Log canonical pairs with conjecturally minimal volume

Louis Esser and Burt Totaro

Consider the problem of finding a complex projective log canonical pair (X, B) with B a nonzero reduced divisor and K_X+B ample such that the volume of K_X+B is as small as possible. This problem arises naturally in attempts to classify stable varieties of general type [13, Remark 7.10]. We know that there is some positive lower bound for the volume in each dimension, by Hacon-M$^{\text{c}}$Kernan-Xu [8, Theorem 1.6].

V. Alexeev and W. Liu constructed a log canonical pair (X, B) of dimension 2 with B a nonzero reduced divisor and $K_X + B$ ample such that $K_X + B$ has volume 1/462 [1, Theorem 1.4]. J. Liu and V. Shokurov showed that this example is not at all arbitrary: it has the smallest possible volume in dimension 2, under the given conditions [13, Theorem 1.4]. (See also Kollár's example in the broader class of log canonical pairs with standard coefficients, Remark 2.3.)

In this paper, we give a simpler description of Alexeev-Liu's example: it is a non-quasi-smooth hypersurface in a weighted projective space, $X_{42} \subset \mathbf{P}^3(21, 14, 6, 11)$, with B the curve $\{x_3 = 0\} \cap X$ (in coordinates $[x_0, x_1, x_2, x_3]$). (This fits into a remarkable number of classification problems in algebraic geometry for which the extreme case is known or conjectured to be a weighted hypersurface [4, 5].) We generalize that construction to produce a log canonical pair (X, B) of any dimension with B a nonzero reduced divisor such that $K_X + B$ is ample and has extremely small volume. We conjecture that our example has the smallest possible volume of $K_X + B$ in every dimension. The volume is doubly exponentially small in terms of the dimension.

A similar story was worked out earlier in the case $B = 0$, where smaller volumes can occur. Namely, Alexeev and Liu constructed a projective klt surface with ample canonical class and volume 1/48983 [1, Theorem 1.4]. Totaro found that their surface is a non-quasi-smooth hypersurface in a weighted projective space, $X_{438} \subset \mathbf{P}^3(219, 146, 61, 11)$. Generalizing that construction, he produced a klt variety of each dimension with ample canonical class and conjecturally minimal volume [14, Theorem 2.1].

We also develop examples for some related extremal problems. Esser constructed a klt Calabi–Yau variety which conjecturally has the smallest minimal log discrepancy in each dimension [3, Conjecture 4.4]. (In particular, this variety has mld 1/13 in dimension 2, 1/311 in dimension 3, and 1/677785 in dimension 4. In dimension 2, we know that 1/13 is the smallest possible mld [5, Proposition 6.1].) However, the properties of Esser's example were not worked out in all dimensions. We now prove the desired properties of Esser's example (in particular, we determine its mld $1/m$), as Theorem 5.1. Using this example, it follows that the "first gap of global lc thresholds" (in Liu-Shokurov's terminology) is at most the same number $1/m$, meaning that there is a klt Calabi–Yau pair $(X, (1 - \frac{1}{m})S)$ with S an irreducible divisor. We present such a "pair" variant of Esser's example explicitly in Theorem 6.1.

Likewise, Wang and the authors constructed a klt Calabi–Yau variety which conjecturally has the largest index in each dimension [5, Conjecture 7.10]. This variety has index 19 in dimension 2, 493 in dimension 3, and 1201495 in dimension 4. (In dimension 2, we know that 19 is the largest possible index [5, Proposition 6.1].) The numerics in this example are extremely similar to Esser's, and so we can now give clearer proofs of many of its properties (section 7). Nevertheless, the precise formula for the index depends on a conjecture that two explicit numbers are relatively prime (Conjecture 7.4), which holds in dimensions at most 30.

In section 8, we compute asymptotics for the small mld and large index examples that provide additional evidence for their optimality.

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1 Notation

Our examples use *Sylvester's sequence*, defined by $s_0 = 2$ and $s_{j+1} = s_j(s_j - 1) + 1$. The sequence begins $2, 3, 7, 43, 1807, \ldots$. We have $s_{j+1} = s_0 \cdots s_j + 1$, and hence the numbers in Sylvester's sequence are pairwise coprime. The key property of this sequence is that

$$\frac{1}{s_0} + \cdots + \frac{1}{s_j} = 1 - \frac{1}{s_{j+1} - 1}.$$

The sequence s_j grows doubly exponentially, with $s_j > 2^{2^{j-1}}$ for all $j \geq 0$.

For positive integers $a_0, \ldots, a_n$, the weighted projective space $Y = \mathbf{P}^n(a_0, \ldots, a_n)$ means the quotient variety $(A^{n+1} - 0)/G_m$ over $\mathbf{C}$, where the multiplicative group G_m acts by $t(x_0, \ldots, x_n) = (t^{a_0}x_0, \ldots, t^{a_n}x_n)$ [9, section 6]. We say that Y is *well-formed* if $\gcd(a_0, \ldots, \widehat{a_j}, \ldots, a_n) = 1$ for each j, which means that the G_m-action is free in codimension 1. For a well-formed weighted projective space Y and an integer m, $O_Y(m)$ is the reflexive sheaf associated to a Weil divisor. The divisor class $O_Y(m)$ is Cartier if and only if m is a multiple of every weight a_j. Well-formedness of Y ensures that the canonical divisor is given by $K_Y = O_Y(-\sum a_j)$.

Let Y be a well-formed weighted projective space. A closed subvariety X of Y is called *quasi-smooth* if its affine cone in A^{n+1} is smooth outside the origin. In particular, a quasi-smooth subvariety has only cyclic quotient singularities and hence is Kawamata log terminal (klt). (A reference for the singularities of the minimal model program such as klt, plt, lc is [11, Definition 2.8].) Also, X is *well-formed* if Y is well-formed and the codimension of $X \cap Y^{\text{sing}}$ in X is at least 2. (For a well-formed weighted projective space Y, the singular locus of Y corresponds to the locus where the G_m-action is not free.) Iano-Fletcher proved the following sufficient criterion for well-formedness [9, Theorem 6.17].

Proposition 1.1. *As long as the degree d is not equal to any of the weights, every quasi-smooth hypersurface of dimension at least 3 in a well-formed weighted projective space is well-formed.*

For a well-formed normal hypersurface X of degree d in a weighted projective space Y, the canonical divisor is given by $K_X = O_X(d - \sum a_j)$ [14, section 1]. Here X need not be quasi-smooth.

A Weil divisor or more generally a $\mathbf{Q}$-divisor is said to be *ample* if some positive multiple is an ample Cartier divisor. The *volume* of a $\mathbf{Q}$-divisor D on a normal projective variety X is

$$\operatorname{vol}(D) := \lim_{m\to\infty} h^0(X, \lfloor mD \rfloor)/(m^n/n!),$$

where $n = \dim(X)$. (The volume in this generality is discussed in [6].) The volume is equal to the intersection number D^n if D is ample. The volume of an ample Cartier divisor is an integer, but that fails in general for an ample Weil divisor. For example, the volume of $O_Y(1)$ on a well-formed weighted projective space $Y = \mathbf{P}^n(a_0, \ldots, a_n)$ is $1/(a_0 \cdots a_n)$.

A *pair* (X, D) in this paper means a normal variety X with a $\mathbf{Q}$-divisor D such that $K_X + D$ is $\mathbf{Q}$-Cartier. A pair is *Calabi–Yau* if D is effective and $K_X + D$ is $\mathbf{Q}$-linearly equivalent to zero. In that case, the *index* of (X, D) is the smallest positive integer m such that $m(K_X + D) \sim 0$. Here the symbol "$\sim$" denotes linear equivalence. A pair has *standard coefficients* if each coefficient of D is of the form $1 - \frac{1}{b}$ with b a positive integer or ∞. For a klt Calabi–Yau pair (X, D) with standard coefficients and index m, the (global) *index-1 cover* of (X, D) is a projective variety Y with canonical Gorenstein singularities such that the canonical class K_Y is linearly equivalent to zero [11, Example 2.47, Corollary 2.51]. Here (X, D) is the quotient of Y by an action of the cyclic group μ_m such that μ_m acts faithfully on $H^0(Y, K_Y) \cong \mathbf{C}$. (Explicitly, D has coefficient $1 - \frac{1}{b}$ on the image of an irreducible divisor on which the subgroup of μ_m that acts as the identity has order b.)

For a pair (X, D) and a proper birational morphism $\pi\colon Y \to X$ with Y normal, there is a uniquely defined $\mathbf{Q}$-divisor D_Y on Y such that $K_Y + D_Y = \pi^*(K_X + D)$. (This is an equality, not just a linear equivalence: for a positive integer m with $m(K_X + D)$ Cartier, $\pi^*(m(K_X + D)) - mK_Y$, viewed as a reflexive sheaf of rank 1, has a *canonical* rational section s given by pulling back differential forms on the smooth locus, and the divisor of zeros of s is mD_Y.) The *log discrepancy* of (X, D) with respect to an irreducible divisor S on Y is 1 minus the coefficient of S in D_Y. The *minimal log discrepancy* (*mld*) of (X, D) is the infimum of all log discrepancies of (X, D) with respect to all irreducible divisors on all birational models of X. Thus (X, D) is klt if and only if its mld is positive, and a pair with smaller mld can be considered more singular.

Berglund-Hübsch-Krawitz mirror symmetry considers weighted-homogeneous polynomials of the following three basic types, *Fermat*, *loop*, and *chain*, as well as combinations of them in disjoint sets of variables [2, section 2.2]:

$$\begin{aligned}
W_{\text{Fermat}} &= x^b,\\
W_{\text{loop}} &= x_1^{b_1}x_2 + x_2^{b_2}x_3 + \cdots + x_{n-1}^{b_{n-1}}x_n + x_n^{b_n}x_1,\\
W_{\text{chain}} &= x_1^{b_1}x_2 + x_2^{b_2}x_3 + \cdots + x_{n-1}^{b_{n-1}}x_n + x_n^{b_n}.
\end{aligned}$$

The corresponding weighted projective hypersurfaces are quasi-smooth for any positive integers b_i, not all 1.

2 A pair with nonzero boundary and small volume

Theorem 2.1. *For each integer n at least 2, in terms of the Sylvester sequence s_n, let*

$$a_{n+1} \coloneqq \begin{cases} \frac{1}{4}(s_n^2 - s_n + 2) & \text{if } n \text{ is even} \\ \frac{1}{4}(s_n^2 - 3s_n + 4) & \text{if } n \text{ is odd.} \end{cases}$$

Let $d \coloneqq s_{n+1} - 1$, and $a_i \coloneqq d/s_i$ for $0 \leq i \leq n$. Then there is a complex hypersurface X of degree d in $\mathbf{P}^{n+1}(a_0, \ldots, a_{n+1})$ that is well-formed and klt. Let $B \coloneqq \{x_{n+1} = 0\} \cap X$. Then the pair (X, B) is purely log terminal (hence log canonical), B is a nonzero reduced divisor, $K_X + B$ is ample, and the volume of $K_X + B$ is

$$\frac{1}{(s_{n+1} - 1)^{n-1} a_{n+1}},$$

which is asymptotic to $4/s_n^{2n}$. In particular, this is less than $1/2^{2^n}$.

The numerology here is similar but not identical to that of the klt variety with ample canonical class and conjecturally minimal volume. In particular, the latter example involves the same weight a_{n+1} [14, Theorem 2.1]. For comparison, the volume of K_X in that example is asymptotic to $2^{2n+2}/s_n^{4n}$, which is much smaller than the volume of $K_X + B$ above. (Requiring B to be a nonzero reduced divisor forces the volume to be bigger, it seems.)

Explicitly, define the variety X in Theorem 2.1 by the equation, for $n \geq 2$ even:

$$0 = x_0^2 + x_1^3 + \cdots + x_n^{s_n} + x_1 \cdots x_n x_{n+1}^2.$$

For $n \geq 3$ odd, define X by

$$0 = x_0^2 + x_1^3 + \cdots + x_n^{s_n} + x_1 \cdots x_{n-1} x_n^2 x_{n+1}^2.$$

Also, define $B \coloneqq \{x_{n+1} = 0\} \cap X$. Since the number of monomials is equal to the number of variables, any linear combination of these monomials with all coefficients nonzero defines a variety isomorphic to X, by scaling the variables. One can check that the monomials shown are all the monomials of degree d, and hence that an open subset of all hypersurfaces of degree d are isomorphic to the one indicated; but we will not need those facts.

Here X is not quasi-smooth. Note that the weights a_i depend on the dimension n as well as on i. For convenience, we write a_i rather than $a_{n,i}$.

Conjecture 2.2. For each integer n at least 2, the pair in Theorem 2.1 has the smallest volume among all projective lc pairs (X, B) of dimension n with B a nonzero reduced divisor and $K_X + B$ ample.

We know that there is some positive lower bound for the volume in each dimension, by Hacon-M$^{\text{c}}$Kernan-Xu [8, Theorem 1.6].

In dimension 2, our example is $X_{42} \subset \mathbf{P}^3(21, 14, 6, 11)$ and $B = \{x_3 = 0\} \cap X$, with $\text{vol}(K_X + B) = 1/462 \doteq 2.2 \times 10^{-3}$. As discussed in the introduction, this is the smallest possible volume for a projective lc pair (X, B) of dimension 2 with B a nonzero reduced divisor and $K_X + B$ ample, by J. Liu and V. Shokurov [13, Theorem 1.4]. This example was found by V. Alexeev and W. Liu, without the description as a hypersurface [1, Theorem 1.4].

In dimension 3, our example is

$$X_{1806} \subset \mathbf{P}^4(903, 602, 258, 42, 431).$$

For $B = \{x_4 = 0\} \cap X$, we have $\operatorname{vol}(K_X + B) \doteq 7.1 \times 10^{-10}$. In dimension 4, our example is

$$X_{3263442} \subset \mathbf{P}^5(1631721, 1087814, 466206, 75894, 1806, 815861).$$

For $B = \{x_5 = 0\} \cap X$, we have $\operatorname{vol}(K_X + B) \doteq 3.5 \times 10^{-26}$.

Remark 2.3. In the broader class of log canonical pairs (X, B) such that B has standard coefficients with $\lfloor B \rfloor \neq 0$ and $K_X + B$ ample, Kollár conjectured the example of smallest volume [10, Example 5.3.1]: for general hyperplanes $H_0, \ldots, H_{n+1}$ in $\mathbf{P}^n$, let

$$(X, B) = (\mathbf{P}^n, \frac{1}{2}H_0 + \frac{2}{3}H_1 + \cdots + \frac{s_n - 1}{s_n}H_n + H_{n+1}).$$

Here $K_X + B$ has volume $1/(s_{n+1} - 1)^n$, which is asymptotic to $1/s_n^{2n}$. In dimension 2, Kollár showed that this example is indeed optimal with these properties, with volume $1/1764$ [10, Remark 6.2.1].

In high dimensions, Kollár's pair has about $1/4$ of the volume in Theorem 2.1, which is extremely close for such small numbers. That is some evidence for the optimality of Theorem 2.1, in the narrower setting of reduced divisors.

Proof. (Theorem 2.1) The weight a_{n+1} is odd, as we showed in [14, proof of Theorem 2.1] (since our example with $B = 0$ included the same weight a_{n+1} as here). For the reader's convenience, here is the argument: s_n is $7 \pmod 8$ if $n \geq 2$ is even and $3 \pmod 8$ if $n \geq 3$ is odd. This is immediate by induction from the recurrence $s_{n+1} = s_n(s_n - 1) + 1$. It follows that $s_n^2 - s_n + 2$ is $4 \pmod 8$ if $n \geq 2$ is even, and that $s_n^2 - 3s_n + 4$ is $4 \pmod 8$ if $n \geq 3$ is odd. So a_{n+1} is odd in both cases.

Next, we show that the weighted projective space $Y = \mathbf{P}^{n+1}(a_0, \ldots, a_{n+1})$ is well-formed. That is, we have to show that $\gcd(a_0, \ldots, \widehat{a_j}, \ldots, a_{n+1}) = 1$ for each j. We have $s_{n+1} - 1 = s_0 \cdots s_n$ with $s_0, \ldots, s_n$ pairwise coprime, which implies that $\gcd(a_0, \ldots, a_n) = 1$. For the rest, it suffices to show that $\gcd(s_{n+1} - 1, a_{n+1}) = 1$. Here $s_{n+1} - 1 = s_n(s_n - 1)$, so it suffices to show that $\gcd(s_n - 1, a_{n+1}) = 1$ and $\gcd(s_n, a_{n+1}) = 1$. The first statement was shown in [14, proof of Theorem 2.1]. For the second, if a prime number p divides s_n and a_{n+1}, then $p > 2$ since a_{n+1} is odd. If n is even, it follows that $a_{n+1} = \frac{1}{4}(s_n^2 - s_n + 2) \equiv \frac{1}{2} \pmod p$, not 0, which is a contradiction. If n is odd, we have $a_{n+1} = \frac{1}{4}(s_n^2 - 3s_n + 4) \equiv 1 \pmod p$, not 0, which is a contradiction. That completes the proof that Y is well-formed.

From the equation for X, the only coordinate linear space of Y contained in X is the point $p \coloneqq [0, \ldots, 0, 1]$. Since that has codimension at least 2 in X, X is well-formed. Also, X is quasi-smooth outside p, so it has only quotient singularities outside p, and so X is klt outside p. At the point p, in coordinates $x_{n+1} = 1$, X is defined by the equation

$$0 = x_0^2 + x_1^3 + \cdots + x_n^{s_n} + x_1 \cdots x_n$$

for n even, resp.

$$0 = x_0^2 + x_1^3 + \cdots + x_n^{s_n} + x_1 \cdots x_{n-1} x_n^2$$

for n odd (in $A^{n+1}/\mu_{a_{n+1}}$). The same hypersurfaces in A^{n+1} appeared in our klt variety of conjecturally minimal volume, and we showed that these hypersurfaces in A^{n+1} have canonical singularities (hence are klt) [14, proof of Theorem 2.1]. The klt, plt, and lc properties are preserved upon dividing by a finite group action that is free in codimension 1 [11, Corollary 2.43]. Therefore, X is klt.

The divisor $B = \{x_{n+1} = 0\} \cap X$ is quasi-smooth and misses the point p. So the pair (X, B) is étale-locally (near each point of B) the quotient of (A^n, A^{n-1}) by a finite group action, free in codimension 1. Since X is klt outside B, it follows that the pair (X, B) is plt (hence lc).

Let $d = s_{n+1} - 1$ be the degree of the hypersurface X. Since X is well-formed and normal, we have

$$\begin{aligned} K_X &= O_X(d - \sum a_j) \\ &= O_X\left((s_{n+1}-1)\left(1 - \frac{1}{s_0} - \cdots - \frac{1}{s_n}\right) - a_{n+1}\right) \\ &= O_X(1 - a_{n+1}). \end{aligned}$$

The divisor B on X is linearly equivalent to $O_X(a_{n+1})$, and so $K_X + B \sim O_X(1)$. It follows that

$$\begin{aligned} \mathrm{vol}(K_X + B) &= \mathrm{vol}(O_X(1)) \\ &= \frac{d}{a_0 \cdots a_{n+1}} \\ &= \frac{(s_{n+1}-1)s_0 \cdots s_n}{(s_{n+1}-1)^{n+1} a_{n+1}} \\ &= \frac{1}{(s_{n+1}-1)^{n-1} a_{n+1}}. \end{aligned}$$

Here s_{n+1} is asymptotic to s_n^2 (with error term on the order of s_n) as n goes to infinity, and $a_{n+1} \sim s_n^2/4$; so the volume of $K_X + B$ is asymptotic to $4/s_n^{2n}$. □

3 Esser's klt Calabi–Yau variety with small mld

Esser constructed a klt Calabi–Yau variety which conjecturally has the smallest mld (roughly $1/2^{2^n}$) in each dimension n [3, Conjecture 4.4]. (For example, this variety has mld $1/13$ in dimension 2, $1/311$ in dimension 3, and mld $1/677785$ in dimension 4.) We know that there is some positive lower bound for this problem in each dimension, by Hacon-M$^{\mathrm{c}}$Kernan-Xu [7, Theorem 1.5].

In version 1 of [3] on the arXiv, the example was worked out completely only in dimensions at most 18. In this paper, we prove the desired properties of Esser's example in all dimensions, in particular confirming Esser's conjectured value for its mld (Theorem 5.1). By Lemma 8.1, this mld is within a constant factor of the conjecturally smallest mld in the broader setting of klt Calabi–Yau pairs with standard coefficients.

We were led to this analysis by constructing a related example among pairs, although in this paper we will prove the properties of Esser's example first. The example among pairs (Theorem 6.1) is a klt Calabi–Yau pair $(X, (1 - \frac{1}{m})S)$ of each dimension $n \geq 2$ with S an irreducible divisor. The number $1/m$ is the same as the mld of Esser's example, and in fact $(X, (1 - \frac{1}{m})S)$ is crepant-birational to Esser's Calabi–Yau variety V/μ_m (Lemma 6.3). (That is, S is the divisor that shows that V/μ_m has mld $1/m$.)

In each dimension $n \geq 2$, Esser's example is the quotient of a hypersurface V in a weighted projective space $\mathbf{P}^{n+1}(a_0, \ldots, a_{n+1})$ by an action of a cyclic group [3, section 4]. (The order of the cyclic group should be the number m defined below, but that will be proved later, in Theorem 5.1. We define the μ_m-action explicitly in the proof of Lemma 6.3.) In odd dimension $n = 2r + 1$, the equation of V has the form

$$0 = x_0^{b_0} x_{2r+2} + x_1^{b_1} x_{2r+1} + \cdots + x_r^{b_r} x_{r+2} + x_{r+1}^{b_{r+1}} x_r + \cdots + x_{2r+1}^{b_{2r+1}} x_0 + x_{2r+2}^{v_{2r+1}} x_{r+1},$$

for exponents b_j and v_{2r+1} defined below. In even dimension $n = 2r$, the equation of V has the form

$$0 = x_0^{b_0} + x_1^{b_1} x_{2r+1} + \cdots + x_r^{b_r} x_{r+2} + x_{r+1}^{b_{r+1}} x_r + \cdots + x_{2r}^{b_{2r}} x_1 + x_{2r+1}^{v_{2r}} x_{r+1}.$$

To define the exponents, we'll use the following notation. For short, given integers $b_{i_1}, \ldots, b_{i_k}$, write $B_{i_1 \cdots i_k}$ for the alternating sum

$$B_{i_1 \cdots i_k} \coloneqq b_{i_1} \cdots b_{i_k} - b_{i_1} \cdots b_{i_{k-1}} + \cdots + (-1)^{k-1} b_{i_1} + (-1)^k.$$

We note for future reference the following symmetry property for alternating sums of B's:

Lemma 3.1. *For any integers $b_{i_1}, \ldots, b_{i_k}$,*

$$B_{i_1 \cdots i_k} - B_{i_2 \cdots i_k} + \cdots + (-1)^{k-1} B_{i_k} = B_{i_k \cdots i_1} - B_{i_{k-1} \cdots i_1} + \cdots + (-1)^{k-1} B_{i_1}.$$

Esser defines the exponents $b_0, \ldots, b_n$ as follows, for $n = 2r + 1$ or $n = 2r$, with $r \geq 1$. For $0 \leq i \leq r$, let $b_i \coloneqq s_i$, the Sylvester number. Then define all but one of the remaining exponents inductively by

$$\begin{aligned} b_{r+i} &\coloneqq 1 + (b_{r+1-i} - 1)^2 B_{r+1, r, r+2, r-1, \ldots, r-1+i, r+2-i} \\ &= 1 + b_0 \cdots b_{r-i} (b_{r+1-i} - 1) B_{r+1, r, r+2, r-1, \ldots, r-1+i, r+2-i} \end{aligned}$$

for $1 \leq i \leq r + 1$ when $n = 2r + 1$, and for $1 \leq i \leq r$ when $n = 2r$. (A symbol B with empty subscript is understood to be 1, and so $b_{r+1} = 1 + (b_r - 1)^2$.) *These exponents b_i, as well as the other numbers defined below, depend on the dimension n as well as on i.* For convenience, we write b_i rather than $b_{n,i}$. Finally, the last exponent is given by

$$\begin{cases} v_{2r+1} \coloneqq B_{r+1, r, r+2, r-1, \ldots, 2r+1, 0} - B_{r, r+2, r-1, \ldots, 2r+1, 0} + \cdots - B_0 & \text{if } n = 2r + 1, \\ v_{2r} \coloneqq 2(B_{r+1, r, r+2, r-1, \ldots, 2r, 1} - B_{r, r+2, r-1, \ldots, 2r, 1} + \cdots - B_1) + 1 & \text{if } n = 2r. \end{cases}$$

The weights $a_0, \ldots, a_{n+1}$ and degree D of V are determined uniquely by the equation of V, given the requirement that $\gcd(a_0, \ldots, a_{n+1}) = 1$. Esser shows that V is a well-formed quasi-smooth Calabi–Yau hypersurface; we write out the details in section 5. In version 1 of [3, Section 4] on the arXiv, Esser *conjectured* that the last weight a_{2r+2} is equal to 1, and he verified this in dimensions at most 18. Using that assumption, he shows that there is an action of the cyclic group μ_m on V, where $m = m_n$ is given by

$$m := \begin{cases} m_{2r+1} = B_{0,2r+1,1,2r,\dots,r,r+1} & \text{if } n = 2r+1, \\ m_{2r} = B_{1,2r,2,2r-1,\dots,r,r+1} & \text{if } n = 2r. \end{cases}$$

The number $m = m_n$ is doubly exponential in the dimension n; in particular, $m_n > 2^{2^n}$ for $n > 2$. Esser's conjecture would also imply that the degree D of V is given by $D = u_{2r+1}$ if $n = 2r+1$ and $D = 2u_{2r}$ if $n = 2r$, where

$$u := \begin{cases} u_{2r+1} = B_{r+1,r,r+2,r-1,\dots,2r+1,0} & \text{if } n = 2r+1, \\ u_{2r} = B_{r+1,r,r+2,r-1,\dots,2r,1} & \text{if } n = 2r. \end{cases}$$

(A connection between dimensions $2r$ and $2r+1$ is that the exponent b_{2r+1} for $n = 2r+1$ satisfies $b_{2r+1} = u_{2r} + 1$.)

Esser's conjecture would imply that V/μ_m is a klt Calabi–Yau variety with mld $1/m$; so this was initially proved in dimensions $n \leq 18$. We prove the conjecture in all dimensions in Theorem 5.1, using the product formulas proved in the next section.

In dimension 2, Esser's hypersurface is:

$$\begin{aligned} V &= V_{22} \subset \mathbf{P}^3(11, 7, 3, 1), \\ 0 &= x_0^2 + x_1^3 x_3 + x_2^5 x_1 + x_3^{19} x_2. \end{aligned}$$

Here V/μ_{13} is a klt Calabi–Yau surface of mld $1/13$, which is the smallest possible [5, Proposition 6.1]. In dimension 3, we have:

$$\begin{aligned} V &= V_{191} \subset \mathbf{P}^4(95, 61, 26, 8, 1), \\ 0 &= x_0^2 x_4 + x_1^3 x_3 + x_2^5 x_1 + x_3^{12} x_0 + x_4^{165} x_2, \end{aligned}$$

with an action of the cyclic group of order 311. In dimension 4, we have:

$$\begin{aligned} V &= V_{925594} \subset \mathbf{P}^5(462797, 308531, 132129, 21445, 691, 1), \\ 0 &= x_0^2 + x_1^3 x_5 + x_2^7 x_4 + x_3^{37} x_2 + x_4^{893} x_1 + x_5^{904149} x_3, \end{aligned}$$

with an action of the cyclic group of order 677785.

4 Product formulas for the small-mld example

Here we prove some product formulas which imply the desired properties of the klt Calabi–Yau variety with small mld [3, equation (6)]. The formulas are also useful for the klt Calabi–Yau variety with large index, because the equations for the two varieties are similar.

Fix a positive integer r. Let $b_0, \dots, b_{2r+1}$ be the numbers defined in section 3. (These are all but the last exponent of Esser's hypersurface V of dimension $2r+1$.) In the notation of that section, define the following related numbers for $0 \leq k \leq r+1$:

$$\begin{aligned} g_k &:= B_{k,2r+1-k,\dots,r,r+1} \\ t_k &:= B_{r+1,r,\dots,2r+1-k,k} \\ w_k &:= (s_k - 1)[B_{r+1,r,\dots,2r+1-k,k} - B_{r,\dots,2r+1-k,k} + \cdots - B_k + 1] - 1. \end{aligned}$$

Proposition 4.1. *For $0 \le k \le r+1$, we have*

$$(s_k - 1)g_k t_k - 1 = b_k \cdots b_{2r+1-k} w_k.$$

For $k = 0$, we'll see in section 5 that this proposition directly implies Esser's conjecture for the small-mld example of dimension $n = 2r+1$; in the notation of section 3, the $k = 0$ statement reads $m_{2r+1}u_{2r+1} - 1 = b_0 \cdots b_{2r+1}v_{2r+1}$. Likewise, for $k = 1$, the proposition reads $2m_{2r}u_{2r} - 1 = b_1 \cdots b_{2r}v_{2r}$ in the notation of section 3; this will imply Esser's conjecture for dimension $2r$. Generalizing these product formulas to Proposition 4.1 makes an inductive proof possible.

Proof. (Proposition 4.1) We prove this by descending induction on $0 \le k \le r+1$. For $k = r+1$, both sides of the equation are equal to $(s_{r+1} - 1) - 1$. Next, assume that $0 \le k \le r$ and the equation holds for $k+1$, meaning that

$$(s_{k+1} - 1)g_{k+1}t_{k+1} - 1 = b_{k+1} \cdots b_{2r-k} w_{k+1}. \tag{4.1}$$

We will prove it for k.

We prove the following lemma at the same time as Proposition 4.1.

Lemma 4.2. *For $0 \le k \le r$,*

$$g_k - s_k g_{k+1} = (s_k - 1)b_{k+1} \cdots b_{2r-k} w_{k+1}.$$

Proof. Given that Proposition 4.1 holds for $k+1$, we prove this lemma for k. By definition of g_k, we have

$$g_k = b_k b_{2r+1-k} g_{k+1} - (b_k - 1).$$

So, noting that $b_k = s_k$ (the Sylvester number), we have

$$g_k - s_k g_{k+1} = s_k(b_{2r+1-k} - 1)g_{k+1} - (s_k - 1).$$

By definition of b_{2r+1-k}, we have $b_{2r+1-k} - 1 = (s_k - 1)^2 t_{k+1}$. So

$$\begin{aligned} g_k - s_k g_{k+1} &= s_k(s_k - 1)^2 g_{k+1}t_{k+1} - (s_k - 1) \\ &= (s_k - 1)[(s_{k+1} - 1)g_{k+1}t_{k+1} - 1] \\ &= (s_k - 1)b_{k+1} \cdots b_{2r-k} w_{k+1}, \end{aligned}$$

using that Proposition 4.1 holds for $k+1$ (equation 4.1). That proves the lemma for k. □

We continue the proof of Proposition 4.1 for k, using that it holds for $k+1$. By definition of t_k, we have

$$t_k - t_{k+1} = (s_k - 1)b_{k+1} \cdots b_{2r+1-k}.$$

Using Lemma 4.2 for the given number k, it follows that

$$\begin{aligned} (s_k - 1)g_k t_k - 1 &= (s_k - 1)[s_k g_{k+1} + (s_k - 1)b_{k+1} \cdots b_{2r-k} w_{k+1}] \\ &\quad \cdot [t_{k+1} + (s_k - 1)b_{k+1} \cdots b_{2r+1-k}] - 1 \\ &= (s_{k+1} - 1)g_{k+1}t_{k+1} - 1 + (s_k - 1)^2 b_k \cdots b_{2r+1-k} g_{k+1} \\ &\quad + (s_k - 1)^2 b_{k+1} \cdots b_{2r-k} t_{k+1} w_{k+1} + (s_k - 1)^3 (b_{k+1} \cdots b_{2r-k})^2 b_{2r+1-k} w_{k+1}. \end{aligned}$$

Since Proposition 4.1 holds for $k+1$ (equation 4.1), it follows that

$$(s_k-1)g_kt_k-1 = b_{k+1}\cdots b_{2r-k}[(s_k-1)^2b_kb_{2r+1-k}g_{k+1}+w_{k+1}+(s_k-1)^2t_{k+1}w_{k+1} \\ +(s_k-1)^3b_{k+1}\cdots b_{2r+1-k}w_{k+1}].$$

By definition, $b_{2r+1-k} = 1+(s_k-1)^2t_{k+1}$. So we can combine the second and third terms in the bracket above into a multiple of b_{2r+1-k}:

$$(s_k-1)g_kt_k-1 = b_{k+1}\cdots b_{2r+1-k}[(s_k-1)^2b_kg_{k+1}+w_{k+1} \\ +(s_k-1)^3b_{k+1}\cdots b_{2r-k}w_{k+1}].$$

Therefore, to prove Proposition 4.1 for k (completing the induction), it suffices to show that:

$$b_kw_k = (s_k-1)^2b_kg_{k+1}+w_{k+1}+(s_k-1)^3b_{k+1}\cdots b_{2r-k}w_{k+1}. \tag{4.2}$$

By definition,

$$w_k = (s_k-1)[B_{r+1,r,\ldots,2r+1-k,k}-B_{r,\ldots,2r+1-k,k}+\cdots-B_k+1]-1$$

and

$$w_{k+1} = s_k(s_k-1)[B_{r+1,r,\ldots,2r-k,k+1}-B_{r,\ldots,2r-k,k+1}+\cdots-B_{k+1}+1]-1.$$

Therefore, subtracting term by term and using that $b_k = s_k$, we have

$$b_kw_k-w_{k+1} = s_k(s_k-1)[(b_k-1)b_{2r+1-k}b_{k+1}b_{2r-k}\cdots b_rb_{r+1} \\ -(b_k-1)b_{2r+1-k}b_{k+1}b_{2r-k}\cdots b_r+\cdots+(b_k-1)b_{2r+1-k}-(b_k-1)+1]-(s_k-1)$$

$$= (s_k-1)^2[b_kb_{2r+1-k}b_{k+1}b_{2r-k}\cdots b_rb_{r+1}-b_kb_{2r+1-k}\cdots b_r+\cdots+b_kb_{2r+1-k}-b_k+1]$$
$$= (s_k-1)^2B_{k,2r+1-k,\ldots,r,r+1}$$
$$= (s_k-1)^2g_k.$$

So the left side of (4.2) is

$$b_kw_k = (s_k-1)^2g_k+w_{k+1}.$$

By Lemma 4.2 for the given number k, we can expand g_k here, giving that

$$b_kw_k = (s_k-1)^2b_kg_{k+1}+w_{k+1}+(s_k-1)^3b_{k+1}\cdots b_{2r-k}w_{k+1}.$$

That proves equation 4.2. Thus Proposition 4.1 holds for k given that it holds for $k+1$. The proposition is proved. $\square$

5 Proof of the properties of Esser's example in all dimensions

Prior to this paper, the properties of Esser's klt Calabi–Yau variety with small mld were known completely only in dimensions at most 18. We now prove the desired properties in all dimensions (Theorem 5.1), using the product formulas in Proposition 4.1.

Theorem 5.1. *In each dimension $n \geq 2$, Esser's hypersurface V defined in section 3 has an action of the cyclic group μ_m (for the number $m = m_n$ defined there) such that V/μ_m is a complex klt Calabi–Yau variety with mld $1/m$. In particular, $m_n > 2^{2^n}$ for $n > 2$.*

We also compute the weights and degree of Esser's hypersurface explicitly, in particular proving Esser's conjecture that the last weight a_{n+1} is equal to 1.

Proof. Let r be a positive integer, and let n be $2r$ or $2r+1$. We first prove the basic properties of Esser's hypersurface V, namely that it is a well-formed quasi-smooth Calabi–Yau hypersurface. Here V was defined in section 3 by writing out its equation. That determines the weights $a_0, \ldots, a_{n+1}$ and degree D of V, given the requirement that $\gcd(a_0, \ldots, a_{n+1}) = 1$. By the form of the equation (a loop for n odd, x_0^2 plus a loop for n even), V is quasi-smooth.

The weighted projective space containing V is well-formed: if a prime number p divides all but one weight a_i, then p divides D since there is a monomial not involving x_i. Then, there is either a monomial of the form $x_j^a x_i$ in the equation for V or n is even and $i = 0$. In the former case, we get that p divides a_i too, contradicting that $\gcd(a_0, \ldots, a_{n+1}) = 1$. The same holds in the latter case unless $p = 2$. If $p = 2$ divides all weights except a_0 in the n even case, then $D \equiv 2 \pmod 4$. The remaining exponents $b_1, \ldots, b_{2r}, v_{2r}$ are odd, so this means the weights a_i would have to alternate between $0 \pmod 4$ and $2 \pmod 4$ moving around the loop part of the equation of X. But the loop has odd length, a contradiction. It follows that V is well-formed by Proposition 1.1, together with a look at the equation for V in dimension 2 (section 3).

Lemma 5.2. *The hypersurface V is Calabi–Yau, in the sense that $D = \sum a_j$.*

Proof. Let A be the $(n+2) \times (n+2)$ matrix that encodes the equation of V (with each row specifying the exponents of one monomial in the equation). Define the *charges* $q_0, \ldots, q_{n+1}$ as the sums of the rows of the matrix A^{-1}. Then the degree D of V is the least common denominator of the charges q_i, and the weights of V are given by $a_i = Dq_i$ [3, section 2.3]. To show that V is Calabi–Yau, we have to show that the sum of the charges is 1, that is, that the sum of the entries of A^{-1} is 1. We use the formula for the inverse of a loop matrix [3, Lemma 2.7]:

Lemma 5.3. *Let A be the loop matrix*

$$A = \begin{pmatrix} e_1 & 1 & & \\ & e_2 & \ddots & \\ & & \ddots & 1 \\ 1 & & & e_k \end{pmatrix}$$

with k odd. Then the inverse matrix A^{-1} is

$$\frac{1}{e_1\cdots e_k+(-1)^{k-1}}\begin{pmatrix} e_2\cdots e_k & -e_3\cdots e_k & \cdots & -e_k & 1\\ 1 & e_3\cdots e_k e_1 & \cdots & e_k e_1 & -e_1\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ e_2\cdots e_{k-2} & \cdots & 1 & e_k e_1\cdots e_{k-2} & -e_1\cdots e_{k-2}\\ -e_2\cdots e_{k-1} & \cdots & -e_{k-1} & 1 & e_1\cdots e_{k-1}\end{pmatrix}.$$

For $n=2r+1$, it follows that A^{-1} is an integer matrix divided by $b_0\cdots b_{2r+1}v_{2r+1}+1$. So we want to show that $b_0\cdots b_{2r+1}v_{2r+1}+1$ minus the sum of the entries of that integer matrix is 0. Analyze this difference by collecting all terms that are multiples of $b_{r+1}\cdots b_{2r+1}v_{2r+1}$, then the remaining terms that are multiples of $b_{r+2}\cdots b_{2r+1}v_{2r+1}$, and so on. The result is:

$$\begin{aligned} &b_{r+1}\cdots b_{2r+1}v_{2r+1}\left[b_0\cdots b_r\left(1-\frac{1}{b_0}-\cdots-\frac{1}{b_r}\right)\right]\\ &\qquad-\sum_{i=1}^{r+1} b_{r+1+i}\cdots b_{2r+1}v_{2r+1}[b_0\cdots b_{r-i}(b_{r+1-i}-1)B_{r+1,r,\ldots,r-1+i,r+2-i}]\\ &\qquad\qquad-[B_{r+1,r,\ldots,2r+1,0}-B_{r,\ldots,2r+1,0}+\cdots-B_0+1]+1.\end{aligned}$$

The first term in brackets is 1, by the properties of the Sylvester numbers $b_0,\ldots,b_r$. By definition of b_{r+i}, the expression in brackets in the term indexed by i (for $1\leq i\leq r+1$) is $b_{r+i}-1$. Finally, the last line is $-v_{2r+1}$. Therefore, the sum telescopes to zero. Thus V is Calabi–Yau in the sense that $D=\sum a_j$.

For $n=2r$, the proof is similar. Again, we have to show that the sum of the entries of the matrix A^{-1} is 1. In this case, the equation of V is x_0^2 plus a loop. By Lemma 5.3, A^{-1} is the block matrix $1/2$ followed by $1/(b_1\cdots b_{2r}v_{2r}+1)$ times an integer matrix C. So we need to show that the sum of the entries of C is $(b_1\cdots b_{2r}v_{2r}+1)/2$. Explicitly, $b_1\cdots b_{2r}v_{2r}+1$ minus 2 times the sum of the entries of C (which we want to be zero) is

$$\begin{aligned} &b_{r+1}\cdots b_{2r}v_{2r}\left[b_1\cdots b_r\left(1-2\left(\frac{1}{b_1}-\cdots-\frac{1}{b_r}\right)\right)\right]\\ &\qquad-\sum_{i=1}^{r} b_{r+1+i}\cdots b_{2r}v_{2r}[2b_1\cdots b_{r-i}(b_{r+1-i}-1)B_{r+1,r,\ldots,r-1+i,r+2-i}]\\ &\qquad\qquad-2[B_{r+1,r,\ldots,2r,1}-B_{r,\ldots,2r,1}+\cdots-B_1+1]+1.\end{aligned}$$

Again, the first term in brackets is 1, the expression in brackets in the term indexed by i is $b_{r+i}-1$, and the last line is $-v_{2r}$. So the sum telescopes to zero. Thus V is Calabi–Yau in the sense that $D=\sum a_j$, in even as well as odd dimensions. Lemma 5.2 is proved. $\square$

By Lemma 5.2 plus the preceding results, V is a well-formed quasi-smooth Calabi–Yau hypersurface. More precisely, $K_V=O_V(D-\sum a_j)=O_V$, and so K_V is linearly equivalent to zero. As a result, K_V is Cartier and V is klt, so V is canonical.

The inverse A^{-1} of a loop matrix is written explicitly in Lemma 5.3. We read off that the charge q_{2r+2} is the alternating sum

$$\begin{aligned} q_{2r+2} &= \frac{B_{0,2r+1,\ldots,r,r+1}}{b_0 \cdots b_{2r+1}v_{2r+1}+1} \\ &= \frac{m_{2r+1}}{b_0 \cdots b_{2r+1}v_{2r+1}+1} \\ &= \frac{1}{u_{2r+1}}, \end{aligned}$$

by Proposition 4.1. The degree D of V is the least common multiple of the denominators of the charges q_i; but in this case of a loop matrix, all the charges have the same denominator [5, Lemma 7.1]. Therefore, the degree D of V is equal to u_{2r+1}. So the weights of V are $a_i = u_{2r+1}q_i$. In particular, the weight a_{2r+2} is equal to 1, as we want.

By the loop equation of V, the absolute value of the determinant of A is given by [2, section 3]:

$$|\det(A)| = b_0 \cdots b_{2r+1}v_{2r+1} + 1.$$

The group $\mathrm{Aut}_T(V)$ of *toric automorphisms* of V is the group of automorphisms of V that are given by diagonal matrices in the given coordinates. This is related to the degree D of V by [2, section 3]:

$$|\det(A)| = D|\mathrm{Aut}_T(V)|.$$

By the computation of the degree D above, it follows that $|\mathrm{Aut}_T(V)| = m_{2r+1}$. Moreover, since the equation of V is a loop, the group $\mathrm{Aut}_T(V)$ is cyclic [2, Proposition 2]. So the group of toric automorphisms of V is the cyclic group μ_m with $m = m_{2r+1}$. (We will describe the μ_m-action explicitly in the proof of Lemma 6.3.)

Also, because the equation of V is a loop, the action of μ_m on V is free in codimension 1 [5, Proposition 7.2]. As a result, V/μ_m is a klt Calabi–Yau variety (not a pair).

Esser showed that the mld of V/μ_m is the smallest charge of the BHK mirror of V [3, Theorem 3.1]. (The mirror is defined to be the hypersurface whose equation is associated to the transpose of the matrix A.) One mirror charge is

$$\begin{aligned} q^{\mathsf{T}}_{2r+2} &= \frac{B_{r+1,r,r+2,r-1,\ldots,2r+1,0}}{b_0 \cdots b_{2r+1}v_{2r+1}+1} \\ &= \frac{u_{2r+1}}{b_0 \cdots b_{2r+1}v_{2r+1}+1} \\ &= \frac{1}{m_{2r+1}}. \end{aligned}$$

Since A^{T} is a loop matrix, all the mirror charges have the same denominator, and so $1/m_{2r+1}$ is the smallest mirror charge. Thus $\mathrm{mld}(V/\mu_m) = 1/m$, as we want. (Another way to compute the mld is given in Lemma 6.3: the variety V/μ_m is crepant-birational to the pair $(X, (1-\frac{1}{m})S)$ discussed later.)

It remains to give the analogous argument in even dimensions. Let $n = 2r$. Because the equation of V is of the form x_0^2 plus a loop, we can write out the inverse matrix A^{-1} using the formula for the inverse of a loop matrix, Lemma 5.3. We read off that the charge q_{2r+1} is the alternating sum

$$\begin{aligned} q_{2r+1} &= \frac{B_{1,2r,\ldots,r,r+1}}{b_1 \cdots b_{2r} v_{2r} + 1} \\ &= \frac{m_{2r}}{b_1 \cdots b_{2r} v_{2r} + 1} \\ &= \frac{1}{2u_{2r}}, \end{aligned}$$

by Proposition 4.1. The degree D of V is the least common multiple of the denominators of the charges q_i; so the degree D is a multiple of $2u_{2r}$, say $D = 2u_{2r}\lambda$ for a positive integer λ. The last weight a_{2r+1} is then $Dq_{2r+1} = \lambda$. Using the loop of monomials (starting with $x_{2r+1}^{v_{2r+1}} x_{r+1}$) in the equation of V, it follows that all weights a_i with $i \neq 0$ are also multiples of λ. By the monomial x_0^2 in the equation of V, we have $a_0 = D/2 = u_{2r}\lambda$, which is also a multiple of λ. Since $\gcd(a_0, \ldots, a_{2r+1}) = 1$, it follows that $\lambda = 1$. Thus the degree D of V is equal to $2u_{2r}$, the weights of V are $a_i = 2u_{2r}q_i$, and the weight a_{2r+1} is equal to 1, as we want.

Since the equation of V for $n = 2r$ is x_0^2 plus a loop, the absolute value of the determinant of A is given by [2, section 3]:

$$|\det(A)| = 2(b_1 \cdots b_{2r} v_{2r} + 1).$$

The group of toric automorphisms is related to the degree D of V by [2, section 3]:

$$|\det(A)| = D|\operatorname{Aut}_T(V)|.$$

By the computation of the degree D above, it follows that $|\operatorname{Aut}_T(V)| = 2m_{2r}$. Moreover, since the equation of V is x_0^2 plus a loop, the group $\operatorname{Aut}_T(V)$ is μ_2 times a cyclic group [2, Proposition 3.2]. So the group of toric automorphisms of V is the product group $\mu_2 \times \mu_m$ with $m = m_{2r}$. (We will describe the μ_m-action explicitly in Lemma 6.3. The μ_2-action changes the sign of x_0.)

Also, because the equation of V is x_0^2 plus a loop, the action of μ_m on $V \cap \{x_0 \neq 0\}$ is free in codimension 1 [5, Proposition 7.2]. Also, a toric automorphism that fixes the divisor $V \cap \{x_0 = 0\}$ can be written as $[x_0, \ldots, x_{n+1}] \mapsto [ex_0, x_1, \ldots, x_{n+1}]$ for some $e \in \mathbf{C}^*$. We have $e^2 = 1$, because the automorphism preserves the equation $x_0^2 + x_1^3 x_{2r+1} + \cdots = 0$ of V. Since $m = m_{2r}$ is odd, it follows that no nontrivial element of μ_m fixes this divisor. So μ_m acts freely in codimension 1 on V, and hence V/μ_m is a klt Calabi–Yau variety (not a pair). Given that, Esser showed that $\operatorname{mld}(V/\mu_m) = 1/m$ [3, section 4]. (Another way to compute the mld is given in Lemma 6.3: the variety V/μ_m is crepant-birational to the pair $(X, (1 - \frac{1}{m})S)$ discussed later.) $\square$

6 On the first gap of global log canonical thresholds

Theorem 6.1. *For every $n \geq 2$, there is a complex klt Calabi–Yau pair $(X, (1 - \frac{1}{m})S)$ of dimension n with S an irreducible divisor, and with the number $m = m_n$ defined in section 3. In particular, $m_n > 2^{2^n}$ for $n > 2$.*

In principle, this follows from Theorem 5.1, with X some birational model of Esser's klt Calabi–Yau variety V/μ_m; but the point of this section is to construct an explicit variety X. This example should be optimal. A bit more strongly, we conjecture:

Conjecture 6.2. For every $n \geq 2$, if $(X, (1-b)S)$ is a complex klt Calabi–Yau pair of dimension n such that S is a nonzero effective Weil divisor, then $b \geq 1/m_n$ for the number m_n in Theorem 6.1.

The conjecture is true in dimension 2, by Liu and Shokurov [13, Theorem 1.1]. They formulate several other extremal problems which have the same bound in dimension 2 (namely, 1/13), and conjecturally in all dimensions. In dimension 1, the bound in Conjecture 6.2 is 1/3, by the example of $(\mathbf{P}^1, \frac{2}{3}S)$ with S equal to 3 points. We know that there is some positive lower bound for this problem in each dimension, by Hacon-M$^{\text{c}}$Kernan-Xu [7, Theorem 1.5].

In dimension 2, our example is:

$$\begin{aligned} X &= X_{30} \subset \mathbf{P}^3(15, 10, 4, 13), \\ 0 &= x_0^2 + x_1^3 + x_2^5 x_1 + x_3^2 x_2. \end{aligned}$$

Here $S = X \cap \{x_3 = 0\}$, and $(X, \frac{12}{13}S)$ is the desired Calabi–Yau pair. In dimension 3, we have:

$$\begin{aligned} X &= X_{360} \subset \mathbf{P}^4(180, 115, 49, 15, 311), \\ 0 &= x_0^2 + x_1^3 x_3 + x_2^5 x_1 + x_3^{12} x_0 + x_4 x_2. \end{aligned}$$

Here $S = X \cap \{x_4 = 0\}$, and $(X, \frac{310}{311}S)$ is the desired Calabi–Yau pair. In dimension 4, we have:

$$\begin{aligned} X &= X_{1387722} \subset \mathbf{P}^5(693861, 462574, 198098, 32152, 1036, 677785), \\ 0 &= x_0^2 + x_1^3 + x_2^7 x_4 + x_3^{37} x_2 + x_4^{893} x_1 + x_5^2 x_3. \end{aligned}$$

Here $S = X \cap \{x_5 = 0\}$, and $(X, \frac{677784}{677785}S)$ is the desired Calabi–Yau pair.

Proof. (Theorem 6.1) We want to construct a klt Calabi–Yau pair $(X, (1 - \frac{1}{m})S)$ of dimension $n \geq 2$ with S an irreducible divisor (and the number $m = m_n$ defined above). We define X as a hypersurface in a weighted projective space $Y = \mathbf{P}^{n+1}(c_0, \ldots, c_{n+1})$. Namely, if $n = 2r+1$, let X have the same equation as V (from section 3) except for the first and last terms:

$$0 = x_0^2 + x_1^{b_1} x_{2r+1} + \cdots + x_r^{b_r} x_{r+2} + x_{r+1}^{b_{r+1}} x_r + \cdots + x_{2r+1}^{b_{2r+1}} x_0 + x_{2r+2} x_{r+1}.$$

We also have formulas for the degree d and the weights c_j of X. (With these definitions, it is straightforward that each monomial above has degree d.) First, $d \coloneqq b_0 \cdots b_{2r+1}$. Next, the last weight c_{2r+2} is $m_{2r+1} = B_{0,2r+1,1,2r,\ldots,r,r+1}$, the index of Esser's example. Finally,

$$c_{r+1+i} \coloneqq b_{r+1-i} \cdots b_r b_{r+1} \cdots b_{r+i} B_{0,2r+1,1,2r,\ldots,r-i}$$

for $0 \leq i \leq r$, and

$$c_{r-i} \coloneqq b_{r+1-i} \cdots b_r b_{r+1} \cdots b_{r+1+i} B_{0,2r+1,1,2r,\ldots,r-1-i,r+2+i}$$

for $0 \le i \le r$. (Since a B with empty subscript is 1, we have $c_0 = b_1 \cdots b_{2r+1}$.)

If $n = 2r$, let X have the same equation as V except for the second and last terms:

$$0 = x_0^2 + x_1^3 + x_2^{b_2} x_{2r} + \cdots + x_r^{b_r} x_{r+2} + x_{r+1}^{b_{r+1}} x_r + \cdots + x_{2r}^{b_{2r}} x_1 + x_{2r+1}^2 x_{r+1}.$$

Again, we have formulas for the degree d and weights c_j of X. First, $d \coloneqq b_0 \cdots b_{2r}$. Next, the last weight c_{2r+1} is $m_{2r} = B_{1,2r,2,2r-1,\ldots,r,r+1}$, the index of Esser's example. Also, $c_0 \coloneqq b_1 \cdots b_{2r}$. Finally,

$$c_{r+1+i} \coloneqq 2b_{r+1-i} \cdots b_r b_{r+1} \cdots b_{r+i} B_{1,2r,2,2r-1,\ldots,r-i}$$

for $0 \le i \le r-1$, and

$$c_{r-i} \coloneqq 2b_{r+1-i} \cdots b_r b_{r+1} \cdots b_{r+1+i} B_{1,2r,2,2r-1,\ldots,r-1-i,r+2+i}$$

for $0 \le i \le r-1$.

Define the divisor S to be $X \cap \{x_{n+1} = 0\}$. From the form of the equation (a chain as in section 1, or x_0^2 plus a chain), it is immediate that X and S are quasi-smooth. (Since S is a hypersurface of dimension at least 1 in a weighted projective space, it follows that S is irreducible.) The weight c_{n+1} (the degree of S) is equal to the index m of Esser's example, Next, we show that $K_X = O_X(d - \sum c_j)$ is equal to $O_X(1-m)$, so that $(X, (1-\frac{1}{m})S)$ is a klt Calabi–Yau pair (assuming that X is well-formed, as we show below). Since $c_{n+1} = m$, it is equivalent to show that $d - \sum_{j=0}^{n} c_j = 1$. Assume that $n = 2r+1$. We compute $d - \sum_{j=0}^{n} c_j$ by collecting all terms (in the formulas above for d and the c_j's) that are multiples of $b_{r+1} \cdots b_{2r+1}$, then the remaining terms that are multiples of $b_{r+2} \cdots b_{2r+1}$, and so on. The result is:

$$\begin{aligned} d - \sum_{j=0}^{2r+1} c_j = b_{r+1} \cdots b_{2r+1} &\left[b_0 \cdots b_r \left(1 - \frac{1}{b_0} - \cdots - \frac{1}{b_r} \right) \right] \\ &- \sum_{i=1}^{r+1} b_{r+1+i} \cdots b_{2r+1} [b_0 \cdots b_{r-i} (b_{r+1-i} - 1) B_{r+1,r,\ldots,r-1+i,r+2-i}]. \end{aligned}$$

The first term in brackets is 1, by the properties of the Sylvester numbers $b_0, \ldots, b_r$. By definition of b_{r+i}, the expression in brackets in the term indexed by i (for $1 \le i \le r+1$) is $b_{r+i} - 1$. Therefore, the sum telescopes, showing that $d - \sum_{j=0}^{2r+1} c_j = 1$, as we want. For $n = 2r$, the proof is similar:

$$\begin{aligned} d - \sum_{j=0}^{2r} c_j = b_{r+1} \cdots b_{2r} &\left[b_0 \cdots b_r \left(1 - \frac{1}{b_0} - \cdots - \frac{1}{b_r} \right) \right] \\ &- \sum_{i=1}^{r} b_{r+1+i} \cdots b_{2r} [2b_1 \cdots b_{r-i} (b_{r+1-i} - 1) B_{r+1,r,\ldots,r-1+i,r+2-i}]. \end{aligned}$$

Again, the expression in brackets in the term indexed by i is $b_{r+i} - 1$. So the sum telescopes, showing that $d - \sum_{j=0}^{2r} c_j = 1$, as we want.

The only property that remains to be checked is that X is well-formed. (We need this in order to justify the formula above for K_X, with X viewed as a variety rather than a pair.) The main point is to show that the weighted projective space $Y = \mathbf{P}^{n+1}(c_0, \ldots, c_{n+1})$ is well-formed. Indeed, from there it follows that X is well-formed when $n \geq 3$, by Proposition 1.1. This general result does not apply in dimension 2, but in that case we can check by hand that our example is well-formed. Namely, the example is $X \subset Y = \mathbf{P}^3(15, 10, 4, 13)$, defined by $0 = x_0^2 + x_1^3 + x_2^5 x_1 + x_3^2 x_2$. This is well-formed, because X does not contain the 1-dimensional stratum $\{x_2 = x_3 = 0\}$ of Y with stabilizer μ_5 or the 1-dimensional stratum $\{x_0 = x_3 = 0\}$ with stabilizer μ_2.

So, for each $n \geq 2$, we need to show that Y is well-formed, meaning that $\gcd(c_0, \ldots, \widehat{c_a}, \ldots, c_{n+1}) = 1$ for each $0 \leq a \leq n+1$. Let p be a prime number dividing c_j for each $j \neq a$; we will derive a contradiction. From looking at the equation of X, we see that it includes a monomial not involving x_a, and so p divides the degree d of X. For $n = 2r+1$, the variable x_a occurs with exponent 1 in some monomial in the equation of X, and so p divides c_a as well. For $n = 2r$, we get the same conclusion unless $a = 0$ or $a = 2r+1$. In those cases, the variable x_a occurs with exponent 2 in some monomial in the equation of X, and so p divides $2c_a$. It follows that p divides c_a unless $p = 2$. To analyze the case where $n = 2r$ and $p = 2$, use that $b_1, \ldots, b_{2r}$ are all odd by their definition. So $c_0 = b_1 \cdots b_{2r}$ is odd and $c_{2r+1} = m_{2r} = B_{1,2r,\ldots,r,r+1} \equiv 1 - 1 + 1 - 1 + \cdots + 1 \equiv 1 \pmod 2$, contradicting that 2 divides all but one of the weights c_j. Thus, if p divides all but one of the weights c_j, then it divides all the weights.

Return to allowing any dimension $n \geq 2$. We now have a prime number p that divides the weight c_j for each $0 \leq j \leq n+1$, and we want to get a contradiction. For $n = 2r+1$, we have $c_0 = b_1 \cdots b_{2r+1}$ and $c_{2r+2} = m_{2r+1}$. By Proposition 4.1, $m_{2r+1}u_{2r+1} - 1 = b_0 \cdots b_{2r+1} v_{2r+1}$, and so c_0 and c_{2r+2} are relatively prime. Next, for $n = 2r$, we have $c_0 = b_1 \cdots b_{2r}$ and $c_{2r+1} = m_{2r}$. By Proposition 4.1, $2m_{2r}u_{2r} - 1 = b_1 \cdots b_{2r} v_{2r}$, and so c_0 and c_{2r+1} are relatively prime. This completes the proof that X is well-formed. Theorem 6.1 is proved. ☐

We remark that X is rational at least in odd dimensions, $n = 2r+1$. Indeed, the variable x_{2r+2} occurs only in one monomial, and it has exponent 1. So the projection from X to $\mathbf{P}^{2r+1}(c_0, \ldots, c_{2r+1})$ is a birational map.

Lemma 6.3. *In each dimension $n \geq 2$, the klt Calabi–Yau pair $(X, (1 - \frac{1}{m})S)$ of Theorem 6.1 is crepant-birational to Esser's klt Calabi–Yau variety V/μ_m with mld $1/m$.*

The proof also gives an explicit formula for the action of μ_m on V, in terms of the weights c_j of X.

Proof. Let $n \geq 2$. Let W be the index-1 cover of the klt Calabi–Yau pair $(X, (1 - \frac{1}{m})S)$ of dimension n from Theorem 6.1. It follows that W has canonical singularities and $K_W \sim 0$. Explicitly, for $n = 2r+1$, W is the Calabi–Yau hypersurface

$$W = W_d \subset \mathbf{P}^{2r+2}(c_0, \ldots, c_{2r+1}, 1),$$
$$0 = x_0^2 + x_1^{b_1} x_{2r+1} + \cdots + x_r^{b_r} x_{r+2} + x_{r+1}^{b_{r+1}} x_r + \cdots + x_{2r+1}^{b_{2r+1}} x_0 + x_{2r+2}^m x_{r+1}.$$

The degree d of W is the same as for X, and the exponents b_j and weights c_j are the same except for $j = 2r+2$. The last weight is changed from $c_{2r+2} = m_{2r+1} = m$

to 1. Let the cyclic group μ_m act on W with weights $(0,\ldots,0,-1)$; then the quotient W/μ_m is the pair $(X,(1-\frac{1}{m})S)$. (The quotient map $W\to X$ is given by $[x_0,\ldots,x_{2r+2}]\mapsto[x_0,\ldots,x_{2r+1},x_{2r+2}^m]$.)

In even dimensions $n=2r$, the index-1 cover W of $(X,(1-\frac{1}{m})S)$ has a similar description. Here W is the Calabi–Yau hypersurface

$$\begin{aligned}&W=W_d\subset \mathbf{P}^{2r+1}(c_0,\ldots,c_{2r},1),\\&0=x_0^2+x_1^3+x_2^{b_2}x_{2r}+\cdots+x_r^{b_r}x_{r+2}+x_{r+1}^{b_{r+1}}x_r+\cdots+x_{2r}^{b_{2r}}x_1+x_{2r+1}^{2m}x_{r+1}.\end{aligned}$$

Again, the last weight has been changed from $c_{2r+1}=m_{2r}=m$ to 1, and the cyclic group μ_m acts on W with weights $(0,\ldots,0,-1)$.

The equations of W and Esser's hypersurface V are the same in the affine charts $x_{n+1}=1$, and the variable x_{n+1} has weight 1 in both cases. Therefore, we get a birational map $\varphi\colon W \dashrightarrow V$ by sending

$$[x_0,\ldots,x_n,1]\mapsto[x_0,\ldots,x_n,1].$$

Since W and V are Calabi–Yau varieties with canonical singularities, φ is automatically crepant.

Moreover, we can define an action of μ_m on V that makes φ μ_m-equivariant. Indeed, we can rewrite the action of μ_m on W as

$$\begin{aligned}\zeta([x_0,\ldots,x_n,x_{n+1}])&=[x_0,\ldots,x_n,\zeta^{-1}x_{n+1}]\\&=[\zeta^{c_0}x_0,\ldots,\zeta^{c_n}x_n,x_{n+1}].\end{aligned}$$

That suggests defining an action of μ_m on V by the formula:

$$\zeta([x_0,\ldots,x_n,x_{n+1}])=[\zeta^{c_0}x_0,\ldots,\zeta^{c_n}x_n,x_{n+1}].$$

To show that this action preserves the hypersurface V, it suffices to check this in the affine chart $x_{n+1}=1$; but there it is clear, because the equation of V in this chart is the same as the equation of W in the corresponding chart. Given that, it is clear that the map φ is μ_m-equivariant. (It suffices to check this in the affine chart $x_{n+1}=1$, where the two μ_m-actions are given by the same formula.)

Therefore, we have a crepant birational map from W/μ_m (viewed as a pair, namely $(X,(1-\frac{1}{m}S)))$ to the klt Calabi–Yau variety V/μ_m. In particular, this shows more explicitly why V/μ_m has mld $1/m$: because the divisor S has log discrepancy $1/m$ with respect to V/μ_m. $\square$

7 Esser-Totaro-Wang's klt Calabi–Yau variety with large index

A major problem on boundedness of Calabi–Yau varieties is the Index Conjecture, which says (in particular) that the index is bounded among all klt Calabi–Yau varieties of a given dimension. In [5, section 7], the authors and Wang constructed a klt Calabi–Yau variety of each dimension $n\geq 2$ which conjecturally has the largest index, roughly 2^{2^n} in dimension n. (For example, this variety has index 19 in dimension 2, 493 in dimension 3, and 1201495 in dimension 4.) In this section, we present this example, give a clearer proof of many of its properties, and provide a conjectural formula for the index. We reduce the problem of computing the index in a given dimension to showing that two explicit numbers are relatively prime, as explained in Proposition 7.3. That holds by a computer check in dimensions at most 30. The expected value of the index is within a constant factor of the conjecturally largest index in the broader setting of klt Calabi–Yau pairs with standard coefficients (Lemma 8.2).

The varieties are again quotients of quasi-smooth Calabi–Yau hypersurfaces in weighted projective space by finite groups; the numerology of these hypersurfaces is extremely similar to that of the small mld examples. Indeed, for $n = 2r+1$ odd, the hypersurface $V' \subset \mathbf{P}^{n+1}(a'_0, \ldots, a'_{n+1})$ has the form

$$0 = x_0^{b_0} x_{2r+2} + x_1^{b_1} x_{2r+1} + \cdots + x_r^{b_r} x_{r+2} + x_{r+1}^{b_{r+1}} x_r + \cdots + x_{2r}^{b_{2r}} x_1 + x_{2r+1}^{b'_{2r+1}} x_0 + x_{2r+2}^{v'_{2r+1}} x_{r+1}.$$

In even dimension $n = 2r$, the equation of V' has the form

$$0 = x_0^{b_0} + x_1^{b_1} x_{2r+1} + \cdots + x_r^{b_r} x_{r+2} + x_{r+1}^{b_{r+1}} x_r + \cdots + x_{2r-1}^{b_{2r-1}} x_2 + x_{2r}^{b'_{2r}} x_1 + x_{2r+1}^{v'_{2r}} x_{r+1}.$$

These equations have the same shape as the small mld examples and share all the same exponents except for the last two, b'_n and v'_n. We define these last two exponents as follows:

$$\begin{cases} b'_{2r+1} := \frac{1}{2}(1 + b_1 \cdots b_{2r} + (s_1 - 1) B_{r+1,r,\ldots,2r,1}) & \text{if } n = 2r+1, \\ b'_{2r} := \frac{1}{3}(1 + 2(s_1 - 1)^2 b_2 \cdots b_{2r-1} + 2(s_2 - 1) B_{r+1,r,\ldots,2r-1,2}) & \text{if } n = 2r. \end{cases}$$

It is straightforward to check that these two expressions are integers. We note the following comparison between b'_n and b_n and define constants $E = E_n$ for future use:

$$\begin{cases} E_{2r+1} := b'_{2r+1} - b_{2r+1} + 1 = \frac{1}{2}(b_1 \cdots b_{2r} + 1) & \text{if } n = 2r+1, \\ E_{2r} := b'_{2r} - b_{2r} + 1 = \frac{1}{3}(8 b_2 \cdots b_{2r-1} + 1) & \text{if } n = 2r. \end{cases} \tag{7.1}$$

The last exponent is given by

$$v'_{2r+1} := B_{1,2r,\ldots,r,r+1} + (s_1 - 1)[B_{r+1,r,\ldots,2r,1} - B_{r,\ldots,2r,1} + \cdots - B_1 + 1] - 1$$

if $n = 2r+1$, and

$$v'_{2r} := (s_1 - 1)^2 B_{2,2r-1,\ldots,r,r+1} + (s_2 - 1)[B_{r+1,r,\ldots,2r-1,2} - B_{r,\ldots,2r-1,2} + \cdots - B_2 + 1] - 1$$

if $n = 2r$. Using the notation above Proposition 4.1, we may write v'_n more concisely as $v'_{2r+1} = g_1 + w_1$ for $n = 2r+1$ and $v'_{2r} = 4g_2 + w_2$ for $n = 2r$.

The weights $a'_0, \ldots, a'_{n+1}$ and degree D' of V' are uniquely determined by the equation of V', given the requirement that $\gcd(a'_0, \ldots, a'_{n+1}) = 1$.

We set the following additional definitions:

$$u' := \begin{cases} u'_{2r+1} = b_1 \cdots b_{2r} + (s_1 - 1) B_{r+1,r,r+2,r-1,\ldots,2r,1} & \text{if } n = 2r+1, \\ u'_{2r} = (s_1 - 1) b_2 \cdots b_{2r-1} + s_1 B_{r+1,r,r+2,r-1,\ldots,2r-1,2} & \text{if } n = 2r. \end{cases}$$

Notice that $u'_{2r+1} = 2b'_{2r+1} - 1$ and $4u'_{2r} = 3b'_{2r} - 1$. Finally, define

$$m' := \begin{cases} m'_{2r+1} := b_0 b'_{2r+1} B_{1,2r,\ldots,r,r+1} - b_0 + 1 & \text{if } n = 2r+1, \\ m'_{2r} := b_1 b'_{2r} B_{2,2r-1,\ldots,r,r+1} - b_1 + 1 & \text{if } n = 2r. \end{cases} \tag{7.2}$$

With the setup in place, we will now prove several properties of V'.

Theorem 7.1. *In each dimension $n \geq 2$, the hypersurface $V' \subset \mathbf{P}(a'_0, \ldots, a'_{n+1})$ defined above is well-formed, quasi-smooth, and Calabi–Yau. The degree D' of V' is given by $D' = u'_{2r+1}$ if $n = 2r+1$ is odd, and $D' = 2u'_{2r}$ if $n = 2r$ is even. The last two weights a'_n and a'_{n+1} of V' equal 1. There is an action on V' by the cyclic group of order $m' = m'_n$, which is free in codimension 1.*

The large index example will then be the quotient $V'/\mu_{m'}$. We expect this quotient to have index m' in every dimension, but this is conditional on the statement in Proposition 7.3.

For $n = 2$, the hypersurface V' is

$$\{x_0^2 + x_1^3 x_3 + x_2^7 x_1 + x_3^9 x_2 = 0\} \subset \mathbf{P}^3(5, 3, 1, 1),$$

with an action of μ_{19}. Here V'/μ_{19} is a klt Calabi–Yau surface of index 19, which is the largest possible [5, Proposition 6.1]. For $n = 3$, the hypersurface V' is

$$\{x_0^2 x_4 + x_1^3 x_3 + x_2^5 x_1 + x_3^{19} x_0 + x_4^{32} x_2 = 0\} \subset \mathbf{P}^4(18, 12, 5, 1, 1),$$

with an action of the cyclic group of order 493. For $n = 4$, the hypersurface V' is

$$\{x_0^2 + x_1^3 x_5 + x_2^7 x_4 + x_3^{37} x_2 + x_4^{1583} x_1 + x_5^{2319} x_3 = 0\} \subset \mathbf{P}^5(1187, 791, 339, 55, 1, 1),$$

with an action of the cyclic group of order 1201495.

Proof of Theorem 7.1. Since the equation is a loop or x_0^2 plus a loop, the hypersurface V' is quasi-smooth.

The weighted projective space containing V' is well-formed: if a prime number p divides all but one weight a'_i, then p divides D' since there is a monomial not involving x_i. Then, there is either a monomial of the form $x_j^a x_i$ in the equation for V' or n is even and $i = 0$. In the former case, we get that p divides a'_i too, contradicting that $\gcd(a'_0, \ldots, a'_{n+1}) = 1$. The same holds in the latter case unless $p = 2$. If $p = 2$ divides all weights except a'_0 in the n even case, then $D' \equiv 2 \pmod 4$. The remaining exponents $b_1, \ldots, b_{2r-1}, b'_{2r}, v'_{2r}$ are odd, so this means the weights a'_i would have to alternate between $0 \pmod 4$ and $2 \pmod 4$ moving around the loop part of the equation of V'. But the loop has odd length, a contradiction. It follows that V' is well-formed by Proposition 1.1, together with a look at the equation for V' in dimension 2 (above).

Lemma 7.2. *The hypersurface V' is Calabi–Yau, in the sense that $D' = \sum a'_j$.*

Proof. Let A be the $(n+2) \times (n+2)$ matrix encoding the equation of V'. As in Lemma 5.2, it will be enough to show that the sum of the entries of A^{-1} is 1. We use the description of the inverse loop matrix from Lemma 5.3. For $n = 2r+1$, A^{-1} is an integer matrix divided by $b_0 \cdots b_{2r} b'_{2r+1} v'_{2r+1} + 1$. We will show that $b_0 \cdots b_{2r} b'_{2r+1} v'_{2r+1}$ minus the sum of entries of this integer matrix is 0. To do this, collect all terms that are multiples of $b_{r+1} \cdots b_{2r} b'_{2r+1} v'_{2r+1}$, then the remaining terms that are multiples of $b_{r+2}\cdots b_{2r}b'_{2r+1}v'_{2r+1}$, and so on. The result is:

$$\begin{aligned}
&b_{r+1}\cdots b_{2r}b'_{2r+1}v'_{2r+1}\left[b_0\cdots b_r\left(1-\frac{1}{b_0}-\cdots-\frac{1}{b_r}\right)\right]\\
&\quad-\sum_{i=1}^{r} b_{r+1+i}\cdots b_{2r}b'_{2r+1}v'_{2r+1}[b_0\cdots b_{r-i}(b_{r+1-i}-1)B_{r+1,r,\ldots,r-1+i,r+2-i}]\\
&\qquad\qquad\qquad\qquad\qquad\qquad-v'_{2r+1}[B_{r+1,r,\ldots,2r,1}]\\
&\qquad\qquad\qquad\qquad\qquad\quad-[B'_{r+1,r,\ldots,2r+1,0}-B'_{r,\ldots,2r+1,0}+\cdots-B'_0+1]+1.
\end{aligned}$$

The notation B' in the last line indicates that we have substituted b'_{2r+1} for b_{2r+1} where appropriate in the alternating sums. We want to show that this whole expression equals 0. So far, the grouping of terms is the same as in the proof of the Calabi–Yau property for the small mld example with $n = 2r + 1$ in section 5.

The bracketed term on the first line is 1, while on the second it is $b_{r+i} - 1$ in the ith summand. On the third line, v'_{2r+1} is multiplied by $b_{2r+1} - 1$ rather than $b'_{2r+1} - 1$, so the terms do not telescope quite as before. Instead, the sum of all but the last line gives $v'_{2r+1}(b'_{2r+1} - b_{2r+1} + 1)$. Using (7.1) and also Lemma 3.1 on the last line, we have therefore reduced to showing

$$\frac{b_1\cdots b_{2r}+1}{2}v'_{2r+1} = [B'_{0,2r+1,\ldots,r,r+1}-B'_{2r+1,\ldots,r,r+1}+\cdots-B'_{r+1}+1]-1. \quad (7.3)$$

On the left-hand side, we can rewrite $(b_1\cdots b_{2r}v'_{2r+1})/2$ (leaving behind the remaining $v'_{2r+1}/2$) as follows, using Proposition 4.1 for $k = 1$:

$$\begin{aligned}
\frac{1}{2}b_1\cdots b_{2r}v'_{2r+1} &= \frac{1}{2}(b_1\cdots b_{2r}g_1 + b_1\cdots b_{2r}w_1)\\
&= \frac{1}{2}(b_1\cdots b_{2r}g_1 + (s_1-1)g_1t_1 - 1) = g_1\left(b'_{2r+1}-\frac{1}{2}\right)-\frac{1}{2}.
\end{aligned}$$

On the other hand, the right-hand side becomes

$$\begin{aligned}
&[B'_{0,2r+1,\ldots,r,r+1}-B'_{2r+1,\ldots,r,r+1}+\cdots-B'_{r+1}+1]-1\\
&= b'_{2r+1}g_1 - b_0 + 1 + 1 + [B_{1,2r,\ldots,r,r+1}-B_{2r,\ldots,r,r+1}+\cdots-B_{r+1}+1]-1\\
&= b'_{2r+1}g_1 + [B_{1,2r,\ldots,r,r+1}-B_{2r,\ldots,r,r+1}+\cdots-B_{r+1}+1]-1.
\end{aligned}$$

Here we've reverted B' to B in sums where the index $2r+1$ does not appear. We can now cancel the terms involving b'_{2r+1} from both sides of (7.3) and apply Lemma 3.1 to the bracketed term on the last line. This gives:

$$\frac{v'_{2r+1}}{2} = \frac{1}{2}g_1 + [B_{r+1,r,\ldots,2r,1}-B_{r,\ldots,2r,1}+\cdots-B_1+1]-\frac{1}{2}.$$

After multiplying by 2, we recover the original definition for v'_{2r+1}, so the proof is complete.

Now we'll show the Calabi–Yau property when $n = 2r$. Again, we need to demonstrate that the sum of entries of the matrix A^{-1} is 1, where A^{-1} is now a block diagonal matrix with $1/2$ in the top left corner followed by a $(n+1)\times(n+1)$ block which is $1/(b_1\cdots b_{2r-1}b'_{2r}v'_{2r}+1)$ times an integer matrix C of inverse loop form (as in Lemma 5.3). Explicitly, $b_1 \cdots b_{2r-1} b'_{2r} v'_{2r} + 1$ minus 2 times the sum of the entries of C (which we want to be zero) is

$$\begin{aligned}
&b_{r+1} \cdots b_{2r-1} b'_{2r} v'_{2r} \left[b_1 \cdots b_r \left(1 - 2 \left(\frac{1}{b_1} - \cdots - \frac{1}{b_r} \right) \right) \right] \\
&\qquad - \sum_{i=1}^{r-1} b_{r+1+i} \cdots b_{2r-1} b'_{2r} v'_{2r} [2 b_1 \cdots b_{r-i} (b_{r+1-i} - 1) B_{r+1,r,\ldots,r-1+i,r+2-i}] \\
&\qquad\qquad\qquad\qquad - v'_{2r} [2(b_1 - 1) B_{r+1,r,\ldots,2r-1,2}] \\
&\qquad\qquad\qquad\qquad\qquad - 2[B'_{r+1,r,\ldots,2r,1} - B'_{r,\ldots,2r,1} + \cdots - B'_1 + 1] + 1.
\end{aligned}$$

So far, the grouping of terms is the same as in the proof of the Calabi–Yau property for the small mld example with $n = 2r$ in section 5. The notation B' in the last line indicates that we have substituted b'_{2r} for b_{2r} where appropriate in the alternating sums.

The bracketed term on the first line is 1, while it is $b_{r+i} - 1$ in the ith summand on the second line. On the third line, v'_{2r} is multiplied by $b_{2r} - 1$ rather than $b'_{2r} - 1$, so the sum does not completely telescope as before. Instead, the sum of all but the last line gives $v'_{2r}(b'_{2r} - b_{2r} + 1)$. Using (7.1) and also Lemma 3.1 on the last line, we have therefore reduced to showing

$$\frac{8 b_2 \cdots b_{2r-1} + 1}{3} v'_{2r} = 2[B'_{1,2r,\ldots,r,r+1} - B'_{2r,\ldots,r,r+1} + \cdots - B'_{r+1} + 1] - 1. \qquad (7.4)$$

On the left-hand side, we can rewrite $(8 b_2 \cdots b_{2r-1} v'_{2r})/3$ (leaving behind the remaining $v'_{2r}/3$) as follows, using Proposition 4.1 for $k = 2$:

$$\begin{aligned}
\frac{8}{3} b_1 \cdots b_{2r} v'_{2r} &= \frac{8}{3}(b_1 \cdots b_{2r} 4 g_2 + b_2 \cdots b_{2r-1} w_2) \\
&= \frac{8}{3}(4 b_2 \cdots b_{2r-1} g_2 + (s_2 - 1) g_2 t_2 - 1) = 4 g_2 \left(b'_{2r} - \frac{1}{3} \right) - \frac{8}{3}.
\end{aligned}$$

On the other hand, the right-hand side becomes

$$\begin{aligned}
&2[B'_{1,2r,\ldots,r,r+1} - B'_{2r,\ldots,r,r+1} + \cdots - B'_{r+1} + 1] - 1 \\
&= 4 b'_{2r} g_2 - 2 b_1 + 2 + 2 + 2[B_{2,2r-1,\ldots,r,r+1} - B_{2r-1,\ldots,r,r+1} + \cdots - B_{r+1} + 1] - 1 \\
&= 4 b'_{2r} g_2 + 2[B_{2,2r-1,\ldots,r,r+1} - B_{2r-1,\ldots,r,r+1} + \cdots - B_{r+1} + 1] - 3.
\end{aligned}$$

Here we've reverted B' to B in sums where the index $2r$ does not appear. We can now cancel the terms involving b'_{2r} from both sides of (7.4) and apply Lemma 3.1 to the bracketed term on the right-hand side. This gives:

$$\frac{v'_{2r}}{3} = \frac{4}{3} g_2 + 2[B_{r+1,r,\ldots,2r-1,2} - B_{r,\ldots,2r-1,2} + \cdots - B_2 + 1] - \frac{1}{3}.$$

After multiplying by 3, we recover the original definition for v'_{2r}. We've shown that V' is Calabi–Yau in every dimension. Lemma 7.2 is proved. $\square$

By Lemma 7.2, $K_{V'} = \mathcal{O}_{V'}(D' - \sum a'_j) = \mathcal{O}_{V'}$. Since the hypersurface V' is klt and has trivial canonical class, it is canonical.

Next, we'll state and prove expressions for the degree D' of V' and the order m' of a cyclic group action on this hypersurface. Along the way, we'll also prove that the last two weights a'_n and a'_{n+1} equal 1. First, we show the following identities involving m'_n and u'_n:

$$\begin{cases} m'_{2r+1}u'_{2r+1} - 1 = b_0 \cdots b_{2r} b'_{2r+1} v'_{2r+1} & \text{if } n = 2r+1, \\ 2m'_{2r}u'_{2r} - 1 = b_1 \cdots b_{2r-1} b'_{2r} v'_{2r} & \text{if } n = 2r. \end{cases} \tag{7.5}$$

First, when $n = 2r + 1$,

$$\begin{aligned} u'_{2r+1}m'_{2r+1} - 1 &= u'_{2r+1}(b_0 b'_{2r+1} B_{1,2r,\ldots,r,r+1} - 1) - 1 \\ &= (b_1 \cdots b_{2r} + (s_1 - 1)B_{r+1,r,\ldots,2r,1}) b_0 b'_{2r+1} B_{1,2r,\ldots,r,r+1} - u'_{2r+1} - 1 \\ &= b_0 \cdots b_{2r} b'_{2r+1} B_{1,2r,\ldots,r,r+1} + b_0 b'_{2r+1}(s_1 - 1) B_{r+1,r,\ldots,2r,1} B_{1,2r,\ldots,r,r+1} - u'_{2r+1} - 1. \end{aligned}$$

We may now apply Proposition 4.1 with $k = 1$ to replace the expression $(s_1 - 1)B_{r+1,r,\ldots,2r,1}B_{1,2r,\ldots,r,r+1} = (s_1 - 1)g_1 t_1$ with $b_1 \cdots b_{2r} w_1 + 1$. Hence

$$\begin{aligned} u'_{2r+1}m'_{2r+1} - 1 &= b_0 \cdots b_{2r} b'_{2r+1}(g_1 + w_1) + b_0 b'_{2r+1} - u'_{2r+1} - 1 \\ &= b_0 b_1 \cdots b_{2r} b'_{2r+1} v'_{2r+1} + b_0 b'_{2r+1} - u'_{2r+1} - 1 \\ &= b_0 b_1 \cdots b_{2r} b'_{2r+1} v'_{2r+1}. \end{aligned}$$

Looking at the form A^{-1} of the loop matrix in Lemma 5.3, we may read off the last charge for the hypersurface V' and apply (7.5) to get:

$$\begin{aligned} q_{2r+2} &= \frac{b_0 b'_{2r+1} b_1 b_{2r} \cdots b_r b_{r+1} - b_0 b'_{2r+1} b_1 b_{2r} \cdots b_r + \cdots + b_0 b'_{2r+1} - b_0 + 1}{b_0 b_1 \cdots b_{2r} b'_{2r+1} v'_{2r+1} + 1} \\ &= \frac{m'_{2r+1}}{b_0 b_1 \cdots b_{2r} b'_{2r+1} v'_{2r+1} + 1} \\ &= \frac{1}{u'_{2r+1}}. \end{aligned}$$

The degree D' of V' is the least common denominator of the charges. All these denominators are the same for a loop potential, so $D' = u'_{2r+1}$ and the weight $a'_{2r+2} = 1$. Since $x_0^2 x_{2r+2}$ is a monomial in the equation for V', we must have $a'_0 = (u'_{2r+1} - 1)/2$. Then, since $x_{2r+1}^{b'_{2r+1}} x_0$ is a monomial also,

$$a'_{2r+1} = \frac{u'_{2r+1} - (u'_{2r+1} - 1)/2}{b'_{2r+1}} = 1.$$

Let $\mathrm{Aut}_T(V')$ be the group of toric automorphisms of V'. The order of this group is related to the degree of the hypersurface and the matrix A by [2, section 3]

$$|\det(A)| = D' |\mathrm{Aut}_T(V')|.$$

Since $\det(A) = b_0 \cdots b_{2r} b'_{2r+1} v'_{2r+1} + 1$, the identity (7.5) implies that $|\mathrm{Aut}_T(V')| = m'_{2r+1}$. The equation of V' is a loop, so $\mathrm{Aut}_T(V')$ is a cyclic group of order $m' = m'_{2r+1}$. Since the equation of V' is a loop, the action of $\mu_{m'}$ is free in codimension 1 [5, Proposition 7.2].

Next, if $n = 2r$, then we prove (7.5) as follows:

$$\begin{aligned}
2u'_{2r+1}m'_{2r+1} - 1 &= 2u'_{2r+1}(b_1 b'_{2r} B_{2,2r-1,\dots,r,r+1} - 2) - 1 \\
&= (2(s_1 - 1)b_2 \cdots b_{2r-1} + 2s_1 B_{r+1,r,\dots,2r-1,2}) b_1 b'_{2r} B_{2,2r-1,\dots,r,r+1} - 4u'_{2r+1} - 1 \\
&= (s_1 - 1)^2 b_1 \cdots b_{2r-1} b'_{2r} B_{2,2r-1,\dots,r,r+1} \\
&\quad + b_1 b'_{2r}(s_2 - 1) B_{r+1,r,\dots,2r-1,2} B_{2,2r-1,\dots,r,r+1} - 4u'_{2r} - 1.
\end{aligned}$$

We may now apply Proposition 4.1 with $k = 2$ to replace the expression $(s_2 - 1)B_{r+1,r,\dots,2r-1,2}B_{2,2r-1,\dots,r,r+1} = (s_2 - 1)g_2 t_2$ with $b_2 \cdots b_{2r-1} w_2 + 1$. Hence

$$\begin{aligned}
2u'_{2r}m'_{2r} - 1 &= b_1 \cdots b_{2r-1} b'_{2r}(4g_2 + w_2) + b_1 b'_{2r} - 4u'_{2r} - 1 \\
&= b_1 \cdots b_{2r-1} b'_{2r} v'_{2r} + b_1 b'_{2r} - 4u'_{2r} - 1 \\
&= b_1 \cdots b_{2r-1} b'_{2r} v'_{2r}.
\end{aligned}$$

For $n = 2r$, the $(n + 2) \times (n + 2)$ matrix A expressing the equation of V' is block diagonal with 2 as the top left entry and an $(n + 1) \times (n + 1)$ loop matrix as the other block. Therefore, we may read off the last charge q_{2r+1} from A^{-1} and apply (7.5) to get:

$$\begin{aligned}
q_{2r+1} &= \frac{b_1 b'_{2r} b_2 b_{2r-1} \cdots b_r b_{r+1} - b_1 b'_{2r} b_2 b_{2r-1} \cdots b_r + \cdots + b_1 b'_{2r} - b_1 + 1}{b_1 \cdots b_{2r-1} b'_{2r} v'_{2r} + 1} \\
&= \frac{m'_{2r}}{b_1 \cdots b_{2r-1} b'_{2r} v'_{2r} + 1} \\
&= \frac{1}{2u'_{2r}}
\end{aligned}$$

Therefore, the degree D' is a multiple of $2u'_{2r}$, say $D' = 2\lambda u'_{2r}$, so the last weight is $a'_{n+1} = \lambda$. Following the monomials around the loop, it follows that every weight a'_i with $i \neq 0$ is a multiple of λ. This contradicts the fact that the weighted projective space containing V' is well-formed, so in fact we have $D' = 2u'_{2r}$ and $a'_{n+1} = 1$. Since $x_1^3 x_{2r+1}$ is a monomial of V', we must have $a'_1 = (D' - 1)/3 = (2u'_{2r} - 1)/3$. Then, since $x_{2r}^{b'_{2r}} x_1$ is a monomial also,

$$a'_{2r} = \frac{2u'_{2r} - (2u'_{2r} - 1)/3}{b'_{2r}} = 1.$$

The determinant of the matrix A in this case is

$$|\det(A)| = 2(b_1 \cdots b_{2r-1} b'_{2r} v'_{2r} + 1)$$

and $|\det(A)| = D'|\mathrm{Aut}_T(V')|$ so (7.5) yields $|\mathrm{Aut}_T(V')| = 2m'_{2r}$. Because the equation is of the form x_0^2 plus a loop, $\mathrm{Aut}_T(V') \cong \mu_2 \times \mu_{m'}$ with $m' = m'_{2r}$. Since the shape of the equation is the same as in the mld example, the same argument from the proof of Theorem 5.1 shows that the $\mu_{m'}$-action is free in codimension 1.

This completes the proof of all the properties listed in Theorem 7.1. $\square$

Since the action of $\mu_{m'}$ on V' is free in codimension 1, the quotient $V'/\mu_{m'}$ is a klt Calabi–Yau variety. The index of the quotient is determined by the induced action of $\mu_{m'}$ on $H^0(V', K_{V'}) \cong \mathbf{C}$. In particular, the index of the quotient is m' if and only if this action is faithful. This in turn is equivalent to a concrete condition involving the exponents.

Proposition 7.3. *The quotient $V'/\mu_{m'}$ defined above has index $m' = m'_n$ in dimension n if and only if* $\gcd(m'_n, E_n) = 1$.

Recall that the constants m' and A were defined in (7.2) and(7.1), respectively.

Proof. Suppose that $n = 2r+1$ is odd. The determinant of the matrix A encoding the equation of V' is $u'_{2r+1}m'_{2r+1}$ by (7.5). The degree of the mirror hypersurface of V' always divides $u'_{2r+1}m'_{2r+1}/u'_{2r+1} = m'_{2r+1}$. By [5, Proposition 7.3], the action of $\mu_{m'}$ on $H^0(V', K_{V'})$ is faithful if and only if the mirror degree actually equals m'. This degree is the least common denominator of the mirror charges of V', which are the sums of columns of A^{-1}.

We may use Lemma 5.3 to write the smallest mirror charge as

$$\begin{aligned}
q^{\mathsf{T}}_{2r+2} &= \frac{b_{r+1}b_r\cdots b_{2r}b_1b'_{2r+1}b_0 - b_{r+1}b_r\cdots b_{2r}b_1b'_{2r+1} + \cdots - b_{r+1} + 1}{u'_{2r+1}m'_{2r+1}} \\
&= \frac{b'_{2r+1}b_1\cdots b_{2r} + B_{r+1,r,\ldots,2r,1}}{u'_{2r+1}m'_{2r+1}} \\
&= \frac{\frac{1}{2}((2b'_{2r+1}-1)b_1\cdots b_{2r} + b_1\cdots b_{2r} + 2B_{r+1,r,\ldots,2r,1})}{u'_{2r+1}m'_{2r+1}} \\
&= \frac{\frac{1}{2}(b_1\cdots b_{2r}+1)u'_{2r+1}}{u'_{2r+1}m'_{2r+1}} \\
&= \frac{E_{2r+1}}{m'_{2r+1}}.
\end{aligned}$$

If $\gcd(E_{2r+1}, m'_{2r+1}) = 1$, then this proves that the mirror degree is m'_{2r+1}. Conversely, if some prime p divides E_{2r+1} and m'_{2r+1}, and the mirror degree were m'_{2r+1}, by following the loop potential, we'd have that p divides every weight of the mirror, a contradiction.

The same reasoning holds when $n = 2r$. In that case, $\mathrm{Aut}_T(V') \cong \mu_2 \times \mu_{m'}$. The μ_2-action (which sends $x_0 \mapsto -x_0$ and leaves the other variables unchanged) is faithful on $H^0(V', K_{V'})$, so the $\mu_{m'}$-action is faithful if and only if the entire group $\mathrm{Aut}_T(V')$ acts faithfully. The determinant of the matrix A encoding the equation of V' is $4m'_{2r}u'_{2r}$. Using [5, Proposition 7.3] again, $\mathrm{Aut}_T(V')$ acts faithfully on $H^0(V', K_{V'})$ if and only if $2u_{2r}D^{\mathsf{T}} = 4m'_{2r}u'_{2r}$, where D^{T} is the mirror degree. This reduces to $D^{\mathsf{T}} = 2m'_{2r}$. We compute the smallest mirror charge as

$$\begin{aligned}
q^{\mathsf{T}}_{2r+1} &= \frac{b_{r+1}b_r\cdots b_{2r-1}b_2b'_{2r}b_1 - b_{r+1}b_r\cdots b_{2r-1}b_2b'_{2r} + \cdots - b_{r+1} + 1}{2u'_{2r}m'_{2r}} \\
&= \frac{(s_1-1)b'_{2r}b_2\cdots b_{2r-1} + B_{r+1,r,\ldots,2r-1,2}}{2u'_{2r}m'_{2r}} \\
&= \frac{\frac{1}{3}((3b'_{2r}-1)(s_1-1)b_2\cdots b_{2r-1} + (s_1-1)b_2\cdots b_{2r-1} + 3B_{r+1,r,\ldots,2r-1,2})}{2u'_{2r}m'_{2r}} \\
&= \frac{\frac{1}{3}(8b_2\cdots b_{2r-1}+1)u'_{2r}}{2u'_{2r}m'_{2r}} \\
&= \frac{E_{2r}}{2m'_{2r}}.
\end{aligned}$$

The constant E_{2r} is odd, so if $\gcd(E_{2r}, m'_{2r}) = 1$, then the mirror degree is $2m'_{2r}$, as required. Conversely, if a prime p (which must be odd) divides both E_{2r} and m'_{2r}, and the mirror degree were $2m'_{2r}$, then the form of the equation of V' implies that all mirror weights would be divisible by p, a contradiction. $\square$

Conjecture 7.4. For each integer $n \geq 2$, the numbers m'_n and E_n defined above are relatively prime.

By Proposition 7.3, Conjecture 7.4 is equivalent to the klt Calabi–Yau variety $V'/\mu_{m'}$ of dimension n having index equal to $m' = m'_n$. By computer calculation, the conjecture holds in dimensions at most 30.

8 Asymptotics of the mld and index

We show in this section that our klt Calabi–Yau varieties of small mld or large index are within a constant factor of the conjecturally optimal examples in the greater generality of klt pairs with standard coefficients.

First, building on examples by Kollár, Jihao Liu constructed a klt Calabi–Yau pair of dimension n with standard coefficients whose mld is $1/(s_{n+1}-1)$ [12, Remark 2.6]. That is conjectured to be the smallest possible mld in this setting. Namely, Liu's pair is

$$(X, D) = \left(\mathbf{P}^n, \frac{1}{2}H_0 + \frac{2}{3}H_1 + \frac{6}{7}H_2 + \cdots + \frac{s_n - 1}{s_n}H_n + \frac{s_{n+1}-2}{s_{n+1}-1}H_{n+1}\right), \quad (8.1)$$

where $H_0, \ldots, H_{n+1}$ are $n+2$ general hyperplanes in $\mathbf{P}^n$.

In the narrower setting of klt Calabi–Yau varieties, it turns out that Esser's example has mld less than 6 times $1/(s_{n+1}-1)$ in odd dimensions, and less than 23 times this number in even dimensions. That is extremely close for such small numbers, and it supports the conjecture that Esser's example has the smallest mld among klt Calabi–Yau varieties.

There is a parallel story for the problem of large index. Wang and the authors constructed a klt Calabi–Yau pair of dimension n with standard coefficients whose index is $(s_n-1)(2s_n-3)$ [5, Theorem 3.3]. That is conjectured to be the largest possible index in this setting. We now show that the conjectural value for the index of the klt Calabi–Yau variety of section 7 is within a constant factor of that number (Lemma 8.2).

More precisely, define a constant

$$\begin{aligned} \alpha &:= 2\prod_{j=1}^{\infty}\left[\frac{s_{j+1}}{(s_j-1)^2}\right]^{2^{j-1}} \\ &\doteq 5.522868. \end{aligned}$$

The convergence of this product is easy from the doubly exponential growth of the Sylvester numbers s_j and the fact that $s_{j+1} = (s_j-1)^2 + s_j$.

Lemma 8.1. *For each integer $n \geq 2$, let $1/m_n$ be the mld of Esser's klt Calabi–Yau variety of dimension n (computed in Theorem 5.1). Then*

$$\frac{1}{m_n} \leq \alpha\frac{1}{s_{n+1}-1}$$

if n is even and

$$\frac{1}{m_n} \leq \frac{3\alpha^2}{4}\frac{1}{s_{n+1}-1}$$

if n is odd.

The ratio $(s_{n+1}-1)/m_n$ actually converges to α for n even and to $3\alpha^2/4 \doteq 22.876556$ for n odd, as n goes to infinity; but we will not need that.

Proof. (Lemma 8.1) Let r be a positive integer, and let n be $2r$ or $2r+1$. Let $b_0, \ldots, b_{n+2}$ be the exponents of Esser's example, listed in section 3. Then b_a is equal to the Sylvester number s_a for $a \leq r$. Therefore,

$$\begin{aligned} b_{r+1} - 1 &= (b_r - 1)^2 \\ &= s_{r+1}\frac{(s_r-1)^2}{s_{r+1}}. \end{aligned}$$

By induction on $1 \leq a \leq r$, it follows that

$$b_{r+a} \geq s_{r+a}\frac{(s_{r+1-a}-1)^2}{s_{r+2-a}}\left[\frac{(s_{r+2-a}-1)^2}{s_{r+3-a}}\right]^{2^0} \cdots \left[\frac{(s_r-1)^2}{s_{r+1}}\right]^{2^{a-2}}.$$

As a result, we have

$$\begin{aligned} m_{2r} &= B_{1,2r,\ldots,r,r+1} \\ &= b_1 b_{2r} \cdots b_r b_{r+1} - b_1 b_{2r} \cdots b_r + \cdots \\ &\geq b_1 \cdots b_r (b_{r+1}-1) b_{r+2} \cdots b_{2r} \\ &\geq s_1 \cdots s_{2r} \left[\frac{(s_1-1)^2}{s_2}\right]^{2^0} \cdots \left[\frac{(s_r-1)^2}{s_{r+1}}\right]^{2^{r-1}} \\ &= (s_{2r+1}-1)\frac{1}{2}\left[\frac{(s_1-1)^2}{s_2}\right]^{2^0} \cdots \left[\frac{(s_r-1)^2}{s_{r+1}}\right]^{2^{r-1}} \\ &\geq \frac{1}{\alpha}(s_{2r+1}-1). \end{aligned}$$

The proof for $n = 2r+1$ is similar. Here $m_{2r+1} = B_{0,2r+1,\ldots,r,r+1} \geq b_0 \cdots b_r (b_{r+1} - 1) b_{r+2} \cdots b_{2r+1}$. The lower bound for b_{r+a} above holds (by induction) for all $1 \leq a \leq r+1$. We deduce that

$$\begin{aligned} m_{2r+1} &\geq s_0 \cdots s_{2r+1}\frac{(s_0-1)^2}{s_1}\left[\frac{(s_1-1)^2}{s_2}\right]^{2^1} \cdots \left[\frac{(s_r-1)^2}{s_{r+1}}\right]^{2^r} \\ &= (s_{2r+2}-1)\frac{1}{3}\left[\frac{(s_1-1)^2}{s_2}\right]^{2^1} \cdots \left[\frac{(s_r-1)^2}{s_{r+1}}\right]^{2^r} \\ &\geq \frac{4}{3\alpha^2}(s_{2r+2}-1). \end{aligned}$$

□

Lemma 8.2. *For each integer $n \geq 2$, let m'_n be the conjectural index of Esser-Totaro-Wang's klt Calabi–Yau variety of dimension n from section 7. Then*

$$m'_n \geq \frac{(s_n-1)(2s_n-3)}{9\alpha/8}$$

if n is even and

$$m'_n \geq \frac{(s_n-1)(2s_n-3)}{6\alpha^2/7}$$

if n is odd.

Here $9\alpha/8 \doteq 6.213227$ and $6\alpha^2/7 \doteq 26.144635$. Thus the expected index of the klt Calabi–Yau variety in section 7 is within a constant factor of the conjecturally largest index among all klt Calabi–Yau pairs with standard coefficients.

Proof. The statement is easy for $n = 2$, and so we can assume that $n > 2$. The index m'_n is defined in terms of the numbers $b_0, \ldots, b_{n-1}$ (the same in the small-mld example) together with b'_n (section 7). For $n = 2r$ with $r > 1$, we have

$$\begin{aligned} b'_{2r} &= \frac{1}{3}[1 + 8b_2 \cdots b_{2r-1} + 12B_{r+1,r,\cdots,2r-1,2}] \\ &\geq \frac{1}{3}[8b_2 \cdots b_{2r-1} + 12(b_2-1)b_3 \cdots b_{2r-1}] \\ &= \frac{128}{3} b_3 \cdots b_{2r-1} \\ &\geq \frac{16}{9} s_{2r} \frac{(s_1-1)^2}{s_2} \left[\frac{(s_2-1)^2}{s_3}\right]^{2^0} \cdots \left[\frac{(s_r-1)^2}{s_{r+1}}\right]^{2^{r-2}}, \end{aligned}$$

using the formula for $b_{r+1} - 1$ and the lower bounds for b_{r+a} from the proof of Lemma 8.1. Therefore,

$$\begin{aligned} m'_{2r} &= B'_{1,2r,\ldots,r,r+1} \\ &\geq b_1 \cdots b_r (b_{r+1}-1) b_{r+2} \cdots b_{2r-1} b'_{2r} \\ &\geq \frac{16}{9} s_1 \cdots s_{2r} \left[\frac{(s_1-1)^2}{s_2}\right]^{2^0} \cdots \left[\frac{(s_r-1)^2}{s_{r+1}}\right]^{2^{r-1}} \\ &\geq \frac{8}{9}(s_{2r}-1)(2s_{2r}-3)\frac{1}{2}\left[\frac{(s_1-1)^2}{s_2}\right]^{2^0} \cdots \left[\frac{(s_r-1)^2}{s_{r+1}}\right]^{2^{r-1}} \\ &\geq \frac{8}{9\alpha}(s_{2r}-1)(2s_{2r}-3). \end{aligned}$$

We omit the similar argument for $n = 2r+1$. ∎

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Department of Mathematics, Princeton University, Fine Hall, Washington Road, Princeton, NJ 08544-1000
esserl@math.princeton.edu

UCLA Mathematics Department, Box 951555, Los Angeles, CA 90095-1555
totaro@math.ucla.edu