here $K_n^{\mathrm{M}}(F)$ is the Milnor K-theory of the field F. This result (first proved by Nesterenko and Suslin [9]) indicates that $K_n^{\mathrm{M}}(F)$ is the 'simplest part' of $K_n(F)$. It would be impossible to state such a precise result without using something like the Chow groups. Also, this result allows us to see some of the standard properties of Milnor K-theory in a more geometric way.

Our proof differs from the proof by Nesterenko and Suslin mainly in its use of explicit rational curves in affine space A_F^{n+1} to verify relations in $\mathrm{CH}^n(F,n)$. In particular, the Steinberg relation in $\mathrm{CH}^2(F,2) \cong K_2^M F$ comes from a specific rational curve in A_F^3. This leads to the hope that the Chow groups of a field can be computed using a very small class of affine algebraic varieties (linear spaces, in the right coordinates), whereas the current definition uses essentially all algebraic cycles in affine space. This hope was partially realized when Bloch ([11], p. 780) wrote down a subcomplex of $\mathrm{CH}^*(F,*)$ related to polylogarithms. The subcomplex is defined using a specific embedding $(P^1 - \{0,1,\infty\})^{r-1} \hookrightarrow A^{2r-1}$. The case $r = 2$ is the rational curve in this paper which gives the Steinberg relation.

1. Definitions

We work throughout with quasi-projective schemes over a field; we assume that all irreducible components have the same dimension unless stated otherwise. For a quasi-projective scheme Y, we define $z^*(Y)$ to be the group of algebraic cycles on Y, i.e., the free-Abelian group (graded by codimension) on the set of irreducible closed subvarieties of Y. If $i: W \to Y$ is a closed subvariety which is a local complete intersection, there is a pullback map $i^*: z^*(Y)' \to z^*(W)$, where $z^*(Y)' \subset z^*(Y)$ is the group generated by subvarieties which meet W *properly*, i.e., in the correct dimension. (Our reference for this and similar facts about algebraic cycles is Fulton [4].)

Bloch's Chow groups $\mathrm{CH}^p(X,n)$ are defined as follows. Let F be a field, and let the 'simplex' Δ^n (isomorphic to the affine space $\mathbf{A}_F^n$) be the hyperplane $\Sigma_{i=0}^n t_i = 1$ in $\mathbf{A}_F^{n+1}$. Given an increasing map $\rho: \{0,\ldots,m\} \to \{0,\ldots,n\}$, we define $\tilde{\rho}: \Delta^m \to \Delta^n$ by $\tilde{\rho}^*(t_i) = \Sigma_{\rho(j)=i} t_j$. If ρ is injective, we say that $\tilde{\rho}(\Delta^m) \subset \Delta^n$ is a *face*. For a quasi-projective scheme X over F, define $z^*(X,n) \subset z^*(X \times_F \Delta^n)$ to be generated by the subvarieties which meet all faces $X \times \Delta^m \subset X \times \Delta^n$ properly. Figure 1 is a picture of such a subvariety (taking $X = \mathrm{pt.}$, $n = 2$).

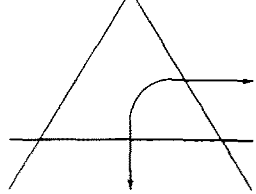

Fig. 1.

We obtain in this way a complex of graded Abelian groups,

$$z^*(X,\cdot)\colon \cdots \to z^*(X,2) \xrightarrow{\Sigma(-1)^i\partial_i} z^*(X,1) \xrightarrow{\Sigma(-1)^i\partial_i} z^*(X,0) \to 0,$$

where, for $i = 0,\ldots,n$, $\partial_i\colon z^*(X,n) \to z^*(X,n-1)$ means pullback along the face map

$$(t_0,\ldots,t_{n-1}) \mapsto (t_0,\ldots,t_{i-1},0,t_i,\ldots,t_{n-1}).$$

The groups $\mathrm{CH}^*(X,n)$ are defined to be the homology groups of this complex.

For example (as Bloch says), $z^*(X,0) = z^*(X)$, and $\mathrm{CH}^*(X,0)$ is defined by killing cycles of the form $Z(0) - Z(1)$, where Z is a cycle in $X \times \mathbf{A}^1$ which meets $X \times \{0\}$ and $X \times \{1\}$ properly. This gives precisely the usual Chow groups, $\mathrm{CH}^*(X) \cong \mathrm{CH}^*(X,0)$, of algebraic cycles on X modulo rational equivalence.

Just as in topology, we can replace simplices by cubes in the definition of Bloch's Chow groups. (The point is that it will be easier to describe the product structure on the Chow groups if we use cubes.)

Namely, given a strictly increasing map $\rho\colon \{1,\ldots,m\} \to \{1,\ldots,n\}$, and given $\varepsilon_i \in \{0,1\}$ for $i \in \{1,\ldots,n\} - \rho(\{1,\ldots,m\})$, the face map $\tilde{\rho}^\varepsilon\colon \mathbf{A}^m \to \mathbf{A}^n$ is given by

$$(\tilde{\rho}^\varepsilon)^*(t_i) = \begin{cases} t_j, & \text{if } i = \rho(j),\\ \varepsilon_i, & \text{if } i \text{ is not in the image of } \rho.\end{cases}$$

We define $c^*(X,n) \subset z^*(X \times_F \mathbf{A}^n)$ as the group generated by those subvarieties which meet all faces of the cube properly. Figure 2 depicts such a subvariety (with $X = \text{pt.}$, $n = 2$).

For $i \in \{1,\ldots,n\}$, $\varepsilon \in \{0,1\}$, let $\partial_i^\varepsilon\colon c^*(X,n) \to c^*(X,n-1)$ be the pullback along the face map

$$(t_1,\ldots,t_{n-1}) \mapsto (t_1,\ldots,t_{i-1},\varepsilon,t_i,\ldots,t_{n-1}).$$

Then we have a complex of graded Abelian groups,

$$c^*(X,n)\colon \cdots \to c^*(X,2) \to c^*(X,1) \to c^*(X,0) \to 0,$$

with boundary maps $d_n\colon c^*(X,n) \to c^*(X,n-1)$ given by

$$d_n = \sum_{i=1}^{n} (-1)^i(\partial_i^0 - \partial_i^1).$$

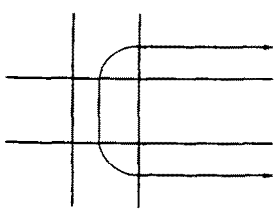

Fig. 2.

The cubical complex $c^i(X,\cdot)$ does not have the right homology groups (as one can see in the case $i=0$); we need to mod out by a subcomplex $d^i(X,\cdot) \subset c^i(X,\cdot)$ of 'degenerate cycles'. Namely, $d^i(X,n)$ is the subgroup of $c^i(X,n)$ generated by those cycles on $X \times A^n$ which are pulled back from some cycle on $X \times A^{n-1}$ via a linear projection of the form

$$(x_1,\ldots,x_n) \mapsto (x_1,\ldots,\hat{x}_k,\ldots,x_n),$$

where $1 \leqslant k \leqslant n$. The homology groups of the complex $c^i(X,\cdot)/d^i(X,\cdot)$ are isomorphic to the Chow groups $\mathrm{CH}^i(X,n)$.

We now make one last change in the definition of $\mathrm{CH}^*(X,n)$. (This change is motivated by the computation of $\mathrm{CH}^1(F,1)$, as I will explain later.) Namely, we observe that $\mathbf{A}^n \cong (\mathbf{P}^1 - \{1\})^n$, via the isomorphism

$$(x_1,\ldots,x_n) \mapsto \left(1-\frac{1}{x_1},\ldots,1-\frac{1}{x_n}\right).$$

(The map $x \mapsto 1 - 1/x$ is an automorphism of $\mathbf{P}^1$ which permutes $1, 0, \infty$.) So we can restate the cubical definition of $\mathrm{CH}^*(X,n)$ in terms of $(\mathbf{P}^1 - \{1\})^n$ with $\{0,1\} \subset \mathbf{A}^1$ replaced by $\{\infty, 0\} \subset (\mathbf{P}^1 - \{1\})^1$. To get the notation straight, here is my last version of this definition. Given a strictly increasing map $\rho\colon \{1,\ldots,m\} \to \{1,\ldots,n\}$, and given $\varepsilon_i \in \{0,\infty\}$ for i not in the image of ρ, the face map $\tilde{\rho}^\varepsilon\colon (\mathbf{P}^1 - \{1\})^m \to (\mathbf{P}^1 - \{1\})^n$ is given by

$$(\tilde{\rho}^\varepsilon)^* t_i = \begin{cases} t_j, & \text{if } i = \rho(j),\\ \varepsilon_i, & \text{if } i \notin \rho(\{1,\ldots,m\}).\end{cases}$$

Then $c^*(X,n) \subset z^*(X \times (\mathbf{P}^1 - \{1\})^n)$ is the group generated by subvarieties meeting all faces of the cube in the correct dimension, and we get a complex as before,

$$c^*(x,\cdot)\colon \cdots \to c^*(X,2) \to c^*(X,1) \to c^*(X,0) \to 0.$$

The boundary maps are

$$d_n = \sum_{i=1}^{n} (-1)^i(\partial_i^\infty - \partial_i^0),$$

where ∂_i^0 is pullback to the face $t_i = 0$ and ∂_i^∞ is pullback to the face $t_i = \infty$. There is a subcomplex of degenerate cycles $d^i(X,n) \subset c^i(X,n)$ as above, and the homology groups of the complex $c^i(X,\cdot)/d^i(X,\cdot)$ are isomorphic to the Chow groups $\mathrm{CH}^i(X,n)$.

The product structure on the Chow groups is given as follows. For any quasi-projective F-schemes X, Y there is an exterior product

$$\mathrm{CH}^p(X,q) \otimes \mathrm{CH}^r(Y,s) \to \mathrm{CH}^{p+r}(X \times Y, q+s),$$

defined via the obvious identification

$$(X \times (\mathbf{P}^1 - \{1\})^q) \times (Y \times (\mathbf{P}^1 - \{1\})^s) \cong (X \times Y) \times (\mathbf{P}^1 - \{1\})^{q+s}.$$

(If we were using the simplicial definition of the Chow groups, we would have to give a triangulation of $\Delta^q \times \Delta^s$ at this point.) If X is smooth, then, following Bloch, we can pull back along the diagonal to get the product structure on X's Chow groups

$$\mathrm{CH}^p(X, q) \otimes \mathrm{CH}^r(X, s) \to \mathrm{CH}^{p+r}(X, q + s).$$

One checks that the isomorphisms between the cubical and the simplicial Chow groups preserve products, so that this product structure on $\mathrm{CH}^*(X, \cdot)$ coincides with that defined by Bloch.

2. Statement of the Theorem. The Map $K_n^{\mathrm{M}}(F) \to \mathrm{CH}^n(F, n)$

THEOREM 1. *If F is a field, then we have*

$$\mathrm{CH}^i(F, n) = 0, \quad \textit{for } i > n,$$
$$\mathrm{CH}^n(F, n) \cong K_n^{\mathrm{M}}(F).$$

Here $\mathrm{CH}^i(F, n) = \mathrm{CH}^i(\mathrm{Spec}\ F, n)$ *and* $K_n^{\mathrm{M}}(F)$ *is the* nth *Milnor K-group of* F.

Proof. $\mathrm{CH}^i(F, n)$ is a group of codimension-i cycles on the n-dimensional variety $(\mathbf{P}_F^1 - \{1\})^n$; so for $i > n$ we certainly have $\mathrm{CH}^i(F, n) = 0$.

We now show that $\mathrm{CH}^n(F, n) \cong K_n^{\mathrm{M}}(F)$. We first define the map $K_n^{\mathrm{M}}(F) \to \mathrm{CH}^n(F, n)$; this will in fact be a ring homomorphism $K_*^{\mathrm{M}}(F) \to \mathrm{CH}^*(X, *)$.

The Milnor ring (Milnor [6]) is defined as the quotient of the tensor algebra on the multiplicative group F^* of F,

$$(\mathbf{Z}, F^*, F^* \otimes F^*, F^* \otimes F^* \otimes F^*, \ldots),$$

by the homogeneous ideal generated by all $\{a, 1 - a\}$, where $a, 1 - a \in F^*$. So the ring homomorphism $K_*^{\mathrm{M}}(F) \to \mathrm{CH}^*(F, *)$ is defined by specifying a group homomorphism

$$\phi\colon F^* \to \mathrm{CH}^1(F, 1)$$

such that $\phi(a)\phi(1 - a) = 0$ in $\mathrm{CH}^2(F, 2)$, when $a, 1 - a \in F^*$.

Now an element of $\mathrm{CH}^n(F, n)$ is represented by a 0-cycle on $(\mathbf{P}^1 - \{1\})^n$ which meets all proper faces of the cube in the correct dimension, i.e., not at all. So it's represented by a sum of closed points in $(\mathbf{P}_F^1 - \{0, 1, \infty\})^n$.

The homomorphism $\phi\colon F^* \to \mathrm{CH}^1(F, 1)$ is defined by sending $\{1\}$ to $0 \in \mathrm{CH}^1(F, 1)$, and $\{a\}$ with $a \in F^* - \{1\}$ to the class of the point $a \in \mathbf{P}_F^1 - \{0, 1, \infty\}$. For short, we say that $\phi(\{a\}) = [a]$, for $a \in F^* - \{1\}$. (If I had not changed coordinates, in the definition of $c^n(F, n)$, from $\mathbf{A}^n$ to $(\mathbf{P}^1 - \{1\})^n$, this map would have to be defined by $\{a\} \mapsto [1/(1 - a)]$.)

To check that this is a group homomorphism, we need to know that

$$[a] + \left[\frac{1}{a}\right] = 0, \quad \text{for } a \in F - \{0, 1\}$$

and

$$[a] + [b] = [ab], \quad \text{for } a, b, ab \in F - \{0, 1\}.$$

To show that a given element of $c^1(F, 1)$ is 0 in $\mathrm{CH}^1(F, 1)$, we have to show that it's the 'boundary' of some element of $c^1(F, 2)$. An element of $c^1(F, 2)$ is a sum of curves in $(\mathbf{P}^1 - \{1\})^2$, meeting the one-dimensional faces of the square in points and not meeting the zero-dimensional faces at all.

Take the rational curve

$$x \mapsto \left(x, \frac{ax - ab}{x - ab}\right)$$

in $(\mathbf{P}^1 - \{1\})^2$, for $a, b \in F - \{0, 1\}$. One checks easily that this curve defines an element of $c^1(F, 2)$. It intersects the one-dimensional faces of the square (i.e., the lines $x = 0$, $x = \infty$, $y = 0$, $y = \infty$ in $(\mathbf{P}^1 - \{1\})^2$) in the points (∞, a), $(b, 0)$, and (ab, ∞) if $ab \neq 1$. If $ab = 1$, it intersects only in (∞, a) and $(1/a, 0)$. This implies that

$$[a] + \left[\frac{1}{a}\right] = 0, \quad \text{for } a \in F - \{0, 1\}$$

and

$$[a] + [b] = [ab], \quad \text{for } a, b, ab \in F - \{0, 1\},$$

as promised.

(Bloch gives a more conceptual proof that $\mathrm{CH}^1(F, 1) \cong F^*$, using the relation between divisors and line bundles, but that proof doesn't seem to generalize to higher codimension, since the relation between higher-codimension subvarieties and vector bundles is more complicated. For that reason I gave a computational proof here, in the spirit of the rest of this paper.)

So we have a homomorphism $\phi\colon K_1^M(F) \to \mathrm{CH}^1(F, 1)$. We now check the Steinberg relation, i.e., that $\phi(a)\phi(1 - a) = 0$ in $\mathrm{CH}^2(F, 2)$, for $a, 1 - a \in F^*$. By my description of products in the Chow groups, this means showing that the point $(a, 1 - a) \in (\mathbf{P}^1 - \{1\})^2 -$ (the square) is the boundary of something in $c^2(F, 3)$.

Take the rational curve

$$x \mapsto \left(x, 1 - x, \frac{a - x}{1 - x}\right)$$

in $(\mathbf{P}^1 - \{1\})^3$. Its only intersection with any codimension-1 face of the cube is the point $(a, 1 - a, 0)$; so we deduce that $[(a, 1 - a)] = 0$ in $\mathrm{CH}^2(F, 2)$.

So we have produced a ring homomorphism

$$K_*^M(F) \to \mathrm{CH}^*(F, *).$$

3. The Map $\mathrm{CH}^n(F, n) \to K_n^{\mathrm{M}}(F)$

Defining the inverse map $\mathrm{CH}^n(F, n) \to K_n^{\mathrm{M}}(F)$ is more complicated. If F is algebraically closed, we can send the point $(x_1, \ldots, x_n) \in (\mathbf{P}^1 - \{1\})^n$–(the cube) (i.e., $(x_1, \ldots, x_n) \in (\mathbf{P}_F^1 - \{0, 1, \infty\})^n$) to $\{x_1, \ldots, x_n\} \in K_n^{\mathrm{M}}(F)$. For a general field F, a closed point p in $(\mathbf{P}_F^1 - \{0, 1, \infty\})^n$ need not be of the form $(x_1, \ldots, x_n)$ with $x_i \in F$; it may have residue field $\kappa(p)$ which is a nontrivial finite extension of F. In any case, we have a map

$$\operatorname{Spec} \kappa(p) \to (\mathbf{P}_F^1 - \{0, 1, \infty\})^n = (\mathbf{A}_F^1 - \{0, 1\})^n$$

which gives n 'coordinate functions' $x_1, \ldots, x_n \in \kappa(p) - \{0, 1\}$, and we have $\kappa(p) = F(x_1, \ldots, x_n)$. The map $\mathrm{CH}^n(F, n) \to K_n^{\mathrm{M}}(F)$ is defined by sending p to

$$N_{\kappa(p)/F}\{x_1, \ldots, x_n\} \in K_n^{\mathrm{M}}(F).$$

Here

$$\{x_1, \ldots, x_n\} \in K_n^{\mathrm{M}}(\kappa(p)) \quad \text{and} \quad N_{E/F}\colon K_n^{\mathrm{M}}(E) \to K_n^{\mathrm{M}}(F)$$

(for any finite extension E/F of fields) is the norm homomorphism defined by Milnor [6], Bass–Tate [1], and Kato [5]. (Kato was the first to show that it was well-defined, independent of a choice of generators for E over F.) For $n = 0$, $K_0^{\mathrm{M}}(E) = K_0^{\mathrm{M}}(F) = \mathbf{Z}$, and the norm is just multiplication by $[E{:}F]$; for $n = 1$, the norm $N_{E/F}\colon E^* \to F^*$ is the usual norm studied in Galois theory.

We have to check that a 0-cycle in $c^n(F, n)$ which is the boundary of a 1-cycle in $c^n(F, n+1)$ maps to 0 in $K_n^{\mathrm{M}}(F)$. It suffices to consider an irreducible curve $C \in c^n(F, n+1)$; that is, C is an irreducible curve in $(\mathbf{P}^1 - \{1\})^{n+1}$ which meets the codimension-1 faces of the $(n+1)$-cube in points, and which does not meet the codimension-2 faces.

Let D be the normalization of C; then D is a smooth curve with a finite map $D \to C$, as is well known. We can forget about C and just consider the finite (in particular, proper) map $D \to (\mathbf{P}^1 - \{1\})^{n+1}$, which again meets the codimension-1 faces of the cube in points and does not meet the codimension-2 faces. It follows from Example 1.2.3, p. 9, in [4] that the 'boundary' of D in $c^n(F, n)$ coincides with the boundary of C; so it suffices to show that the boundary of D maps to 0 in $K_n^{\mathrm{M}}(F)$.

The map $D \to (\mathbf{P}^1 - \{1\})^{n+1}$ can be described by $n+1$ rational functions $g_1, \ldots, g_{n+1}$ on D. The pevious paragraph implies that no g_i is identically equal to 0 or ∞, and that any $w \in D$ with $g_i(w) = 0$ or ∞ has $g_j(w) \notin \{0, \infty\}$ for all $j \neq i$.

Let $P(D)$ be the unique smooth compactification of D; $P(D)$ is also a curve over F. Then for each closed point $w \in P(D)$, there is an associated valuation on the field K of rational functions on $P(D)$ (or D) and, hence, (by [6]) a map $\partial_w\colon K_{n+1}^{\mathrm{M}}(K) \to K_n^{\mathrm{M}}(\kappa(w))$. In this situation, Suslin's reciprocity law ([8]) asserts that for any $x \in K_{n+1}^{\mathrm{M}}(K)$,

$$\sum_{w \in P(D)} N_{\kappa(w)/F}\, \partial_w x = 0 \quad \text{in } K_n^{\mathrm{M}}(F).$$

(*Note:* Suslin originally proved this modulo torsion, but thanks to Kato's proof in [5] that the norm is well-defined, Suslin's proof gives this result. See [10] for details.)

Now we have $g_1, \ldots, g_{n+1} \in K^*$, so $\{g_1, \ldots, g_{n+1}\} \in K^M_{n+1}(K)$ and Suslin's reciprocity law says that

$$\sum_{w \in P(D)} N_{\kappa(w)/F}\, \partial_w\{g_1, \ldots, g_{n+1}\} = 0.$$

If $w \in P(D) - D$, i.e., w is a 'point at infinity', then one of the g_i must have $g_i(w) = 1$. (If not, the map $D \to (\mathbf{P}^1 - \{1\})^{n+1}$ would not be proper.) And no matter what the other functions g_j do at w, Milnor's definition of ∂_w shows that $\partial_w\{g_1, \ldots, g_{n+1}\} = 0$ if some $g_i(w) = 1$. So we can write the sum without the points at infinity:

$$\sum_{w \in D} N_{\kappa(w)/F}\, \partial_w\{g_1, \ldots, g_{n+1}\} = 0.$$

For $w \in D$, Milnor's definition of $\partial_w\{g_1, \ldots, g_{n+1}\}$ shows that it's 0 if the g_i are all holomorphic and nonzero at w. And if some g_i does have a zero or pole at $w \in D$, then all the others are holomorphic and nonzero, so Milnor's definition gives

$$\begin{aligned}\partial_w\{g_1, \ldots, g_{n+1}\} &= (-1)^{i-1} \operatorname{ord}_w g_i \{g_1(w), \ldots, \widehat{g_i(w)}, \ldots, g_{n+1}(w)\} \\ &\in K^M_n(\kappa(w)).\end{aligned}$$

This implies that

$$\sum_{w \in D} N_{\kappa(w)/F}\, \partial_w\{g_1, \ldots, g_{n+1}\}$$

is the element of $K^M_n(F)$ defined by the boundary of D in $c^n(F, n)$. Suslin's reciprocity law asserts that this element is 0. So the map $\mathrm{CH}^n(F, n) \to K^M_n(F)$ is well-defined.

4. The Composition $\mathrm{CH}^n(F, n) \to K^M_n(F) \to \mathrm{CH}^n(F, n)$

From our definitions it is clear that the compositions $K^M_n(F) \to \mathrm{CH}^n(F, n) \to K^M_n(F)$ is the identity. We now show that the composition $\mathrm{CH}^n(F, n) \to K^M_n(F) \to \mathrm{CH}^n(F, n)$ is the identity. That is, for a closed point $p \in (\mathbf{P}^1_F - \{0, 1, \infty\})^n$, one has to show that

$$[p] = [N_{\kappa(p)/F}(p)] \quad \text{in } \mathrm{CH}^n(F, n),$$

where I am using an obvious shorthand (the norm gives an element of $K^M_n(F)$, hence (after a choice) a sum of F-rational points in $(\mathbf{P}^1_F - \{0, 1, \infty\})^{n+1}$). Of course, this is trivial if p is itself F-rational, hence in particular, if F is algebraically closed. In general, it will suffice to show that any element of $\mathrm{CH}^n(F, n)$ is equivalent to a sum of F-rational points.

Here is the proof for $\mathrm{CH}^1(F, 1)$. We are given $p \in \mathbf{P}^1_F - \{0, 1, \infty\}$ and, hence, a generator $x \in \kappa(p) - \{0, 1\}$. Let

$$\pi(x) = x^d - a_{d-1}x^{d-1} + \cdots + (-1)^d a_0$$

be the monic irreducible polynomial of x over F; clearly $a_0 \in F^*$ is the classical norm of x. Define a polynomial in two variables by

$$p(x, y) = \pi(x) - (x-1)^{d-1}(x - a_0)y.$$

Then $p(x, y) = 0$ defines a curve in $(\mathbf{P}^1 - \{1\})^2$, in fact an element of $c^2(F, 2)$. The curve intersects $y = 0$ only in the given closed point p and $y = \infty$ only at a_0. Furthermore, as a polynomial in x, $p(x, y)$ has leading coefficient $1 - y$ and constant term $(-1)^d a_0(1 - y)$, so it doesn't intersect $x = \infty$ or $x = 0$ at all. (Keep in mind that we are working in $(\mathbf{P}^1 - \{1\})^2$.) So

$$[p] = [a_0] = [N_{\kappa(p)/F}(p)] \quad \text{in } \mathrm{CH}^1(F, 1),$$

as desired.

The proof for $\mathrm{CH}^n(F, n)$ is as follows, for an infinite field F. (The proof for a finite field is given in the next section.) Let S_k be the set of points $p \in (\mathbf{P}^1_F - \{0, 1, \infty\})^n$ such that the residue field $\kappa(p)$ is generated over F by the first k coordinates of p; that is, such that $\kappa(p) = F(x_1, \ldots, x_k)$. Then $S_0 \subset \cdots \subset S_n$, S_0 is the set of F-rational points in $(\mathbf{P}^1_F - \{0, 1, \infty\})^n$, and S_n is the set of all closed points in the scheme $(\mathbf{P}^1_F - \{0, 1, \infty\})^n$.

LEMMA 2. *If $p \in S_k$, then p is equivalent in $CH^n(F, n)$ to a sum of points in S_{k-1}.*

(Applying Lemma 2 repeatedly, we find that every element of $\mathrm{CH}^n(F, n)$ is equivalent to a sum of F-rational points, as desired.)

Proof. In what follows, I will consistently write $x_1, \ldots, x_n$ to denote the 'coordinates' of p, so that $x_1, \ldots, x_n \in \kappa(p) - \{0, 1\}$. I will use $t_1, \ldots, t_n$ as variables.

Every element of S_k can be defined by n polynomial equations over F of the form:

$$\begin{aligned} p_1(t_1) &= 0 \\ &\vdots \\ p_k(t_1, \ldots, t_k) &= 0 \\ p_{k+1}(t_1, \ldots, t_k) &= t_{k+1} \\ &\vdots \\ p_n(t_1, \ldots, t_k) &= t_n. \end{aligned}$$

In fact, for $i = 1, \ldots, k$ we can choose p_i so that $p_i(x_1, \ldots, x_{i-1}, t_i)$ is the monic irreducible polynomial for x_i over $F(x_1, \ldots, x_{i-1})$. (Of course, each element of $F(x_1, \ldots, x_{x-1})$ can be written as a polynomial over F in $x_1, \ldots, x_{i-1}$.) In particular, this is true for p_k; let d be the degree of x_k over $F(x_1, \ldots, x_{k-1})$, i.e., the degree of p_k as a polynomial in t_k. If $d = 1$ then the lemma is true; so we assume $d \geqslant 2$. And for $i = k+1, \ldots, n$, we choose p_i so that $p_i(x_1, \ldots, x_k) = x_i$. By the division algorithm for polynomials in one variable over the field $F(x_1, \ldots, x_{k-1})$, we can assume that $p_{k+1}, \ldots, p_n$ have t_k-degree less than d.

The idea is that we can replace p by a point in S_{k-1}, in which x_k is replaced by the classical norm $N_{F(x_1, \ldots, x_k)/F(x_1, \ldots, x_{k-1})}(x_k)$, along with points in S_k for which the degree of x_k over $F(x_1, \dots, x_{k-1})$ is less than d. If this works, then Lemma 2 is proved, by induction on d. (For brevity, I will write $N(x_k)$ instead of $N_{F(x_1,\dots,x_k)/F(x_1,\dots,x_{k-1})}(x_k)$.)

Let $(-1)^d a_0(t_1, \dots, t_{k-1})$ be the constant term of $p_k(t_1, \dots, t_k)$, viewed as a polynomial in t_k. Define

$$q(t_1, \dots, t_k, u) = p_k(t_1, \dots, t_k) - (t_k - 1)^{d-1}(t_k - a_0(t_1, \dots, t_{k-1}))u.$$

Then consider the curve C in $(\mathbf{P}^1 - \{1\})^{n+1}$ defined by the following n equations.

$$\begin{aligned} p_1(t_1) &= 0 \\ &\vdots \\ p_{k-1}(t_1, \dots, t_{k-1}) &= 0 \\ q(t_1, \dots, t_k, u) &= 0 \\ p_{k+1}(t_1, \dots, t_k) &= t_{k+1} \\ &\vdots \\ p_n(t_1, \dots, t_k) &= t_n \end{aligned}$$

We now describe the intersections of this curve with the codimension-1 faces of the $(n+1)$-cube. By definition of $p_1, \dots, p_{k-1}$, the curve C does not intersect $t_1 = 0$, $t_1 = \infty, \dots, t_{k+1} = 0$, or $t_{k+1} = \infty$. (Recall that $x_1, \dots, x_n \in \kappa(p) - \{0\}$.) As a polynomial in t_k, $q(t_1, \dots, t_k, u)$ has leading coefficient $1 - u$ and constant term $(-1)^d a_0(t_1, \dots, t_{k-1})(1-u)$. Also, we know that $a_0(x_1, \dots, x_{k-1}) \neq 0$ in $\kappa(p)$, since $p_k(x_1, \dots, x_{k-1}, t_k)$ is the irreducible polynomial for x_k over $F(x_1, \dots, x_{k-1})$ and $x_k \neq 0$. So C doesn't intersect $t_k = 0$ or $t_k = \infty$. Finally, since $t_1, \dots, t_k$ are never equal to ∞ on C, the equations involving $t_{k+1}, \dots, t_n$ show that C doesn't intersect $t_{k+1} = \infty, \dots,$ or $t_n = \infty$.

So C can only intersect the faces $u = 0$, $u = \infty$, $t_{k+1} = 0, \dots,$ and $t_n = 0$. Furthermore, it intersects each codimension-1 face in the correct dimension. In fact, it intersects $u = 0$ only at the given point p, $u = \infty$ only at the point $(x_1, \dots, x_{k-1}, N(x_k), \dots, p_i(x_1, \dots, x_{k-1}, N(x_k)), \dots)$, and intersects $t_i = 0$ $(k+1 \leqslant i \leqslant n)$ only in points because $x_i \neq 0$ in $\kappa(p)$. So if C meets no codimension-2 face of the cube, then C defines an element of $c^n(F, n+1)$, and we can conclude that the 'boundary' of C in $c^n(F, n)$ represents 0 in $\mathrm{CH}^n(F, n)$. This would mean that the given point $p \in S_k$ is equivalent in $\mathrm{CH}^n(F, n)$ to the sum of

$$(x_1, \dots, x_{k-1}, N(x_k), \dots, p_i(x_1, \dots, x_{k-1}, N(x_k)), \dots) \in S_{k-1}$$

and points in S_k with the degree of the new x_k over $F(x_1, \dots, x_{k-1})$ smaller than for p. (For this we have to observe that p_i, for $k+1 \leqslant i \leqslant n$, has t_k-degree less than the t_k-degree of p_k. Also, the equation $q(t_1, \dots, t_k, u) = 0$ is of degree 1 in u, so it can be solved for u as a rational function (over F) in $x_1, \dots, x_{k-1}$, and the new x_k.)

The proof would thus be finished, except that C may in fact meet some codimension-2 face of the cube. The rest of the proof consists of using general-position arguments to avoid this problem; this is why we require here that F be an infinite field. (Finite fields do not have enough elements for general position to work well.)

Here $C \cap \{u = 0\}$ ($=$our original point p) intersects no other face, so the only trouble occurs when C intersects two of the faces $u = \infty, t_{k+1} = 0, \ldots, t_n = 0$ in the same point. That is, there is trouble (1) when $p_i(x_1, \ldots, x_{k-1}, t_k)$ and $p_j(x_1, \ldots, x_{k-1}, t_k)$ (for some $i \neq j, i, j \in \{k+1, \ldots, n\}$) are not relatively prime as polynomials in t_k over the field $F(x_1, \ldots, x_{k-1})$, or (2) when $p_i(x_1, \ldots, x_{k-1}, N(x_k)) = 0$ for some $i \in \{k+1, \ldots, n\}$.

We first show that every point $p \in S_k$ is equivalent to a sum of points in S_k, with the same degree d, for which the problem (1) does not occur.

First, we use a version of the earlier proof that $[a] + [b] = [ab]$ in $\mathrm{CH}^1(F, 1)$ to show that p is equivalent in $\mathrm{CH}^n(F, n)$ to a sum of points of the same form for which each $p_i(x_1, \ldots, x_{k-1}, t_k)$ $(k+1 \leqslant i \leqslant n)$ is either monic and irreducible as a polynomial in t_k, or constant as a polynomial in t_k.

Then problem (1) only comes up if two (or more) of the p_i's, $k+1 \leqslant i \leqslant n$, are equal and of degree > 0. In this case, we imitate the proof that $\{a, a\} = \{-1, a\}$ in Milnor K-theory, using the Chow groups instead. Namely, by a version of the Steinberg relation $(a, 1-a) = 0$ in $\mathrm{CH}^2(F, 2)$ which I proved earlier, we deduce that

$$(a, a) = \left(a \cdot \frac{1 - 1/a}{1 - a}, a\right) = (-1, a).$$

Applying this result repeatedly, we deduce that p is equivalent to a sum of points of the same form for which problem (1) does not occur.

For problem (2), it seems natural to change x_k appropriately. To be precise, modulo S_{k-1} we can replace x_k by cx_k, for all but finitely many $c \in F^*$. I claim that there are only finitely many c for which problem (2) arises; and this completes the proof for an infnite field F.

The simplest situation that shows how one can replace x_k by cx_k and simpler terms, is that if x is a point in $\mathbf{P}^1_F - \{0, 1, \infty\}$ of degree d (i.e, the residue field of x has degree d over F), and $c \in F^*$, then $(cx) = (c^d) + (x)$ in $\mathrm{CH}^1(F, 1)$. This follows from our earlier results, but I won't write out the proof, since we need a more general version in the next paragraph.

amely, we consider the following curve in $(\mathbf{P}^1 - \{1\})^{n+1}$.

$$\begin{aligned} p_1(t_1) &= 0 \\ &\vdots \\ p_{k-1}(t_1, \ldots, t_{k-1}) &= 0 \\ (t_k - c^d)p_k(t_1, \ldots, t_k) - c^d(t_k - 1)p_k(t_1, \ldots, t_{k-1}, t_k/c)u &= 0 \\ p_{k+1}(t_1, \ldots, t_k) &= t_{k+1} \\ &\vdots \\ p_n(t_1, \ldots, t_k) &= t_n \end{aligned}$$

Here we know that for $k+1 \leqslant i \leqslant n$,

$$p_i(x_1, \dots, x_k) = x_i \neq 0;$$

so, for all but finitely many c,

$$p_i(x_1, \dots, x_{k-1}, cx_k) \neq 0 \quad \text{and} \quad p_i(x_1, \dots, x_{k-1}, c^d) \neq 0.$$

This means that for all but finitely many c, this curve is in $c^n(F, n+1)$. For such c, we deduce that $p \in c^n(F, n)$ is equivalent (in $\mathrm{CH}^n(F, n)$) to the sum of $(x_1, \dots, x_{k-1}, cx_k, \dots, p_i(x_1, \dots, cx_k), \dots)$, a point in S_{k-1}, and points in S_k with smaller d.

We now verify that the significant point here, with x_k replaced by cx_k, will (for almost all c) solve problem (2). (Note that problem (1) remains solved, since the polynomials $p_{k+1}, \dots, p_n$ are unchanged.) Here (for $k+1 \leqslant i \leqslant n$) $N(x_k)$ is replaced by $N(cx_k) = c^d N(x_k)$, so problem (2) occurs when

$$p_i(x_1, \dots, x_{k-1}, c^d N(x_k)) = 0.$$

Now $d \geqslant 1$, $N(x_k) \neq 0$, and the polynomial $p_i(x_1, \dots, x_{k-1}, t_k)$ is not identically zero, since $p_i(x_1, \dots, x_k) = x_i \neq 0$ in $\kappa(p)$. So there are only finitely many $c \in F^*$ for which $p_i(x_1, \dots, x_{k-1}, c^d N(x_k)) = 0$, i.e., for which problem (2) arises.

Therefore, for F infinite, there exists a $c \in F^*$ for which problems (1) and (2) do not arise. $\square$

5. Finite Fields

For F finite, $\kappa(p)$ will also be finite, so the multiplicative group $\kappa(p)^*$ is cyclic; let z be a generator. Then the given point $p = (x_1, \dots, x_n)$, with $x_i \in \kappa(p)^*$, can be written as $(z^{i_1}, \dots, z^{i_n})$. Let $\pi(u)$ be the irreducible polynomial for z over F; then p is the projection of a point in $(\mathbf{P}_F^1 - 1)^{n+1}$ defined by the following equations.

$$\begin{aligned} \pi(u) &= 0 \\ t_1 &= u^{i_1} \\ &\vdots \\ t_n &= u^{i_n} \end{aligned}$$

(We project this point into $(\mathbf{P}^1 - \{1\})^n$ by throwing out the u-coordinate.)

Then we can consider a curve in $(\mathbf{P}^1 - \{1\})^{n+1}$ which is itself the projection of the following curve in $(\mathbf{P}^1 - \{1\})^{n+2}$. Here we define a_0 to be the norm $N_{\kappa(p)/F}(z)$, so that $\pi(u)$ has constant term $(-1)^d a_0$.

$$\begin{aligned} \pi(u) - (u - a_0)(u-1)^{d-1} v &= 0 \\ t_1 &= u^{i_1} \\ &\vdots \\ t_n &= u^{i_n} \end{aligned}$$

(This curve is projected into $(\mathbf{P}^1 - \{1\})^{n+1}$ by ignoring u.)

The curve in $(\mathbf{P}^1 - \{1\})^{n+1}$ intersects the faces of the $(n+1)$-cube only for $v = 0$, giving the original point p, and for $v = \infty$, giving $(a_0^{i_1}, \ldots, a_0^{i_n})$ (or 0, if one of the powers of a_0 equals 1). This shows that p is equivalent in $\mathrm{CH}^n(F, n)$ to an F-rational point. This completes the proof for finite fields F.

I should mention that the Milnor K-groups $K_n^{\mathrm{M}}F$ of a finite field F are actually 0 for $n \geqslant 2$ by [6], and so the groups $\mathrm{CH}^n(F, n)$ are also 0 in this case.

6. Comments

The difficulties in the last part of the proof (proving that the composition $\mathrm{CH}^n(F, n) \to K_n^{\mathrm{M}}(F) \to \mathrm{CH}^n(F, n)$ is the identity) result from the complexity of the definition of the norm map in Milnor K-theory. The norm map on K_1^{M} has a well-known explicit description, but Milnor and Bass and Tate were only able to construct the general norm map by an inductive procedure similar to my argument above.

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