CALABI-YAU VARIETIES OF LARGE INDEX

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1. INTRODUCTION

Call a normal projective variety X *Calabi-Yau* if its canonical divisor K_X is $\mathbb{Q}$-linearly equivalent to zero. The smallest positive integer m with mK_X linearly equivalent to zero is called the *index* of X. A major conjecture on the classification of Calabi-Yau varieties predicts that, under suitable assumptions on singularities, the index is bounded in each dimension. More generally, the following conjecture predicts an index bound for Calabi-Yau pairs (X, D); such a pair consists of a normal projective variety X and an effective $\mathbb{Q}$-divisor D on X such that $K_X + D \sim_{\mathbb{Q}} 0$. Conjectures in this direction go back to Alexeev and McKernan-Prokhorov, and Y. Xu gave a formulation close to what follows [1], [33, Conjecture 3.8], [36].

Conjecture 1.1 (Index Conjecture). *Let n be a positive integer and $I \subset [0, 1]$ a set of rational numbers that satisfies the descending chain condition (DCC). Then there is a positive integer $c(n, I)$ with the following property. Let (X, D) be a complex klt Calabi-Yau pair of dimension n such that the coefficients of D belong to I. Then $c(n, I)(K_X + D) \sim 0$.*

In dimension at most 3, the conjecture is true, and a similar boundedness statement holds even for slc Calabi-Yau pairs [36, Theorem 1.13], [22, Corollary 1.6]. For klt Calabi-Yau pairs (X, D) with $D \neq 0$, the conjecture also holds in dimension 4 [36, Theorem 1.14]. An important result in any dimension is that all the Calabi-Yau pairs in Conjecture 1.1 (with given dimension and coefficients in a given DCC set I) actually have coefficients only in some finite subset of I, by Hacon-McKernan-Xu [20, Theorem 1.5]. Finally, Birkar proved Conjecture 1.1 when X is rationally connected [8, Theorem 1.6].

For several classes of Calabi-Yau varieties or pairs, we construct examples which we conjecture have the largest index in each dimension, with supporting evidence in low dimensions. In our examples, the index grows doubly exponentially with dimension, and may be written in terms of *Sylvester's sequence*, defined recursively by $s_0 = 2$ and $s_n = s_{n-1}(s_{n-1} - 1) + 1$ for $n \geq 1$.

First, we produce terminal Calabi-Yau varieties with large index. The key idea is to apply mirror symmetry (Remark 3.8) to our Calabi-Yau varieties with an ample Weil divisor of small volume [18, Proposition 3.1].

Theorem 1.2 (Corollary 4.1). *For each positive integer n, there is a complex terminal Calabi-Yau n-fold with index $(s_{n-1} - 1)(2s_{n-1} - 3)$. In particular, this is larger than $2^{2^{n-1}}$.*

This example should have the largest possible index among all canonical Calabi-Yau varieties of dimension n, and also among all terminal Calabi-Yau varieties of dimension n (Conjecture 4.2). For example, our construction gives a known Calabi-Yau 3-fold of index 66, the largest possible [32, Corollary 5], and a new terminal Calabi-Yau 4-fold of index 3486.

Next, we turn to klt Calabi-Yau pairs (X, D) with standard coefficients, meaning that all coefficients of D are of the form $1 - \frac{1}{b}$ for positive integers b. Pairs with standard coefficients naturally arise as quotients by finite groups of varieties Y with $\mathbb{Q}$-Cartier canonical class, in the sense that $\pi\colon Y \to X$ has $K_Y = \pi^*(K_X + D)$. For Calabi-Yau pairs, the index can be somewhat bigger than for varieties. (In fact, the example in Theorem 1.2 is based on the same construction used to define the pair below.) Our example has a simple description as a pair on a weighted projective space; it can also be viewed as a hypersurface in an ill-formed fake weighted projective space.

Theorem 1.3 (Theorem 3.3). *For each positive integer n, there is a complex klt Calabi-Yau pair of dimension n with standard coefficients that has index $(s_n - 1)(2s_n - 3)$. In particular, this is larger than 2^{2^n}.*

This should be the largest possible index among such pairs (Conjecture 3.4).

Jihao Liu recently constructed klt Calabi-Yau pairs with small minimal log discrepancy (mld) [31, Remark 2.6]. We write out the details as Theorem 3.1. This is related to several examples by Kollár [25, Example 8.16], [24]. By definition, the klt property for a pair means that the mld is positive. It follows from Hacon-McKernan-Xu's work that there is a positive lower bound for the mld's of all klt Calabi-Yau pairs with standard coefficients in a given dimension (Proposition 2.1). Liu's example is conjectured to achieve the minimum (Conjecture 3.2).

The pairs in Theorem 1.3 and Theorem 3.1 above may be viewed as quotients of certain Calabi-Yau weighted projective hypersurfaces. These are also notable for their extreme topological properties. We prove that the total rank of the orbifold cohomology of these hypersurfaces grows doubly exponentially with dimension.

Theorem 1.4 (Theorem 5.1(ii)). *For every positive integer n, there is a quasi-smooth Calabi-Yau hypersurface of dimension n whose sum of orbifold Betti numbers is*

$$H = 2(s_0 - 1) \cdots (s_n - 1).$$

In particular, this is larger than 2^{2^n} for $n > 1$.

This should be the largest possible sum of orbifold Betti numbers among all projective varieties with quotient singularities and trivial canonical class (Conjecture 5.3).

Finally, we conjecture the largest index for a klt Calabi-Yau variety (rather than a pair) in each dimension (Conjecture 7.10). We prove that conjecture in dimension 2: the largest index of a klt Calabi-Yau surface is 19 (Proposition 6.1). Also, the smallest mld of a klt Calabi-Yau surface is $\frac{1}{13}$.

2. NOTATION AND BACKGROUND

We work over the complex numbers. For the singularities of the minimal model program, such as terminal, canonical, Kawamata log terminal (klt), or log canonical (lc), some introductions are [26, 35].

For the terminology of weighted projective spaces such as well-formedness or quasi-smoothness, see [18, section 2] for a quick summary, or [21]. A *fake weighted projective space* means a projective toric variety whose divisor class group has rank 1, or equivalently the projective toric variety associated to a lattice simplex [23]. Such a variety is the quotient of a weighted projective space by a finite abelian group.

Here are the properties of Sylvester's sequence needed for our examples; a bit more detail is given in [18, section 3]. The sequence is defined by $s_0 = 2$ and $s_n = s_{n-1}(s_{n-1} - 1) + 1$ for $n \geq 1$. The first few terms are 2, 3, 7, 43, 1807. We have $s_n > 2^{2^{n-1}}$ for all n. Also, $s_n = s_0 \cdots s_{n-1} + 1$, and hence the numbers in the sequence are pairwise coprime. Finally, the key point for our applications is that the sum of the reciprocals tends very quickly to 1. Namely:

$$\frac{1}{s_0} + \frac{1}{s_1} + \cdots + \frac{1}{s_{n-1}} = 1 - \frac{1}{s_n - 1}.$$

We recall the definition of the minimal log discrepancy (mld). Let X be a normal variety with a $\mathbb{Q}$-divisor D such that $K_X + D$ is $\mathbb{Q}$-Cartier. For a proper birational morphism $\mu \colon X' \to X$ with X' normal and an irreducible divisor $E \subset X'$, the *log discrepancy* of E with respect to (X, D), written $a_E(X, D)$, is the coefficient of E in the $\mathbb{Q}$-divisor $K_{X'} + E - \mu^*(K_X + D)$. The *center* of E in X is the image $\mu(E)$. The log discrepancy and the center depend only on the valuation defined by E on the function field of X; that allows us to identify some irreducible divisors on different birational models of X. For a pair (X, D) and a point x of the scheme X, the *minimal log discrepancy* of (X, D) at x is the infimum

$$\mathrm{mld}_x(X, D) \coloneqq \inf\{a_E(X, D) \colon \mathrm{center}_X(E) = \overline{x}\},$$

and the (global) *minimal log discrepancy* of (X, D) is

$$\mathrm{mld}(X, D) \coloneqq \inf_{x \in X} \mathrm{mld}_x(X, D).$$

Furthermore, when (X, D) is lc (meaning that its mld is nonnegative), the mld can be computed using the finitely many irreducible divisors that appear on a log resolution of (X, D) (and so it is rational) [28, Definition 7.1].

By definition, a pair (X, D) is klt if and only if its mld is positive. Furthermore, in each dimension, there is a positive lower bound for the mld of every klt Calabi-Yau pair with standard coefficients. More generally:

Proposition 2.1. *Let n be a positive integer and $I \subset [0, 1]$ a DCC set. Then there is a positive number ϵ such that every klt Calabi-Yau pair (X, D) of dimension n with coefficients of D in I has mld at least ϵ.*

Proof. We repeat the argument of [15, Lemma 3.13] for pairs rather than varieties. Suppose by way of contradiction that there is a sequence of klt Calabi-Yau pairs (X_i, D_i) of dimension n with coefficients in I such that the sequence $\epsilon_i := \mathrm{mld}(X_i, D_i)$ converges to 0. Replacing this with a subsequence, we may suppose that the sequence ϵ_i is decreasing and that each term is less than 1. For each positive integer i, define a new klt pair as follows: choose a point $x_i \in X_i$ with $\mathrm{mld}_{x_i}(X_i, D_i) = \epsilon_i$. If the closure Y of x_i has codimension 1 in X_i, then Y has coefficient $1 - \epsilon_i$ in D_i. In this case, let (X_i', D_i') be (X_i, D_i), i.e., leave the original pair unchanged. If the point $x_i \in X_i$ has codimension greater than 1, then choose an exceptional divisor E_i over X with log discrepancy ϵ_i. Since ϵ_i is less than 1, [28, Corollary 1.39] gives that there is a projective birational morphism $\mu \colon X_i' \to X_i$ for which X_i' is $\mathbb{Q}$-factorial and E_i is the only exceptional divisor. Let $D_i' = \mu_*^{-1} D_i + (1 - \epsilon_i) E_i$. Then (X_i', D_i') is a new klt pair which is still Calabi-Yau. Indeed,

$$K_{X_i'} + D_i' = \mu^*(K_{X_i} + D_i) \sim_{\mathbb{Q}} 0.$$

Thus, for every positive integer i, we've constructed a klt Calabi-Yau pair of dimension n that includes the coefficient $1 - \epsilon_i$. The coefficients of these pairs are in the union J of the set I with the numbers $1 - \epsilon_i$. Since J satisfies the descending chain condition, this conclusion contradicts Hacon-McKernan-Xu's result [20, Theorem 1.5]. $\square$

For a klt Calabi-Yau pair (X, D) with standard coefficients, let m be the index of (X, D). Then the (global) *index-1 cover* of (X, D) is a projective variety Y with canonical Gorenstein singularities such that the canonical class K_Y is linearly equivalent to zero [28, Example 2.47, Corollary 2.51]. Here (X, D) is the quotient of Y by an action of the cyclic group μ_m such that μ_m acts faithfully on $H^0(Y, K_Y) \cong \mathbb{C}$. (Explicitly, D has coefficient $1 - \frac{1}{b}$ on the image of an irreducible divisor on which the subgroup of μ_m that acts as the identity has order b.)

Finally, we note the existence of equivariant terminalizations.

Proposition 2.2. *Let a finite group G act on a complex variety X. Then there is a G-equivariant projective birational morphism $\pi \colon Y \to X$ with Y terminal and K_Y nef over X. If X is canonical, then $K_Y = \pi^*(K_X)$.*

Proof. Using a canonical resolution procedure, there is a G-equivariant resolution of singularities $Z \to X$ [27, section 3.4.1]. By the method of Birkar-Cascini-Hacon-McKernan, we can run a G-equivariant minimal model program for Z over X [7], [34, section 4.3]. That gives a G-equivariant birational contraction $Z \dashrightarrow Y$ over X (meaning that $Y \to X$ is a birational morphism) with Y terminal, Y projective over X, and K_Y nef over X. If X is canonical, then $K_Y = \pi^*(K_X) + \sum a_i E_i$ for some rational numbers $a_i \geq 0$. Since K_Y is nef over X, we have $a_i = 0$ for all i by the negativity lemma [26, Lemma 3.39]. $\square$

Without the group action, we can also arrange that Y is $\mathbb{Q}$-factorial, but that is generally impossible in this equivariant setting. A variety Y produced by Proposition 2.2 is "$G\mathbb{Q}$-factorial", meaning that every G-invariant Weil divisor is $\mathbb{Q}$-Cartier; but that does not imply that Y is $\mathbb{Q}$-factorial [34, Example 1.1.1].

3. Calabi-Yau pairs with small mld or large index

In this section, we'll construct a Calabi-Yau pair of large index, Theorem 1.3. We first present Theorem 3.1, Liu's example with small mld [31, Remark 2.6].

Theorem 3.1. *For each positive integer n, there is a complex klt Calabi-Yau pair of dimension n with standard coefficients whose mld is $1/(s_{n+1} - 1)$. In particular, this is less than $1/2^{2^n}$.*

Proof. Kollár constructed a klt pair (X, D) with standard coefficients such that $K_X + D$ is ample and $K_X + D$ has small volume, conjecturally the smallest in each dimension [24], [19, Introduction]. A small change gives Liu's Calabi-Yau pair. Namely, define a pair (X, D) by

$$(X, D) := \left(\mathbb{P}^n, \frac{1}{2}H_0 + \frac{2}{3}H_1 + \frac{6}{7}H_2 + \cdots + \frac{s_n - 1}{s_n}H_n + \frac{s_{n+1} - 2}{s_{n+1} - 1}H_{n+1} \right),$$
(3.1)

where $H_0, \ldots, H_{n+1}$ are $n + 2$ general hyperplanes in $\mathbb{P}^n$. (Here the last coefficient of Kollár's pair has been changed.) Then (X, D) is klt, it has standard coefficients, and $K_X + D$ is $\mathbb{Q}$-linearly equivalent to zero. It is clear from the last coefficient that (X, D) has mld $1/(s_{n+1} - 1)$, as we want. $\square$

Conjecture 3.2. *Let (X, D) be the pair defined in (3.1). Then the mld $1/(s_{n+1} - 1)$ of (X, D) is the smallest possible among all klt Calabi-Yau pairs with standard coefficients of dimension n.*

In dimension 1, the example above is the klt Calabi-Yau pair $(\mathbb{P}^1, \frac{1}{2}H_0 + \frac{2}{3}H_1 + \frac{5}{6}H_2)$, with mld $\frac{1}{6}$, and in dimension 2 we have $(\mathbb{P}^2, \frac{1}{2}H_0 + \frac{2}{3}H_1 + \frac{6}{7}H_2 + \frac{41}{42}H_3)$, with mld $\frac{1}{42}$. Conjecture 3.2 is true in these cases; in dimension 2, we prove this as Proposition 6.8.

The klt Calabi-Yau pair (X, D) above has index $s_{n+1} - 1$, which is quite good (doubly exponential in n); but we now give a better example for that problem.

Namely, we produce a klt Calabi-Yau pair with standard coefficients that has conjecturally maximal index. We first give a simple construction: the properties are easy to check, but the origin of the example is hidden. Our pair (X, D) has index 6 in dimension 1, 66 in dimension 2, and 3486 in dimension 3.

We found this example by applying mirror symmetry (Remark 3.8) to our canonical Calabi-Yau variety of conjecturally minimal volume [18, Proposition 3.1]. With that approach, our pair arises as the quotient of a Calabi-Yau hypersurface in weighted projective space by the action of a cyclic group, or equivalently as a hypersurface in an ill-formed fake weighted projective space. We give those descriptions later in this section.

Theorem 3.3. *For an integer $n \geq 2$, let $d = 2s_n - 2$ and let X be the weighted projective space $\mathbb{P}^n(d^{(n-1)}, d-1, 1)$, with coordinates $y_1, \ldots, y_{n+1}$. For $1 \leq i \leq n$, let D_i be the divisor $\{y_i = 0\}$ on X. Let D_0 be the divisor $\{y_1 + \cdots + y_{n-1} + y_n y_{n+1} + y_{n+1}^d = 0\}$ on X. Let*

$$D = \frac{1}{2}D_0 + \frac{2}{3}D_1 + \cdots + \frac{s_{n-1} - 1}{s_{n-1}}D_{n-1} + \frac{d-2}{d-1}D_n.$$

Then (X, D) is a klt Calabi-Yau pair of dimension n with standard coefficients and with index $(s_n - 1)(2s_n - 3)$.

For $n = 1$, let $X = \mathbb{P}^1$ with 3 distinct complex points p_1, p_2, p_3. Then $(X, \frac{1}{2}p_1 + \frac{2}{3}p_2 + \frac{5}{6}p_3)$ is a klt Calabi-Yau pair with standard coefficients and with index $(s_1 - 1)(2s_1 - 3) = 6$.

Proof. The statement for $n = 1$ is straightforward, so assume that $n \geq 2$. Then X is a well-formed weighted projective space. We have $K_X = O_X(-(n-1)d - (d-1) - 1) = O_X(-nd)$. Also, we have linear equivalences $D_i \sim O_X(d)$ for $0 \leq i \leq n-1$, and $D_n \sim O_X(d-1)$. So $K_X + D$ is $\mathbb{Q}$-linearly equivalent to $O_X(a)$, where

$$a = -nd + d - 2 + \sum_{i=0}^{n-1} d\left(1 - \frac{1}{s_i}\right)$$

$$= -2 + d\left(1 - \sum_{i=0}^{n-1} \frac{1}{s_i}\right)$$

$$= -2 + \frac{d}{s_n - 1}$$

$$= 0.$$

So (X, D) is a Calabi-Yau pair. The coefficients of D are standard.

Since K_X and the divisors D_i are integral multiples of $O_X(1)$ in the divisor class group of X, the index of (X, D) is the least common multiple of the denominators of the coefficients of D. Since the numbers in Sylvester's sequence are pairwise coprime, this lcm is $\mathrm{lcm}(s_n - 1, d - 1) = \mathrm{lcm}(s_n - 1, 2s_n - 3) = (s_n - 1)(2s_n - 3)$, as we want.

It remains to show that (X, D) is klt. Since the klt property is preserved under taking quotients by finite groups which act freely in codimension 1 [28, Corollary 2.43], it suffices to show this in the coordinate chart $y_i = 1$ for each i. Equivalently, it suffices to show that the affine cone $(\mathbb{A}^{n+1}, F)$ over (X, D) is klt outside the origin. Here F is a linear combination of irreducible divisors $F_0, \ldots, F_n$ with coefficients less than 1, and so it suffices to show that $F_0, \ldots, F_n$ are smooth and transverse outside the origin. This is clear for $F_1 = \{y_1 = 0\}, \ldots, F_n = \{y_n = 0\}$.

Next, $F_0 = \{y_1 + \cdots + y_{n-1} + y_n y_{n+1} + y_{n+1}^d = 0\}$ in $\mathbb{A}^{n+1}$. All intersections of subsets of $F_0, \ldots, F_n$ are smooth of the expected dimension except possibly for $F_0 \cap \cdots \cap F_{n-1}$ and $F_0 \cap \cdots \cap F_n$. Here $F_0 \cap \cdots \cap F_{n-1}$ (as a scheme) is the curve $y_n y_{n+1} + y_{n+1}^d = 0$ in $\mathbb{A}^2$, which is smooth outside the origin, as we want. And $F_0 \cap \cdots \cap F_n$ is the origin, as a set. So all intersections are transverse outside the origin in $\mathbb{A}^{n+1}$, and hence (X, D) is klt. $\qquad\square$

This implies Theorem 1.3, using that $s_n > 2^{2^{n-1}}$ for all n. We conjecture that this is the example of largest index.

Conjecture 3.4. *Let (X, D) be the pair in Theorem 3.3. Then the index $(s_n - 1)(2s_n - 3)$ of (X, D) is the largest possible index among all klt Calabi-Yau pairs of dimension n with standard coefficients.*

Proposition 3.5. *Conjecture 3.4 is true in dimensions at most 2. Moreover, there is exactly one klt Calabi-Yau pair with standard coefficients that has index 6 in dimension 1 (up to isomorphism), and there are exactly two (related by a blow-up) of index 66 in dimension 2.*

Proof. This is elementary in dimension 1. (The index-1 cover of (X, D) is the unique elliptic curve over $\mathbb{C}$ with automorphism group of order 6.) So let (X, D) be a klt Calabi-Yau pair of dimension 2 with standard coefficients and index m. Let Y be the index-1 cover of (X, D) (section 2). Thus Y is a projective surface with canonical singularities such that K_Y is trivial. Here (X, D) is the quotient of Y by an action of the cyclic group μ_m which is purely non-symplectic, meaning that μ_m acts faithfully on $H^0(Y, K_Y) \cong \mathbb{C}$.

Let Z be the minimal resolution of Y; then K_Z is trivial, and so Z is an abelian surface or a K3 surface. The action of μ_m lifts to Z by Proposition 2.2. Clearly the action of μ_m on Z is still purely non-symplectic. When Z is a K3 surface, Nikulin showed that $m \leq 66$, and Machida and Oguiso showed that for $m = 66$, Z and the action of μ_{66} are unique up to isomorphism and automorphisms of μ_{66} [32, Main Theorem 1]. When Z is an abelian surface, μ_m acts faithfully on $H^0(Z, K_Z) \subset H^2(Z, \mathbb{C}) = \Lambda^2 H^1(Z, \mathbb{C})$, so it acts faithfully on $H^1(Z, \mathbb{Z}) \cong \mathbb{Z}^4$, which implies that $m \leq 12$.

There remains the problem of classifying the contractions $Z \to Y$ of the K3 surface Z above. Machida and Oguiso show that Z has Picard rank 2, with cone of curves spanned by one (-2)-curve C and one elliptic curve with self-intersection zero [32, section 2]. It follows that C is the only (-2)-curve in Z.

So the only canonical K3 surfaces with minimal resolution Z are Z itself and the surface Y with a node obtained by contracting C. So a klt Calabi-Yau pair with standard coefficients, dimension 2, and index 66 is isomorphic to Z/μ_{66} or to Y/μ_{66}. The surface X of Theorem 3.3 is the latter one, since its index-1 cover (Proposition 3.6) has a node.   □

We now describe the example of Theorem 3.3 as a quotient of a hypersurface in weighted projective space.

Proposition 3.6. *For a positive integer n, let $d := 2s_n - 2 = 2s_0 \cdots s_{n-1}$. Then the hypersurface X' of degree d in the weighted projective space $Y' := \mathbb{P}(d/s_0, \ldots, d/s_{n-1}, 1, 1)$ defined by the equation $x_0^2 + x_1^3 + \cdots + x_{n-1}^{s_{n-1}} + x_n^{d-1}x_{n+1} + x_{n+1}^d = 0$ is quasi-smooth of dimension n, canonical, and has $K_{X'}$ linearly equivalent to zero.*

Proof. Since the weighted projective space Y' has two weights equal to 1, it is well-formed. One can verify that the hypersurface X' is quasi-smooth, meaning that its affine cone is smooth outside the origin in $\mathbb{A}^{n+2}$. The sum of the weights is $d/s_0 + \cdots + d/s_{n-1} + 2 = d(1 - 1/(s_n - 1)) + 2 = d$, and so $K_{X'}$ is linearly equivalent to zero. Since X' is quasi-smooth, it has quotient singularities, hence is klt. Since $K_{X'}$ is linearly equivalent to zero, it is Cartier. It follows that X' is canonical.   □

Proposition 3.7. *With the same notation as above, let $m := (s_n - 1)(2s_n - 3)$ and let the group μ_m of mth roots of unity act on the weighted projective space Y' as follows. For any $\zeta \in \mu_m$,*

$$\zeta[x_0 : \cdots : x_{n+1}] = [\zeta^{d/(2s_0)}x_0 : \zeta^{d/(2s_1)}x_1 : \cdots : \zeta^{d/(2s_{n-1})}x_{n-1} : x_n : \zeta^{d/2}x_{n+1}].$$
$$(3.2)$$

The hypersurface X' is invariant under this group action. The quotient of X' by μ_m is a klt Calabi-Yau pair (X, D) with standard coefficients and index m, isomorphic to the pair in Theorem 3.3.

We omit the proof, since this is just a different description of the same example.

Remark 3.8.

(1) The index $m := (s_n - 1)(2s_n - 3) = (d - 1)d/2$ in this example coincides with the degree of the hypersurface which we conjecture is the canonical Calabi-Yau n-fold with an ample Weil divisor of minimum volume [18, Conjecture 1.2]. Indeed, the hypersurface defined in Proposition 3.6 is related to the small volume example by mirror symmetry. Specifically, the hypersurface X' in Proposition 3.6 above is the Berglund-Hübsch-Krawitz (BHK) mirror of the hypersurface

$$\widehat{X'_m} \subset \mathbb{P}(m/s_0, \ldots, m/s_{n-1}, s_n - 1, s_n - 2)$$

of degree m defined by the equation $x_0^2 + x_1^3 + \cdots + x_{n-1}^{s_{n-1}} + x_n^{d-1} + x_n x_{n+1}^d = 0$. That is, the equations for X' and $\widehat{X'}$, each with exactly

$n + 2$ monomials in $n + 2$ variables, are the transposes of each other, in terms of the explicit BHK recipe for mirror symmetry [2, Section 2.3].

(2) Section 5 describes the extreme topological behavior of these Calabi-Yau hypersurfaces.

(3) Another striking feature of Proposition 3.6 is that the weighted projective space $\mathbb{P}(d/s_0, \ldots, d/s_{n-1}, 1, 1)$ has the largest anticanonical volume of any canonical toric Fano $(n + 1)$-fold [5, Corollary 1.3].

For possible future use, we give one last description of the klt Calabi-Yau pair (X, D) of Theorem 3.3, now as a hypersurface in an ill-formed fake weighted projective space. (Ill-formedness means that we have a toric pair, not just a toric variety.) It is convenient to use the language of toric Deligne-Mumford stacks [10]. Just as there is a projective $\mathbb{Q}$-factorial toric variety associated to a complete simplicial fan in a lattice N, we can associate a toric Deligne-Mumford stack to the data of a *stacky fan* $\boldsymbol{\Sigma}$. A stacky fan $\boldsymbol{\Sigma}$ is a triple (N, Σ, β), where N is a finitely generated abelian group, Σ is a complete simplicial fan in $N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$ with r rays, and β is a set of r elements of N that span the rays of Σ. (They need not be the smallest lattice points on these rays.) In our case, $N \cong \mathbb{Z}^{n+1}$ is torsion-free, which means that the associated stack has trivial generic stabilizer. If the chosen point of a given ray ρ is a positive integer b times the smallest lattice point, then the associated toric stack has stabilizer group μ_b along the irreducible divisor D associated to ρ, and the associated pair has coefficient $(b - 1)/b$ along D.

Define a stacky fan $\boldsymbol{\Sigma}$ using the following distinguished points generating rays of a simplicial fan in $N \cong \mathbb{Z}^{n+1}$, where $e_0, \ldots, e_n$ is a basis for N:

$$v_i = \begin{cases} s_i e_i, & i = 0, \ldots, n-1, \\ (d-1)e_n, & i = n, \\ -d(e_0 + \cdots + e_{n-1}) - (d-1)e_n, & i = n+1. \end{cases}$$

Most of these lattice points are not primitive on the respective rays, hence the stacky behavior. Note also that $(d/s_0)v_0 + \cdots (d/s_{n-1})v_{n-1} + v_n + v_{n+1} = 0$. Therefore, these vectors define the weighted projective space $\mathbb{P}(d/s_0, \ldots, d/s_{n-1}, 1, 1)$ in the lattice spanned by $v_0, \ldots, v_{n+1}$. The resulting stack will thus be a quotient of this weighted projective space, namely the quotient in Proposition 3.7. The pair (X, D) is the pair associated to a hypersurface in this stack.

4. Terminal Calabi-Yau varieties of large index

Building on the klt Calabi-Yau pair of Theorem 3.3, we now construct terminal Calabi-Yau varieties with large index, conjecturally optimal.

Corollary 4.1. *Let n be an integer at least 2. Let X' be the Calabi-Yau hypersurface of dimension $n - 1$ with an action of the cyclic group of order $m := (s_{n-1} - 1)(2s_{n-1} - 3)$ from Proposition 3.7. Let Z be a μ_m-equivariant*

terminalization of X' (Proposition 2.2). Let E be a smooth elliptic curve, and let $p \in E(\mathbb{C})$ be a point of order m. Let μ_m act on $Z \times E$ by

$$\zeta(x, y) = (\zeta(x), y + p),$$

where ζ is a primitive mth root of unity. Then the quotient $S = (Z \times E)/\mu_m$ is a terminal Calabi-Yau n-fold that has index m.

Proof. The variety X' is canonical, by Proposition 3.6. So $K_Z = \pi^*(K_{X'})$ by Proposition 2.2, and hence Z is a terminal projective variety with K_Z trivial. Since E is a complex elliptic curve, it has a point p of order m. Translation by an non-identity point has no fixed points on E, so μ_m acts freely on E. It follows that μ_m acts freely on $Z \times E$, and so the quotient $S = (Z \times E)/\mu_m$ is a Calabi-Yau variety (rather than a pair). Since Z is terminal and μ_m acts freely on $Z \times E$, the quotient $S = (Z \times E)/\mu_m$ is also terminal.

It remains to see that S has index m. The group μ_m acts trivially on $H^0(E, K_E)$ since global sections of K_E are translation-invariant. Therefore, an element of μ_m acts trivially on $H^0(Z \times E, K_{Z \times E}) = H^0(Z, K_Z) \otimes_{\mathbb{C}} H^0(E, K_E)$ if and only if it acts trivially on $H^0(Z, K_Z)$. So the index of S is equal to the index of the quotient stack $[Z/\mu_m]$ (or the associated klt pair), namely m. $\square$

We conjecture that this example is optimal.

Conjecture 4.2. *Let n be an integer at least 2 and let X be the Calabi-Yau variety of Corollary 4.1. Then the index $m = (s_{n-1} - 1)(2s_{n-1} - 3)$ of X is the largest possible index among all terminal Calabi-Yau varieties of dimension n, and also among all canonical Calabi-Yau varieties of dimension n.*

This conjecture holds in dimensions at most 3. Indeed, in dimension 2, terminal singularities are smooth, and the largest possible index of a smooth Calabi-Yau surface is $6 = (s_1 - 1)(2s_1 - 3)$, which occurs for a "bielliptic" surface $(W \times E)/\mu_6$ as above [6, Corollary VIII.7]. Likewise, the largest possible index of a canonical or terminal Calabi-Yau 3-fold is $66 = (s_2 - 1)(2s_2 - 3)$ [32, Corollary 5].

For n at most 3, there is in fact a *smooth* Calabi-Yau n-fold with index $(s_{n-1} - 1)(2s_{n-1} - 3)$ [32, Corollary 5]. To see this in dimension 3, let X' be the canonical Calabi-Yau surface from Proposition 3.7 with an action of μ_{66}. Let $Z \to X'$ be a μ_{66}-equivariant terminalization of X' (Proposition 2.2). In dimension 2, Z is smooth and unique, known as the minimal resolution of X'. Also, $K_Z = \pi^*(K_{X'})$, and so K_Z is trivial (more precisely, Z is a smooth K3 surface). Then the quotient $(Z \times E)/\mu_{66}$ as in Corollary 4.1 is a smooth Calabi-Yau 3-fold of index 66. It would be interesting to construct smooth Calabi-Yau varieties of large index in high dimensions.
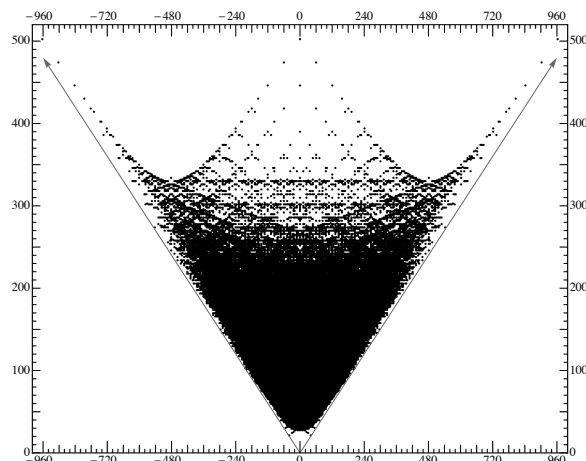
FIGURE 1. *Hodge numbers of the Calabi-Yau 3-folds found by Kreuzer and Skarke; x-axis is $\chi = 2(h^{1,1} - h^{2,1})$, y-axis is $h^{1,1} + h^{2,1}$.*

5. CALABI-YAU VARIETIES WITH LARGE BETTI NUMBERS

The Calabi-Yau hypersurfaces in section 3 are also notable for their topological properties. In dimension 3, the family of hypersurfaces $X'_{84} \subset \mathbb{P}^4(42, 28, 12, 1, 1)$ of Proposition 3.6 is an extreme in Kreuzer and Skarke's famous list of Calabi-Yau 3-folds [30, section 3]. (They found more than 400 million canonical 3-folds with trivial canonical class.) In particular, a crepant resolution of X' has the most negative known Euler characteristic, -960. (The special hypersurface in Proposition 3.6 with an action of μ_{3486} seems to be new, though.) The BHK mirror to the special hypersurface X' is the small-volume example

$$\widehat{X'_{3486}} \subset \mathbb{P}^4(1743, 1162, 498, 42, 41).$$

That has the largest known Euler characteristic among K-trivial 3-folds, 960 (for a crepant resolution).

A third hypersurface, $X'_{1806} \subset \mathbb{P}^4(903, 602, 258, 42, 1)$, has the same sum of Betti numbers (of a crepant resolution) as in the previous two examples, namely 1008. All three of these hypersurfaces are visible at the top of the graph of all pairs of Hodge numbers from Kreuzer and Skarke's list, shown in Figure 1 [14]. In this section, we'll show that these observations in dimension 3 fit into a pattern that holds for all dimensions n.

In higher dimensions, we do not know whether our weighted projective hypersurfaces have crepant resolutions. Instead, we'll consider Chen-Ruan's *orbifold cohomology* for these hypersurfaces [16]. For a projective variety with Gorenstein quotient singularities, orbifold cohomology (as a Hodge structure) agrees with the cohomology of a crepant resolution if one exists [37, Corollary 1.5]. Assuming Gorenstein quotient singularities, the orbifold Hodge numbers also agree with Batyrev's *stringy Hodge numbers* [37, Remark 1.4].

We'll make use of the following formula for orbifold Hodge numbers, building on Vafa's ideas (see [4, (4.2)]). Given a quasi-smooth hypersurface X of degree d in $\mathbb{P}(a_0, \ldots, a_{n+1})$, where $a_0 + \cdots + a_{n+1} = d$, we write: $q_i := a_i/d$ and $\tilde{\theta}_i(\ell)$ is the fractional part of $\ell a_i/d$. Then the orbifold Hodge number $h_{\mathrm{orb}}^{p,q}$ of the mirror of X is the coefficient of $t^p \bar{t}^q$ in the expression

$$P(t, \bar{t}) = \sum_{\ell=0}^{d-1} \left[\prod_{\tilde{\theta}_i(\ell)=0} \frac{1 - (t\bar{t})^{1-q_i}}{1 - (t\bar{t})^{q_i}} \prod_{\tilde{\theta}_i(\ell)\neq 0} (t\bar{t})^{\frac{1}{2}-q_i} \left(\frac{t}{\bar{t}} \right)^{\tilde{\theta}_i(\ell)-\frac{1}{2}} \right]. \qquad (5.1)$$

In [4], $P(t, \bar{t})$ is defined as the part of the above expression with integer exponents, but it will be useful to keep track of the parts with fractional exponents as well.

For each dimension n, there are three Calabi-Yau hypersurfaces which we expect to have the "most extreme cohomology":

$$X_1^{(n)} := \{x_0^{s_0} + x_1^{s_1} + \cdots + x_{n-1}^{s_{n-1}} + x_n^{d-1} x_{n+1} + x_{n+1}^d = 0\}$$
$$\subset \mathbb{P}(d/s_0, \ldots, d/s_{n-1}, 1, 1) \text{ of degree } d = 2s_n - 2,$$

$$X_2^{(n)} := \{x_0^{s_0} + \cdots + x_n^{s_n} + x_{n+1}^d = 0\}$$
$$\subset \mathbb{P}(d/s_0, \ldots, d/s_n, 1) \text{ of degree } d = s_{n+1} - 1, \text{ and}$$

$$X_3^{(n)} := \{x_0^{s_0} + x_1^{s_1} + \cdots + x_{n-1}^{s_{n-1}} + x_n^{2s_n-3} + x_n x_{n+1}^{2s_n-2} = 0\}$$
$$\subset \mathbb{P}(d/s_0, \ldots, d/s_{n-1}, s_n - 1, s_n - 2) \text{ of degree } d = (s_n - 1)(2s_n - 3).$$

Note that any quasi-smooth hypersurface with the same degree and weights as one of the examples above would have identical cohomology, but we pick specific hypersurfaces to highlight the connection to previous sections. The first and third examples form a mirror pair and are the source of our large index and small volume results, respectively (see Remark 3.8). Hypersurfaces of degree $d = s_{n+1} - 1$ in $\mathbb{P}(d/s_0, \ldots, d/s_n, 1)$ first appeared in [18, Lemma 3.10]. The specific Fermat hypersurface $X_2^{(n)}$ is self-mirror and can be viewed as the source of Liu's conjecturally optimal small mld example. Indeed, the pair (X, D) in Theorem 3.1 is the quotient of this hypersurface by an action of $\mu_{s_{n+1}-1}$.

Theorem 5.1. *Let n be a positive integer. Then the following properties hold:*

(i) $h_{\mathrm{orb}}^{p,q}(X_i^{(n)}) = 0$ *for $i = 1, 2, 3$ unless $p = q$ or $p + q = n$.*

(ii) *The sum of the orbifold Betti numbers of $X_i^{(n)}$ for $i = 1, 2, 3$ is*

$$H = 2(s_0 - 1)\cdots(s_n - 1).$$

This equals the orbifold Euler characteristic of X if n is even.

(iii) *Suppose that n is odd. Then the dimension of the middle orbifold cohomology group is*

$$\dim(H_{\mathrm{orb}}^n(X_i^{(n)}, \mathbb{Q})) = \begin{cases} (s_0 - 1)\cdots(s_{n-1} - 1)(2s_n - 4), & i = 1, \\ (s_0 - 1)\cdots(s_n - 1), & i = 2, \\ 2(s_0 - 1)\cdots(s_{n-1} - 1), & i = 3. \end{cases}$$
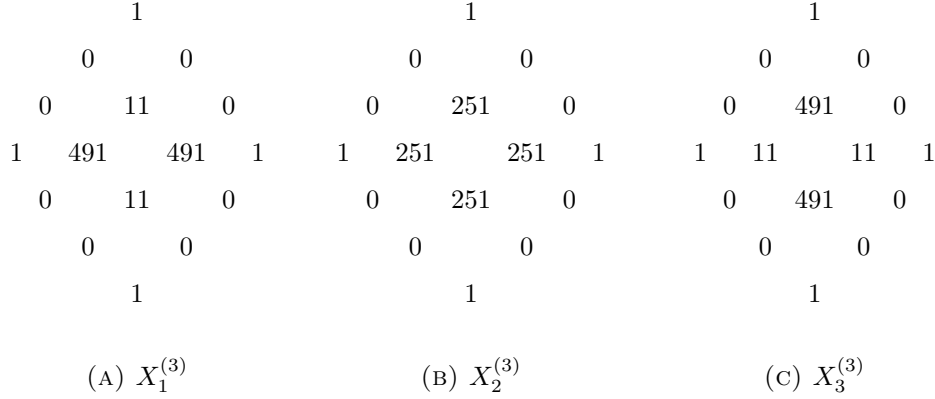
$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 11 & & 0 & \\
1 & & 491 & & 491 & & 1 \\
 & 0 & & 11 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 251 & & 0 & \\
1 & & 251 & & 251 & & 1 \\
 & 0 & & 251 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 491 & & 0 & \\
1 & & 11 & & 11 & & 1 \\
 & 0 & & 491 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

(A) $X_1^{(3)}$ (B) $X_2^{(3)}$ (C) $X_3^{(3)}$

FIGURE 2. The orbifold Hodge diamonds of three extreme Calabi-Yau 3-folds.

The orbifold Euler characteristic is

$$
\chi_{\mathrm{orb}}(X_i^{(n)}) = \begin{cases} -(s_0 - 1)\cdots(s_{n-1} - 1)(2s_n - 6), & i = 1, \\ 0, & i = 2, \\ (s_0 - 1)\cdots(s_{n-1} - 1)(2s_n - 6), & i = 3. \end{cases}
$$

Remark 5.2. The proof will actually show a slight generalization of (i): the stated vanishing of orbifold Hodge numbers holds for any quasi-smooth Calabi-Yau weighted hypersurface $X_d \subset \mathbb{P}(a_0, \ldots, a_{n+1})$ satisfying the conditions that a) each a_i divides d and b) the quotients d/a_i that are strictly smaller than d are pairwise coprime. Mirror symmetry preserves the vanishing, so it will also hold for mirrors of such hypersurfaces, e.g., $X_3^{(n)}$. However, it does not hold in general. A simple counterexample is the fourfold $X_{12} \subset \mathbb{P}(3^{(3)}, 1^{(3)})$, which has $h^{1,2} = 3$.

The sum $H = 1008$ of Betti numbers common to our three examples in dimension 3 is the largest known among all Calabi-Yau threefolds (a crepant resolution always exists in this case). The situation in dimension 3 motivates the following conjecture:

Conjecture 5.3. *In dimension n, the largest possible sum of orbifold Betti numbers for a projective variety with quotient singularities and trivial canonical class is*

$$
H = 2(s_0 - 1)\cdots(s_n - 1).
$$

For odd n, the smallest possible orbifold Euler characteristic is $-(s_0-1)\cdots(s_{n-1}-1)(2s_n - 6)$, and the largest is $(s_0 - 1)\cdots(s_{n-1} - 1)(2s_n - 6)$.

The individual orbifold Hodge numbers are harder to compute, but the values for our examples in dimensions 3 and 4 are shown in Figure 2 and Figure 3, respectively.
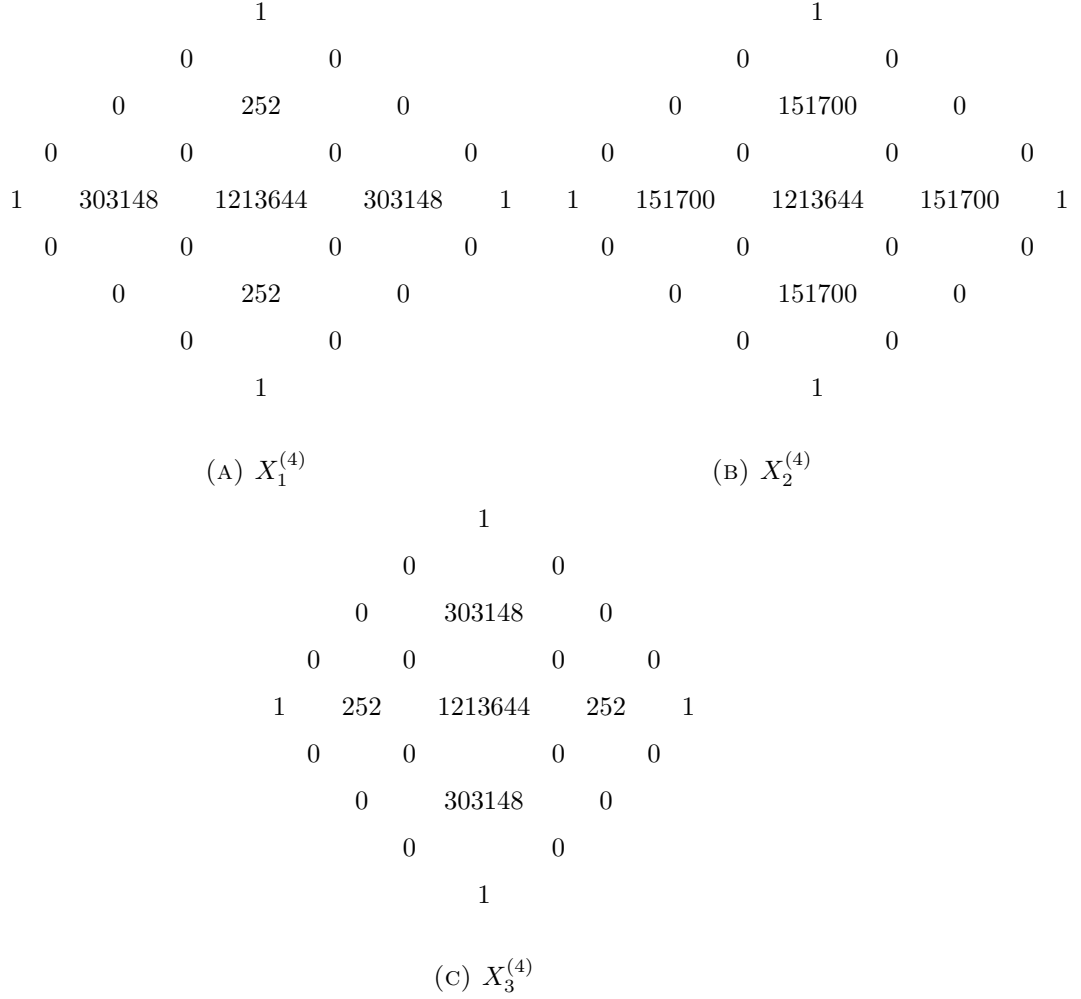
$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 252 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
1 & & 303148 & & 1213644 & & 303148 & & 1 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 252 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}
$$

(A) $X_1^{(4)}$

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 151700 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
1 & & 151700 & & 1213644 & & 151700 & & 1 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 151700 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}
$$

(B) $X_2^{(4)}$

$$
\begin{array}{ccccc}
 & & 1 & & \\
 & 0 & & 0 & \\
0 & & 303148 & & 0 \\
\end{array}
$$

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 303148 & & 0 & \\
0 & & 0 & & 0 & & 0 \\
1 & 252 & & 1213644 & & 252 & 1 \\
0 & & 0 & & 0 & & 0 \\
 & 0 & & 303148 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

(C) $X_3^{(4)}$

FIGURE 3. The orbifold Hodge diamonds of three extreme Calabi-Yau 4-folds.

Proof of Theorem 5.1. The properties of $X_1^{(n)}$ will imply those for $X_3^{(n)}$ by mirror symmetry [17, Theorem 4]; so we'll focus on the first two examples. For the moment, let X denote any quasi-smooth hypersurface with the property that all weights a_i divide the degree d.

First, we'll make the substitutions $t = x^d$, $\bar{t} = y^d$ to eliminate fractional exponents in (5.1) and define a polynomial

$$
Q(x,y) := P(x^d, y^d) = \sum_{\ell=0}^{d-1} \left[\prod_{\tilde{\theta}_i(\ell)=0} \frac{1 - (xy)^{d-a_i}}{1 - (xy)^{a_i}} \prod_{\tilde{\theta}_i(\ell)\neq0} (xy)^{\frac{d}{2}-a_i} \left(\frac{x}{y}\right)^{d\tilde{\theta}_i(\ell)-\frac{d}{2}} \right].
$$

Write

$$
Q_\ell(x,y) := \prod_{\tilde{\theta}_i(\ell)=0} \frac{1 - (xy)^{d-a_i}}{1 - (xy)^{a_i}} \prod_{\tilde{\theta}_i(\ell)\neq0} (xy)^{\frac{d}{2}-a_i} \left(\frac{x}{y}\right)^{d\tilde{\theta}_i(\ell)-\frac{d}{2}} \tag{5.2}
$$

for the ℓth term in the sum.

Since each a_i divides d, we have that
$$\frac{1 - (xy)^{d-a_i}}{1 - (xy)^{a_i}} = 1 + (xy)^{a_i} + \cdots + (xy)^{d-2a_i}.$$

The orbifold Hodge numbers of X correspond to coefficients of monomials in $Q(x, y)$ of the form $x^a y^b$ with both a and b divisible by d. First, we claim that for any monomial $x^a y^b$ with a nonzero coefficient in $Q(x, y)$, d divides a if and only if d divides b. This is because $Q(x, y)$ is a polynomial in xy with the exception of the term $\left(\frac{x}{y}\right)^{d\tilde{\theta}_i(\ell) - \frac{d}{2}}$. The difference of the exponents of x and y in this expression is $2d\tilde{\theta}_i(\ell) - d$. But for any fixed ℓ, we have
$$\sum_{i, \tilde{\theta}_i(\ell) \neq 0} (2d\tilde{\theta}_i(\ell) - d) \equiv \sum_{i, d \nmid \ell a_i} (2\ell a_i - d) \equiv \ell \sum_i 2a_i \equiv 0 \bmod d.$$

Here the second-to-last congruence holds because adding the terms $2\ell a_i$ for which $d | \ell a_i$ does not change the value modulo d (removing extraneous d terms also, of course, does not change this value). Therefore, each summand $Q_\ell(x, y)$ is a polynomial in xy multiplied by a monomial with the powers of x and y differing by a multiple of d. This proves the claim. In particular, the sum of all orbifold Hodge numbers is the same as the sum of coefficients of powers of x^d in
$$q(x) := Q(x, 1) = \sum_{\ell=0}^{d-1} \left[\prod_{\tilde{\theta}_i(\ell)=0} (1 + x^{a_i} + \cdots + x^{d-2a_i}) \prod_{\tilde{\theta}_i(\ell) \neq 0} x^{d\tilde{\theta}_i(\ell) - a_i} \right].$$

To isolate powers of x^d, we can sum the values of this polynomial over dth roots of unity. More precisely, let H be the sum of Hodge numbers and ζ a primitive dth root of unity. Then:
$$H = \frac{1}{d} \sum_{j=0}^{d-1} q(\zeta^j) = \sum_{j=0}^{d-1} \sum_{\ell=0}^{d-1} \left[\prod_{\tilde{\theta}_i(\ell)=0} (1 + \zeta^{ja_i} + \cdots + \zeta^{j(d-2a_i)}) \prod_{\tilde{\theta}_i(\ell) \neq 0} \zeta^{j(d\tilde{\theta}_i(\ell) - a_i)} \right].$$

We may simplify this further by noticing, as before, that the sum of all the $d\tilde{\theta}_i(\ell)$ is divisible by d for fixed ℓ. We can therefore drop the $d\tilde{\theta}_i(\ell)$ term in the last exponent. Further, since the sum of the a_i is d, we have
$$\prod_{\tilde{\theta}_i(\ell) \neq 0} \zeta^{j(-a_i)} = \prod_{\tilde{\theta}_i(\ell)=0} \zeta^{ja_i}.$$

After making this substitution, both products are now indexed over the weights a_i for which $\tilde{\theta}_i(\ell) = 0$ (i.e., for which $d | \ell a_i$). Thus, we may combine them into one:
$$H = \frac{1}{d} \sum_{j=0}^{d-1} \sum_{\ell=0}^{d-1} \prod_{i, d | \ell a_i} (\zeta^{ja_i} + \zeta^{j(2a_i)} + \cdots + \zeta^{j(d-a_i)}).$$

The quantity $\zeta^{ja_i} + \zeta^{j(2a_i)} + \cdots + \zeta^{j(d-a_i)}$ equals -1 if $d \nmid ja_i$ and equals $\frac{d}{a_i} - 1$ if $d|ja_i$, since all terms will be equal to 1 in this case. We'll switch sums and write this as

$$H = \frac{1}{d} \sum_{\ell=0}^{d-1} \sum_{j=0}^{d-1} \prod_{i,d|\ell a_i} \begin{cases} \frac{d}{a_i} - 1, & d|ja_i, \\ -1, & d \nmid ja_i \end{cases}.$$

Given a positive integer c, we'll use the notation

$$f_c(j) := \begin{cases} c - 1, & c|j, \\ -1, & c \nmid j \end{cases}.$$

In this notation,

$$H = \frac{1}{d} \sum_{\ell=0}^{d-1} \sum_{j=0}^{d-1} \prod_{i,d|\ell a_i} f_{\frac{d}{a_i}}(j). \tag{5.3}$$

The same reasoning shows that the contribution to H coming from $Q_\ell(x,y)$ is $\frac{1}{d} \sum_{j=0}^{d-1} \prod_{i,d|\ell a_i} f_{\frac{d}{a_i}}(j)$. The following lemma will help to evaluate such sums.

Lemma 5.4. *Let C be a nonempty finite set of pairwise coprime positive integers and d a positive integer divisible by every element of C. Then*

$$\sum_{j=0}^{d-1} \prod_{c \in C} f_c(j) = 0.$$

Proof. We'll prove the statement by induction on the size of the set C. Suppose for the base case that $C = \{c_1\}$. Then there are $\frac{d}{c_1}$ values of j divisible by c and $d - \frac{d}{c_1}$ which aren't. The expression in the lemma therefore equals

$$\frac{d}{c_1}(c_1 - 1) + \left(d - \frac{d}{c_1}\right)(-1) = d - \frac{d}{c_1} - d + \frac{d}{c_1} = 0.$$

Now, suppose that the statement is true for sets of size m and set $C = \{c_1, \ldots, c_m, c_{m+1}\}$. Let d be an integer divisible by each element of C. Then we can break up the expression into two pieces based on whether c_{m+1} divides j:

$$\sum_{j=0}^{d-1} \prod_{i=1}^{m+1} f_{c_i}(j) = \sum_{\substack{1 \le j < d, \\ c_{m+1}|j}} \prod_{i=1}^{m+1} f_{c_i}(j) + \sum_{\substack{1 \le j < d, \\ c_{m+1} \nmid j}} \prod_{i=1}^{m+1} f_{c_i}(j).$$

Next, we may factor the $i = m + 1$ term out of each product to obtain:

$$(c_{m+1}-1) \sum_{\substack{1 \le j < d, \\ c_{m+1}|j}} \prod_{i=1}^{m} f_{c_i}(j) + (-1) \sum_{\substack{1 \le j < d, \\ c_{m+1} \nmid j}} \prod_{i=1}^{m} f_{c_i}(j) = c_{m+1} \sum_{\substack{1 \le j < d, \\ c_{m+1}|j}} \prod_{i=1}^{m} f_{c_i}(j) - \sum_{j=0}^{d-1} \prod_{i=1}^{m} f_{c_i}(j).$$

In the sum over multiples of c_{m+1}, we can replace each j by j/c_{m+1} without changing the value of the other f_{c_i} because the c_i are pairwise coprime. Therefore, the last expression may be rewritten:

$$c_{m+1} \sum_{k=0}^{\frac{d}{c_{m+1}}-1} \prod_{i=1}^{m} f_{c_i}(k) - \sum_{j=0}^{d-1} \prod_{i=1}^{m} f_{c_i}(j).$$

Since both d and d/c_{m+1} are positive integers divisible by each element of the pairwise coprime set $\{c_1, \ldots, c_m\}$, the two sums above both equal zero by the inductive hypothesis. This completes the proof. $\qquad\square$

Using this lemma, we prove the properties of Theorem 5.1. Let

$$S_\ell := \sum_{j=0}^{d-1} \prod_{i,d|\ell a_i} f_{\frac{d}{a_i}}(j)$$

be the ℓth term of the sum (5.3), so that $H = \frac{1}{d}\sum_{j=0}^{d-1} S_\ell$. Notice that the product is indexed by a set depending only on ℓ (and not j). The following lemma will show that S_ℓ is equal to zero for many values of ℓ in our examples.

Lemma 5.5. *Let X be a quasi-smooth Calabi-Yau hypersurface of degree d with weights $a_0, \ldots, a_{n+1}$ all dividing d. Suppose that the set $C := \{d/a_i : a_i \neq 1\}$ is pairwise coprime. Then in the notation above: if $\ell \neq 0$ and c divides ℓ for some $c \in C$, then $S_\ell = 0$.*

Proof. Let $c_i = d/a_i$ for every element $d/a_i \in C$. Assuming that $\ell \neq 0$, we have $d \nmid \ell$ so the sum S_ℓ may be written

$$S_\ell = \sum_{j=0}^{d-1} \prod_{i,d|\ell a_i} f_{\frac{d}{a_i}}(j) = \sum_{j=0}^{d-1} \prod_{c_i \in C, c_i | \ell} f_{c_i}(j).$$

Any subset of C is pairwise coprime and all elements of C divide d, so by Lemma 5.4, $S_\ell = 0$ whenever the product is nonempty. This occurs precisely when some c_i divides ℓ. $\qquad\square$

Let X satisfy the condition of Lemma 5.5. (For instance, $X = X_2^{(n)}$ or $X_1^{(n)}$; in these cases, $C = \{s_0, \ldots, s_n\}$ or $C = \{s_0, \ldots, s_{n-1}\}$, respectively.) Then, S_ℓ is only nonzero if: a) $\ell = 0$, in which case the second product in $Q_\ell(x, y)$ is empty, or b) no s_i divides ℓ, in which case the first product in $Q_\ell(x, y)$ is empty. In case a), Q_0 is a polynomial in xy. In case b), $Q_\ell(x, y)$ is a monomial $x^a y^b$ with

$$a + b = 2\sum_{i=0}^{n+1} \left(\frac{d}{2} - a_i\right) = 2\left(\frac{(n+2)d}{2} - d\right) = nd.$$

Therefore, the only orbifold Hodge numbers $h_{\text{orb}}^{p,q}$ of the mirror of X that could be nonzero satisfy $p = q$ or $p + q = n$. Taking the mirror just switches these two types of Hodge numbers, so the vanishing in Theorem 5.1(i) holds for both X and

its mirror. This is enough to prove the first part of the theorem. We also note that, in odd dimensions, all contributions to the middle orbifold cohomology of X (i.e., to Hodge numbers $h^{p,p}$ of the mirror) come from $Q_0(x, y)$.

To prove parts (ii) and (iii) of Theorem 5.1, we'll consider $X_2^{(n)}$ and $X_1^{(n)}$ separately:

Proof for $X_2^{(n)}$: (hypersurface of degree $d = s_{n+1} - 1$ in $\mathbb{P}(d/s_0, \ldots, d/s_n, 1)$)

The sum of all orbifold Betti numbers (equivalently, the sum of all orbifold Hodge numbers) is $H = \frac{1}{d}(A + B)$, where $A := S_0$ and B is the sum of all S_ℓ such that $s_i \nmid \ell$, $i = 0, \ldots n$. First, we compute

$$A = S_0 = \sum_{j=0}^{d-1} f_{s_0}(j) \cdots f_{s_n}(j) f_d(j)$$

$$= (s_0 - 1) \cdots (s_n - 1)(d - 1) + (-1) \sum_{j=1}^{d-1} f_{s_0}(j) \cdots f_{s_n}(j).$$

Here S_0 is divided into two pieces by explicitly evaluating the function f_d. By Lemma 5.4,

$$\sum_{j=1}^{d-1} f_{s_0}(j) \cdots f_{s_n}(j) = -f_{s_0}(0) \cdots f_{s_n}(0) = -(s_0 - 1) \cdots (s_n - 1).$$

Therefore, A simplifies to $d(s_0 - 1) \cdots (s_n - 1)$. The number of values ℓ summed in B is

$$d\left(1 - \frac{1}{s_0}\right) \cdots \left(1 - \frac{1}{s_n}\right) = (s_0 - 1) \cdots (s_n - 1).$$

Since the product in S_ℓ is empty when no s_i divides ℓ, each S_ℓ term in B equals d; the total is $B = d(s_0 - 1) \cdots (s_n - 1)$. Therefore, $H = \frac{1}{d}(A + B) = 2(s_0 - 1) \cdots (s_n - 1)$, which proves property (ii) for this example. In odd dimensions, the middle orbifold cohomology has dimension $\frac{1}{d}A = (s_0 - 1) \cdots (s_n - 1)$, which is half the total, as expected. Note that $X_2^{(n)}$ is self-mirror, so its orbifold Euler characteristic vanishes.

Proof for $X_1^{(n)}$: (hypersurface of degree $d = 2s_n - 2$ in $\mathbb{P}(d/s_0, \ldots, d/s_{n-1}, 1, 1)$)

The idea of the proof is similar in this case. Once again, let $A := S_0$ and B be the sum of the S_ℓ with $s_i \nmid \ell$, $i = 0, \ldots n - 1$. The number of terms in B is

$$d\left(1 - \frac{1}{s_0}\right) \cdots \left(1 - \frac{1}{s_{n-1}}\right) = 2(s_0 - 1) \cdots (s_{n-1} - 1)$$

and each of these satisfies $S_\ell = d$. So $B = 2d(s_0 - 1) \cdots (s_{n-1} - 1)$. Similarly,

$$A = S_0 = \sum_{j=0}^{d-1} f_{s_0}(j) \cdots f_{s_{n-1}}(j)(f_d(j))^2$$

$$= (s_0 - 1) \cdots (s_{n-1} - 1)(d-1)^2 + (-1)^2 \sum_{j=1}^{d-1} f_{s_0}(j) \cdots f_{s_{n-1}}(j).$$

Applying Lemma 5.4 again, the last sum equals $-f_{s_0}(0) \cdots f_{s_{n-1}}(0) = (s_0 - 1) \cdots (s_{n-1} - 1)$. Thus,

$$A = (s_0 - 1) \cdots (s_{n-1} - 1)d(d - 2).$$

The total sum is

$$H = \frac{1}{d}(A + B) = \frac{1}{d}(d(d-2)(s_0 - 1) \cdots (s_{n-1} - 1) + 2d(s_0 - 1) \cdots (s_{n-1} - 1))$$
$$= (s_0 - 1) \cdots (s_{n-1} - 1)d = 2(s_0 - 1) \cdots (s_n - 1).$$

The dimension of the middle orbifold cohomology in odd dimensions is $\frac{1}{d}A = (d-2)(s_0 - 1) \cdots (s_{n-1} - 1) = (s_0 - 1) \cdots (s_{n-1} - 1)(2s_n - 4)$. The third example is the mirror of the first, so properties (i) and (ii) also hold for $X_3^{(n)}$. The sums A and B switch under mirror symmetry, so the dimension of $H_{\mathrm{orb}}^n(X_3^{(n)}, \mathbb{Q})$ is $\frac{1}{d}B = 2(s_0 - 1) \cdots (s_{n-1} - 1)$. The orbifold Euler characteristics of $X_1^{(n)}$ and $X_3^{(n)}$ are $\frac{1}{d}(B - A)$ and $\frac{1}{d}(A - B)$, respectively, using the vanishing result (i). □

6. KLT CALABI-YAU SURFACES WITH LARGE INDEX OR SMALL MLD

We have conjectured the klt Calabi-Yau pairs with standard coefficients of largest index in each dimension, and also the terminal or canonical Calabi-Yau varieties of largest index. In section 7, we conjecture the answer to a natural intermediate problem: find the klt Calabi-Yau *varieties* of largest index. In this section, we prove that conjecture in dimension 2, finding the klt Calabi-Yau surfaces with the largest index. We also find the smallest minimal log discrepancy for klt Calabi-Yau surfaces. The main tool is Brandhorst-Hofmann's classification of finite group actions on K3 surfaces, extending many earlier results [12].

Proposition 6.1. *The largest index of any klt Calabi-Yau surface is* 19. *The smallest mld of any klt Calabi-Yau surface is* $\frac{1}{13}$.

For comparison, the largest index of a klt Calabi-Yau pair with standard coefficients of dimension 2 is 66 (Proposition 3.5), and the smallest mld of a klt Calabi-Yau pair with standard coefficients of dimension 2 is $\frac{1}{42}$ (Proposition 6.8, below).

Proof. For any klt Calabi-Yau surface X, let Y be the index-1 cover of X (section 2). Thus Y is a projective surface with canonical singularities such that K_Y is trivial. Writing m for the index of X, we have $X = Y/\mu_m$, where μ_m acts purely

non-symplectically, meaning that it acts faithfully on $H^0(Y, K_Y) \cong \mathbb{C}$. Also, μ_m acts freely in codimension 1 on Y, because X is a Calabi-Yau surface rather than a pair.

Let $\pi \colon Z \to Y$ be the minimal resolution. Since $K_Z = \pi^*(K_Y)$, Z is a smooth projective surface with K_Z linearly equivalent to zero. By Proposition 2.2, the action of μ_m on Y lifts to an action on Z. By the classification of smooth projective surfaces with trivial canonical bundle, Z is a $K3$ surface or an abelian surface. If Z is an abelian surface, then m is at most 6 [9, Theorem C], [38, Theorem 4.1]. In that case, X has mld at least $\frac{1}{6}$.

Since we are trying to find the largest index or smallest mld among klt Calabi-Yau surfaces, we can assume from now on that Z is a smooth K3 surface, where more extreme values can occur. Note that Y may be a nontrivial contraction of Z. Let $W = Z/\mu_m$, and let (W, D) be the pair with standard coefficients associated to the μ_m-action on Z. Denote by $\sigma \colon W \to X$ the quotient of the morphism $\pi \colon Z \to Y$ by the action of μ_m, so that we have:

$$
\begin{array}{ccc}
Z & \xrightarrow{\ \pi\ } & Y \\
\downarrow & & \downarrow \\
W & \xrightarrow{\ \sigma\ } & X.
\end{array}
$$

Kondo found that the group μ_{19} acts purely non-symplectically on a certain smooth projective K3 surface Z [29, section 7]. In fact, the K3 surface and the action are unique up to isomorphism and automorphisms of μ_{19} [11, Theorem 1.1]. Also, μ_{19} acts freely in codimension 1 on Z [12, Table 37]. So $W = Z/\mu_{19}$ is a klt Calabi-Yau surface of index 19. In the spirit of this paper, we can describe Z as the minimal resolution of the hypersurface $Y_{10} \subset \mathbb{P}^3(5, 3, 1, 1)$ defined by $0 = x_0^2 + x_1^3 x_3 + x_3^9 x_2 + x_2^7 x_1$. Here μ_{19} acts on Y by

$$\zeta[x_0 : x_1 : x_2 : x_3] = [\zeta x_0 : \zeta^7 x_1 : \zeta^2 x_2 : x_3],$$

Y has an A_2 singularity, and Y/μ_{19} is also a klt Calabi-Yau surface of index 19.

In the other direction, Appendix B in [12] shows that for each purely non-symplectic action of a cyclic group of order greater than 19 on a smooth K3 surface, there is a smooth curve of positive genus that is fixed by a nontrivial subgroup. This curve cannot be contracted by $\pi \colon Z \to Y$ (in the notation above), and so μ_m cannot act freely in codimension 1 on Y. This contradicts that $X = Y/\mu_m$ is a klt Calabi-Yau surface (rather than a Calabi-Yau pair). So 19 is the largest index among all klt Calabi-Yau surfaces.

Next, we will find the smallest possible mld for a klt Calabi-Yau surface X. We know that X has index $m \leq 19$. As above, let Y be the index-1 cover of X, Z its minimal resolution, and (W, D) the quotient of Z by μ_m.

Lemma 6.2.
$$\mathrm{mld}(X) = \mathrm{mld}(W, D).$$

Proof. Consider the commutative square above. Since Y has canonical singularities, we have $K_Z = \pi^* K_Y$. Taking quotients by μ_m, we have that K_Y is the pullback of K_X and K_Z is the pullback of $K_W + D$. So $K_W + D$ and $\sigma^* K_X$ both pull back to K_Z on Z. Since they differ by an explicit $\mathbb{Q}$-divisor on W, it follows that $K_W + D = \sigma^* K_X$. Since the mld of X can be computed on a log resolution of (W, D), we conclude that $\mathrm{mld}(X) = \mathrm{mld}(W, D)$. $\square$

Given a pair (X, D), we'll use the notation $\mathrm{mld}_{\geq 2}(X, D)$ to denote the minimal log discrepancy over all points $x \in X$ with codimension at least 2 (corresponding to exceptional divisors over X, rather than all divisors over X). We have the following general bound on discrepancies for quotients by finite groups.

Proposition 6.3. *Let (Z, F) be a quasi-projective terminal pair with a faithful action of a finite group G. Let $W = Z/G$, and let D be the $\mathbb{Q}$-divisor on W such that $K_Z + F$ is the pullback of $K_W + D$. Then*

$$\mathrm{mld}_{\geq 2}(W, D) > \frac{1}{|G|}.$$

If $Z = (Z, 0)$ is a terminal Gorenstein variety, then

$$\mathrm{mld}_{\geq 2}(W, D) \geq \frac{2}{|G|}.$$

We need to exclude points of codimension 1 in these inequalities. For example, if a cyclic group $G = \mu_m$ fixes an irreducible divisor S in a variety Z, then the image D_1 of S in W occurs in D with coefficient $(m-1)/m$, and so the generic point w of D_1 has $\mathrm{mld}_w(W, D) = 1/m = 1/|G|$.

Proof. By [28, Corollary 2.43], we have the inequalities

$$\mathrm{mld}_{\geq 2}(W, D) \leq \mathrm{mld}_{\geq 2}(Z, F) \leq |G| \, \mathrm{mld}_{\geq 2}(W, D).$$

(The notation "$\mathrm{discrep}(W, D)$" of [28] means the infimum of all discrepancies of exceptional divisors over X, which is the same as $\mathrm{mld}_{\geq 2}(W, D) - 1$.) By definition, the terminality of (Z, F) means that $\mathrm{mld}_{\geq 2}(Z, F) > 1$. It follows that

$$\mathrm{mld}_{\geq 2}(W, D) \geq \frac{1}{|G|} \mathrm{mld}_{\geq 2}(Z, F) > \frac{1}{|G|}.$$

If $Z = (Z, 0)$ is a terminal Gorenstein variety, then $|G|(K_W + D)$ is Cartier. It follows that the discrepancy of every divisor over W is a multiple of $1/|G|$, and so $\mathrm{mld}_{\geq 2}(W, D) \geq 2/|G|$. $\square$

By [12, Table 37], there is a smooth K3 surface Z with a purely non-symplectic action of the cyclic group μ_{13}; the fixed locus consists of a smooth rational curve C and 9 other points. Since K_Z is trivial, the adjunction formula gives that C is a (-2)-curve. Let Y be the canonical K3 surface obtained by contracting C. Then the cyclic group μ_{13} acts freely in codimension 1 on Y. So $X := Y/\mu_{13}$ is a klt Calabi-Yau surface of index 13.

For an explicit example, start with the canonical K3 surface $S_{11} \subset \mathbb{P}^3(5, 3, 2, 1)$ defined by $0 = x_0^2 x_3 + x_1^3 x_2 + x_2^3 x_0 + x_3^8 x_1$, on which there is a purely non-symplectic action of μ_{13}:

$$\zeta[x_0 : x_1 : x_2 : x_3] = [\zeta x_0 : \zeta^2 x_1 : \zeta^{-4} x_2 : x_3].$$

This action is free in codimension 1, and so S/μ_{13} is a klt Calabi-Yau surface of index 13. One checks that the minimal resolution Z of S contains a smooth rational curve C fixed by μ_{13} over the A_4 singularity $[1 : 0 : 0 : 0]$ of S, in agreement with [12, Table 37].

Lemma 6.4. *There is a klt Calabi-Yau surface of index 13. Every such surface has mld $\frac{1}{13}$.*

Proof. We have shown that there is a klt Calabi-Yau surface X of index 13. As throughout this section, let Y be the index-1 cover of X, Z the minimal resolution of Y, and $(W, D) = Z/\mu_{13}$. By Lemma 6.2, $\mathrm{mld}(X) = \mathrm{mld}(W, D)$. Here $13(K_W + D)$ is Cartier, and so $\mathrm{mld}(W, D)$ is a positive integer divided by 13. Also, D is $\frac{12}{13}$ times the image of the curve C above, and so (W, D) has log discrepancy $\frac{1}{13}$ at the generic point of this curve. Thus $\mathrm{mld}(W, D) = \frac{1}{13}$. $\square$

Lemma 6.5. *The mld of a klt Calabi-Yau surface of index 14 is $\frac{1}{7}$.*

Remark 6.6. Although it is not needed for the proof of Proposition 6.1, we remark that there is a klt Calabi-Yau surface of index 14. Start with the canonical K3 surface $S_7 \subset \mathbb{P}^3(3, 2, 1, 1)$ defined by $0 = x_0^2 x_3 + x_1^3 x_2 + x_1 x_2^5 + x_3^7$, on which there is a purely non-symplectic action of μ_{14}:

$$\zeta[x_0 : x_1 : x_2 : x_3] = [\zeta x_0 : \zeta x_1 : \zeta^{-3} x_2 : \zeta^{-2} x_3].$$

The curve $C = \{0 = x_1 = x_3\} \cong \mathbb{P}^1(3, 1)$ is contained in S and fixed by the subgroup μ_2. Here $C^2 = -4/3 < 0$, and so C can be contracted, by Artin's contraction theorem applied to the minimal resolution of S [3, Theorem 2.7]. This yields another canonical K3 surface Y on which μ_{14} acts freely in codimension 1. Then $X = Y/\mu_{14}$ is a klt Calabi-Yau surface of index 14.

Proof. (Lemma 6.5) Let X be a klt Calabi-Yau surface of index 14. In the notation above, Z must be a smooth K3 surface with a purely non-symplectic action of μ_{14}, and every curve in Z fixed by a nontrivial subgroup of μ_{14} must have genus zero. By [12, Table 15, 0.14.0.4], there is exactly one deformation type of smooth K3 surfaces Z with such an action, up to automorphisms of μ_{14}. The table shows that there is no curve in Z fixed by the whole group μ_{14}, but there are three smooth curves of genus zero fixed by the subgroup μ_2 and one smooth curve of genus zero fixed by μ_7.

Let $(W, D) = Z/\mu_{14}$. We have $\mathrm{mld}(X) = \mathrm{mld}(W, D)$ by Lemma 6.2. Here D has coefficients $6/7$ and $1/2$. So the mld of (W, D) is $1/7$ at one point of codimension 1 and greater than $1/7$ at every other point of codimension 1. Also, by Proposition 6.3, since (W, D) is the quotient of a smooth K3 surface by μ_{14}, the mld of (W, D) at any closed point is at least $2/14 = 1/7$. So the mld of X is $1/7$. $\square$

Lemma 6.7. *For m equal to 17 or 19, the mld of a klt Calabi-Yau surface of index m is at least $\frac{2}{m}$.*

Proof. Let X be a klt Calabi-Yau surface of index m equal to 17 or 19. Let Y be the index-1 cover of X, Z the minimal resolution of Y, and $(W, D) = Z/\mu_m$. By [12, Appendix B], for m equal to 17 or 19, every purely non-symplectic action of μ_m on a smooth K3 surface is free in codimension 1; so $D = 0$. Then Lemma 6.2 gives that $\mathrm{mld}(X) = \mathrm{mld}(W)$. Since W is a normal variety (rather than a pair), $\mathrm{mld}(W)$ is equal to the minimum of its mld's at closed points, and these are at least $\frac{2}{m}$ by Proposition 6.3. □

We now show that among all klt Calabi-Yau surfaces X, the smallest mld is $1/13$. We know by Lemma 6.4 that $1/13$ does occur. Here X must be a quotient Y/μ_m as discussed above, in particular with μ_m acting freely in codimension 1. The mld of X is a positive integer divided by m. So if X has mld less than $1/13$, it must have index $m \geq 14$. Since the minimal resolution $Z \to Y$ contracts only smooth rational curves, every curve in Z fixed by a nontrivial subgroup of μ_m must have genus zero. By [12, Appendix B], if a smooth K3 surface Z has a purely non-symplectic action of the cyclic group μ_m with m at least 14 and not equal to 14, 17, or 19, then there is a curve of genus > 0 fixed by a nontrivial subgroup. So m must be 14, 17, or 19. Therefore, Lemmas 6.5 and 6.7 give that the smallest mld of a klt Calabi-Yau surface is $1/13$. Proposition 6.1 is proved. □

We conclude with the analogous bound for pairs.

Proposition 6.8. *The smallest mld of any klt Calabi-Yau pair with standard coefficients of dimension 2 is $\frac{1}{42}$.*

Proof. By Theorem 3.1, there is a klt Calabi-Yau pair with standard coefficients of dimension 2 that has mld $1/42$.

Suppose that there is a klt Calabi-Yau pair (X, E) with standard coefficients of dimension 2 that has mld less than $1/42$. Let m be the index of (X, E). The mld is a positive integer divided by m, and so we must have $m > 42$. By Proposition 3.5, we have $m \leq 66$. Let Y be the index-1 cover of (X, E), so that $(X, E) = Y/\mu_m$ with μ_m acting purely non-symplectically. Let Z be the minimal resolution of Y, and let $(W, D) = Z/\mu_m$. Here Z is a K3 surface. The proof of Lemma 6.2 shows that $\mathrm{mld}(X, E) = \mathrm{mld}(W, D)$.

By Brandhorst and Hofmann's classification of purely non-symplectic cyclic group actions on K3 surfaces, m must be 44, 48, 50, 54, or 66 [12, Appendix B]. In each case, the classification shows that the subspace of Z fixed by G has dimension zero. Equivalently, each curve in Z is fixed by a proper subgroup of G. So the coefficients of D have the form $(b-1)/b$ with $b \leq m/2$. Equivalently, the mld of (W, D) at each point of codimension 1 in X is at least $2/m$.

By Proposition 6.3, the mld of (W, D) at each closed point is also at least $2/m$. So $\mathrm{mld}(W, D)$ is at least $2/m$, hence at least $1/33$, contradicting that it is less than $1/42$. $\qquad\square$

7. Higher-dimensional klt Calabi-Yau varieties with large index

In this section, we find klt Calabi-Yau varieties in every dimension at least 2, which we conjecture have the largest possible index. In dimension 2, the construction reproduces the index 19 example mentioned in the proof of Proposition 6.1. Each of our varieties is constructed as a quotient of a weighted projective hypersurface by a cyclic group action. To define these actions, we'll use the techniques of Artebani-Boissière-Sarti [2] to write down hypersurfaces with an action of a large cyclic group. We only manage to compute the index of our example exactly in dimensions at most 30. So, at least in low dimensions, we construct klt Calabi-Yau varieties of extremely large index.

First, we prove some general properties about hypersurfaces defined by loop potentials, some of which appear in [2]. A *potential* is a sum of n monomials in n variables. A *loop potential* has the form

$$W \coloneqq x_1^{b_1} x_2 + \cdots + x_{n-1}^{b_{n-1}} x_n + x_n^{b_n} x_1 \tag{7.1}$$

for some integers $b_i \geq 2$. Suppose that for some positive integer weights $a_1, \ldots, a_n$ on the variables, W is homogeneous of degree $d := \sum_i a_i$. Then the hypersurface $X \coloneqq \{W = 0\}$ in $\mathbb{P}(a_1, \ldots, a_n)$ is quasi-smooth and Calabi-Yau. Writing W in the form

$$W = \sum_{i=1}^{n} \prod_{i=1}^{n} x_i^{a_{ij}},$$

define an associated matrix $A = (a_{ij})$:

$$A = \begin{pmatrix} b_1 & 1 & & & \\ & b_2 & 1 & & \\ & & \ddots & \ddots & \\ & & & b_{n-1} & 1 \\ 1 & & & & b_n \end{pmatrix}.$$

Let $\mathrm{Aut}(W)$ be the group of diagonal automorphisms of $\mathbb{C}^n$ preserving the potential W. By the results of [2], we may identify $\mathrm{Aut}(W)$ with $A^{-1}\mathbb{Z}^n/\mathbb{Z}^n$. That is, the columns of A^{-1} generate $\mathrm{Aut}(W)$ modulo $\mathbb{Z}^n$, where an element $(c_1, \ldots, c_n) \in \mathrm{Aut}(W)$ acts on $\mathbb{C}^n$ by multiplication by the diagonal matrix $\mathrm{diag}(e^{2\pi i c_1}, \ldots, e^{2\pi i c_n})$.

This action descends to an action on the weighted projective hypersurface X, giving a surjective homomorphism $\alpha \colon \mathrm{Aut}(W) \to \mathrm{Aut}_T(X)$ to the group of toric automorphisms of X.

Lemma 7.1 (cf. [2, Proposition 2]). *The group* $\mathrm{Aut}(W)$ *has order* $\Gamma := \det(A) = (-1)^{n+1} + b_1 \cdots b_n$. *The last column of* A^{-1} *is*

$$v_n = \frac{1}{\Gamma} \begin{pmatrix} (-1)^{n-1} \\ (-1)^{n-2}b_1 \\ (-1)^{n-3}b_1 b_2 \\ \vdots \\ b_1 \cdots b_{n-1} \end{pmatrix}.$$

The other columns v_i *of* A^{-1} *satisfy the following relations modulo* $\mathbb{Z}^n$: $v_i = (-1)^{n-i} b_n b_{n-1} \cdots b_{i+1} v_n$, $i = 1, \ldots, n-1$. *In particular,* $\mathrm{Aut}(W)$ *is cyclic with generator* $\varphi := v_n$ *(or any one of the other columns). The first row of* A^{-1} *is*

$$w_1 = \frac{1}{\Gamma} \begin{pmatrix} b_2 \cdots b_n & -b_3 \cdots b_n & b_4 \cdots b_n & \cdots & (-1)^{n-1} \end{pmatrix}.$$

The other rows w_i *of* A^{-1} *satisfy the following relations modulo* $\mathbb{Z}^n$: $w_i = (-1)^{i-1} b_1 b_2 \cdots b_{i-1}$, $i = 2, \ldots n$.

The kernel of $\alpha \colon \mathrm{Aut}(W) \to \mathrm{Aut}_T(X)$ is a subgroup $J_W \subset \mathrm{Aut}(W)$ of order d generated by the vector of charges $(q_1, \ldots, q_n)$, where each *charge* q_i is the sum of the entries of the ith row of A^{-1}. The charges have the property $q_i = a_i/d$. Lemma 7.1 implies that

$$q_1 = \frac{1}{\Gamma} \left((-1)^{n-1} + (-1)^{n-2} b_n + \cdots + b_2 \cdots b_n \right).$$

Modulo $\mathbb{Z}^n$, the other charges are multiples of this one, so that the degree d equals the denominator of q_1:

$$d = \frac{\Gamma}{\gcd(\Gamma, (-1)^{n-1} + (-1)^{n-2} b_n + \cdots + b_2 \cdots b_n)}. \tag{7.2}$$

The same reasoning shows that the degree d^T of the mirror hypersurface (which is defined by the potential W^T associated to the matrix A^T) is

$$d^\mathsf{T} = \frac{\Gamma}{\gcd(\Gamma, (-1)^{n-1} + (-1)^{n-2} b_1 + \cdots + b_1 \cdots b_{n-1})}. \tag{7.3}$$

Loop potentials are useful for constructing cyclic group actions which are free in codimension 1, in light of the following proposition.

Proposition 7.2. *If a hypersurface* X *with* $\dim(X) \geq 3$ *is defined by a loop potential, the action of* $\mathrm{Aut}_T(X)$ *is free in codimension 1.*

Proof. First, we reduce to only checking the stabilizers of coordinate hyperplanes. Indeed, $\mathrm{Aut}_T(X)$ is a subgroup of the torus action on $\mathbb{P}(a_1, \ldots, a_n)$, so the only nontrivial stabilizers of the action of $\mathrm{Aut}_T(X)$ on weighted projective space occur on coordinate strata in the complement of the open torus. Further, we claim that the intersection of X with a toric stratum S is only codimension 1 in X if S is codimension 1 in $\mathbb{P}(a_1, \ldots, a_n)$. Indeed, if X contained S of codimension 2 in weighted projective space, then the potential $W = x_1^{b_1} x_2 + \cdots + x_{n-1}^{b_{n-1}} x_n + x_n^{b_n} x_1$ would become identically zero after setting some pair of variables x_i and x_j equal

to zero. Since $\dim(X) \geq 3$, we have $n \geq 5$, and no choice of i and j makes the loop potential zero.

Thus, it's enough to compute the stabilizers of each $H_i := X \cap \{x_i = 0\}$. Without loss of generality we'll just show the stabilizer of H_n is trivial. To this end, let $H \subset \operatorname{Aut}_T(W)$ be the subgroup fixing H_n and let $G = \alpha^{-1}(H)$. Then G is a cyclic subgroup of $\operatorname{Aut}(W)$. We want to show that in fact $G = J_W$.

For an element $g := (c_1, \ldots, c_n)$ in $\operatorname{Aut}(W)$, the statement that the element $\alpha(g)$ fixes H_n is equivalent to the following statement in terms of coordinates on $\mathbb{C}^n$: for all $x_1, \ldots, x_{n-1} \in \mathbb{C}$, there exists a $t \in \mathbb{C}^*$ such that

$$g \cdot (x_1, \ldots, x_{n-1}, 0) = t \cdot (x_1, \ldots, x_{n-1}, 0),$$

where the $\mathbb{C}^*$ action is $t \cdot (y_1, \ldots, y_n) = (t^{a_1} y_1, \ldots, t^{a_n} y_n)$. Now, the action of $\operatorname{Aut}(W)$ is by diagonal matrices whose entries are unit complex numbers, so it's immediate that $|t| = 1$ and in fact that $t = e^{i\theta}$ for θ a rational multiple of π. In order to compute in terms of elements of $(\mathbb{Q}/\mathbb{Z})^n$, it will be convenient to express the element $t \in \mathbb{C}^*$ as $r(q_1, \ldots, q_n) \in (\mathbb{Q}/\mathbb{Z})^n$, where $(q_1, \ldots, q_n)$ is the vector of charges and r is some rational number. (Remember that we use the convention that each vector $(c_1, \ldots, c_n) \in (\mathbb{Q}/\mathbb{Z})^n$ corresponds to the action by the diagonal matrix $\operatorname{diag}(e^{2\pi i c_1}, \ldots, e^{2\pi i c_n})$.)

There is a point of H_n with all coordinates $x_1, \ldots, x_{n-1}$ nonzero. Otherwise, since H_n is codimension 1 in X, it must contain some entire stratum $\{x_i = x_n = 0\}$, which we saw above cannot occur when $\dim(X) \geq 3$. Thus, if the element g fixes H_n, there is an $r \in \mathbb{Q}$ such that the following congruences hold:

$$rq_i \equiv c_i \bmod \mathbb{Z}, 1 \leq i \leq n-1. \tag{7.4}$$

We already know that the vector of charges itself generates J_W, so if $r \in \mathbb{Z}$, $r(q_1, \ldots, q_n)$ will equal an element of J_W. We will derive a contradiction from the following assumption: there exists $g \in \operatorname{Aut}(W)$ and r *not* an integer such that the congruences (7.4) hold.

First, reduce to the case that $r = 1/A$ for a positive integer $A \geq 2$. Indeed, we can always replace r and g by the same integer multiple of each and the congruences (7.4) will still hold; we can also add an integer to r and multiply g by the corresponding element of J_W. Using such adjustments, any rational number $r = B/A$ in lowest terms can be turned into $1/A$. Consider the first two coordinates c_1 and c_2 of g. We know that $c_2 \equiv -b_1 c_1 \bmod \mathbb{Z}$, since this relation holds for the generator φ of the group $\operatorname{Aut}(W)$ and g is some multiple of φ. But $c_i \equiv rq_i \bmod \mathbb{Z}$, so we have $rq_2 \equiv -b_1 rq_1 \bmod \mathbb{Z}$. Therefore, $s := b_1 rq_1 + rq_2$ is an integer. Substituting $r = 1/A$ and using the definition of charges $q_i = a_i/d$, we have

$$b_1 a_1 + a_2 = Ads.$$

However, our loop potential is homogeneous of degree d and contains the monomial $x_1^{b_1} x_2$ so $b_1 a_1 + a_2 = d$. This would mean that $As = 1$, contradicting the assumption that $A \geq 2$ and s is an integer.

Therefore, we have shown that there exists an integer $r \in \mathbb{Z}$ such that $rq_i \equiv c_i \bmod \mathbb{Z}, 1 \leq i \leq n-1$. To conclude, we must also prove that $rq_n \equiv c_n \bmod \mathbb{Z}$, so that $(c_1, \ldots, c_n) = r(q_1, \ldots, q_n) \in (\mathbb{Q}/\mathbb{Z})^n$ is an element of J_W. To do this, use that the monomial $x_{n-1}^{b_{n-1}} x_n$ is preserved by any diagonal automorphism of the loop potential. Hence

$$rq_n \equiv -b_{n-1} r q_{n-1} \equiv -b_{n-1} c_{n-1} \equiv r c_n \bmod \mathbb{Z}.$$

This completes the proof. $\square$

Next, we'll state a criterion for whether the induced action of $\mathrm{Aut}_T(X)$ on $H^0(X, K_X) \cong \mathbb{C}$ is faithful. This in turn implies that the quotient $X/\mathrm{Aut}_T(X)$ has index $|\mathrm{Aut}_T(X)| = \Gamma/d$. Let $\mathrm{SL}(W) \subset \mathrm{Aut}(W)$ be the subgroup with determinant 1. Since X is Calabi-Yau, we have $J_W \subset \mathrm{SL}(W)$. The image $\alpha(\mathrm{SL}(W)) \subset \mathrm{Aut}_T(X)$ is the kernel of the action on $H^0(X, K_X)$. By [2, Corollary 1], $|\mathrm{SL}(W)| = \Gamma/d^{\mathsf{T}}$. Therefore, we have the following criterion:

Proposition 7.3. *The group* $\mathrm{Aut}_T(X)$ *acts faithfully on* $H^0(X, K_X)$ *if and only if* $d \cdot d^{\mathsf{T}} = \Gamma$.

Proof. The kernel of α is J_W, so the action is faithful if and only if $\mathrm{SL}(W) = J_W$. Since $J_W \subset \mathrm{SL}(W)$, this holds exactly when the two groups have the same order. But $|J_W| = d$, so this is equivalent to $d = \Gamma/d^{\mathsf{T}}$. $\square$

Nearly identical arguments apply to any X with $\dim(X) \geq 3$ defined by a polynomial of the form $W = x_0^{b_0} + f_{\mathrm{loop}}$, where f_{loop} is of the form of (7.1). The corresponding matrix A is block diagonal, so there is a subgroup $G \cong \mathbb{Z}/\Gamma\mathbb{Z}$ of $\mathrm{Aut}(X)$ with index b_0 whose image in $\mathrm{Aut}_T(X)$ acts freely in codimension 1. The quotient of X by this cyclic group is therefore a klt Calabi-Yau variety (rather than a pair). We can check whether this action is purely non-symplectic using a criterion very similar to Proposition 7.3.

Next, we define several sequences of numbers based on Sylvester's sequence for each dimension n. These will be used to define the exponents and weights for our example of large index.

Definition 7.4. For $n = 2r + 1$, $r \geq 1$ or $n = 2r$, $r \geq 1$, define integers $b_0, \ldots, b_{n-1}$ as follows. Set $b_i := s_i$ when $0 \leq i \leq r$, and define b_{r+i} by the inductive formula:

$$b_{r+i} := 1 + (b_{r+1-i} - 1)^2[1 + (b_r - 1)b_{r+1}$$
$$+ (b_{r-1} - 1)b_r b_{r+1} b_{r+2} + \cdots + (b_{r+2-i} - 1)b_{r+3-i} \cdots b_{r+i-1}]$$

for $1 \leq i \leq r$ when $n = 2r + 1$ or $1 \leq i \leq r - 1$ when $n = 2r$.

Definition 7.5. For $n = 2r + 1$, $r \geq 1$, define an integer d by:

$$d := (b_0 - 1)^2 b_1 \cdots b_{2r} + (b_1 - 1)[1 + (b_r - 1)b_{r+1}$$
$$+ (b_{r-1} - 1)b_r b_{r+1} b_{r+2} + \cdots + (b_1 - 1)b_2 \cdots b_{2r}].$$

For $n = 2r$, $r \geq 1$, define an integer d by:
$$d := (b_1 - 1)^2 b_2 \cdots b_{2r-1} + (b_2 - 1)[1 + (b_r - 1)b_{r+1}$$
$$+ (b_{r-1} - 1)b_r b_{r+1} b_{r+2} + \cdots + (b_2 - 1)b_3 \cdots b_{2r-1}].$$

Definition 7.6. For $n = 2r + 1$, $r \geq 1$, define $b_{2r+1} := \frac{d+1}{2}$.

For $n = 2r$, $r \geq 1$, define $b_{2r} := \frac{2d+1}{3}$.

Definition 7.7. For $n = 2r + 1$, $r \geq 1$, define a_j for $0 \leq j \leq 2r + 2$ by the following inductive formulas. They are positive integers.

$a_{2r+1} = a_{2r+2} := 1,$

$$a_{2r-i} := a_{2r-i+1}\left(\frac{1}{s_{i+1}}\right) + (s_{r+1} - 1)\left(1 - \frac{1}{s_{i+1}}\right)\left[a_{2r-i+1}\left(1 - \frac{1}{s_{i+1}}\right) + \cdots\right.$$
$$\left. + a_{2r}\left(1 - \frac{1}{s_2}\right) + a_{2r+1}\left(1 - \frac{1}{s_1}\right) + a_{2r+2}\left(1 - \frac{1}{s_0}\right)\right]b_{r+1} \cdots b_{2r-i-1} \text{ for } 0 \leq i \leq r - 1,$$

$a_i := d - a_{2r+1-i}b_{2r+1-i}$ for $0 \leq i \leq r$.

For $n = 2r$, $r \geq 1$, define a_j for $0 \leq j \leq 2r + 1$ by the inductive formulas:

$a_{2r} = a_{2r+1} := 1$

$$a_{2r-1-i} := a_{2r-i}\left(\frac{1}{s_{i+2}}\right) + (s_{r+1} - 1)\left(1 - \frac{1}{s_{i+2}}\right)\left[a_{2r-i}\left(1 - \frac{1}{s_{i+2}}\right) + \cdots\right.$$
$$\left. + a_{2r-1}\left(1 - \frac{1}{s_3}\right) + a_{2r}\left(1 - \frac{1}{s_2}\right) + a_{2r+1}\left(1 - \frac{1}{s_1}\right)\right]b_{r+1} \cdots b_{2r-i-2} \text{ for } 0 \leq i \leq r - 2,$$

$a_i := d - b_{2r+1-i}a_{2r+1-i}$ for $0 \leq i \leq r$,

$$a_0 := \frac{d}{2}.$$

Definition 7.8. For $n = 2r + 1$, $r \geq 1$, we define an integer $b_{2r+2} := d - a_{r+1}$.

For $n = 2r$, $r \geq 1$, we define an integer $b_{2r+1} := d - a_{r+1}$.

Definition 7.9. For $n = 2r + 1$, $r \geq 1$, define an integer m by:
$$m := b_0 \cdots b_{2r+1} - b_0 \cdots b_r b_{r+2} \cdots b_{2r+1}$$
$$+ b_0 \cdots b_{r-1} b_{r+2} \cdots b_{2r+1} - b_0 \cdots b_{r-1} b_{r+3} \cdots b_{2r+1} + \cdots - b_0 + 1.$$

For $n = 2r$, $r \geq 1$, define an integer m by:
$$m := b_1 \cdots b_{2r} - b_1 \cdots b_r b_{r+2} \cdots b_{2r} + b_1 \cdots b_{r-1} b_{r+2} \cdots b_{2r} - \cdots + 1.$$

We were led to the choice of weights a_i and exponents b_i in Definition 7.7 and Definition 7.4 by an inductive procedure. First we chose the general form of the equation in (7.5) or (7.6) below. Since each monomial in the equation must have degree d, we get a relation involving d and some a_i and b_i. We use the relations given by all monomials and the relation that all the weights a_i sum to d to express d as $r + 1$ linear combinations of $a_{r+i}, \ldots, a_{2r+2}$ for $1 \leq i \leq r + 1$ and express each a_{r+i} as a linear combination of $a_{r+i+1}, \ldots, a_{2r+2}$ for $1 \leq i \leq r$.

During each step, we choose one weight as big as possible which determines our choice of the corresponding exponent. In each step, we can increase i by one. This process yields the formulas above.

We now define our example. When $n = 2r + 1$ for $r \geq 1$, let X_d be the hypersurface in $\mathbb{P}(a_0, a_1, \ldots, a_{2r+1}, a_{2r+2})$ of degree d given by

$$x_0^{b_0} x_{2r+2} + x_1^{b_1} x_{2r+1} + \cdots + x_r^{b_r} x_{r+2} + x_{r+1}^{b_{r+1}} x_r + x_{r+2}^{b_{r+2}} x_{r-1} + \cdots + x_{2r+1}^{b_{2r+1}} x_0 + x_{2r+2}^{b_{2r+2}} x_{r+1} = 0. \tag{7.5}$$

This is a loop potential, but with the variables written in a different order than in (7.1) above; one can check that it is homogeneous of weighted degree d. The group $G = \mathrm{Aut}_T(X)$ is cyclic of order m and acts freely in codimension 1 on X by Proposition 7.2. Similarly, when $n = 2r$ for $r \geq 1$, let X_d be the hypersurface in $\mathbb{P}(a_0, a_1, \ldots, a_{2r}, a_{2r+1})$ of degree d given by

$$x_0^{b_0} + x_1^{b_1} x_{2r+1} + x_2^{b_2} x_{2r} + \cdots + x_r^{b_r} x_{r+2} + x_{r+1}^{b_{r+1}} x_r + x_{r+2}^{b_{r+2}} x_{r-1} + \cdots + x_{2r}^{b_{2r}} x_1 + x_{2r+1}^{b_{2r+1}} x_{r+1} = 0. \tag{7.6}$$

This potential has the form $x_0^2 + f_{\mathrm{loop}}$. There is a subgroup G of index 2 inside $\mathrm{Aut}_T(X)$ corresponding to f_{loop} which is cyclic of order m and which acts freely in codimension 1 on X (this does not follow from Proposition 7.2 when $n = 2$, but nevertheless is still true in this case).

We conjecture that the subgroup G acts faithfully on $H^0(X, K_X) \cong \mathbb{C}$, so that the index of X/G is in fact m. By Proposition 7.3, proving this conjecture would amount to a gcd calculation involving the exponents b_i. This is true in dimensions at most 30 by computer verification, and so we have klt Calabi-Yau varieties of extremely large index at least in low dimensions. The number m has $\log m$ asymptotic to $\log s_{n+1}$, and so this conjecture would imply that the index of our klt Calabi-Yau n-fold is comparable to that of the klt Calabi-Yau pair in Theorem 3.3, and much bigger than the index of our terminal Calabi-Yau variety in Corollary 4.1. More precisely, we conjecture:

Conjecture 7.10. *The quotient X/G above has index m. Moreover, in each dimension $n \geq 2$, this is the largest possible index for a klt Calabi-Yau variety.*

For $n = 2$, the hypersurface X is

$$\{x_0^2 + x_1^3 x_3 + x_2^7 x_1 + x_3^9 x_2 = 0\} \subset \mathbb{P}^3(5, 3, 1, 1),$$

with an action of the cyclic group G of order 19 that is free in codimension 1 and faithful on $H^0(X, K_X)$. So X/G is a klt Calabi-Yau surface of index 19, which is the largest possible, by Proposition 6.1. For $n = 3$, the hypersurface X is

$$\{x_0^2 x_4 + x_1^3 x_3 + x_2^5 x_1 + x_3^{19} x_0 + x_4^{32} x_2 = 0\} \subset \mathbb{P}^4(18, 12, 5, 1, 1).$$

The cyclic group of order 493 acts on X with quotient a klt Calabi-Yau 3-fold of index 493. That is well above the index 66 of the terminal Calabi-Yau 3-fold in Corollary 4.1.

For $n = 4$, the hypersurface X is

$$\{x_0^2 + x_1^3 x_5 + x_2^7 x_4 + x_3^{37} x_2 + x_4^{1583} x_1 + x_5^{2319} x_3 = 0\} \subset \mathbb{P}^5(1187, 791, 339, 55, 1, 1).$$

The cyclic group of order 1201495 acts on X with quotient a klt Calabi-Yau 4-fold of index 1201495. In dimensions 3 and 4, our example has the largest index among all quotients by toric automorphisms of quasi-smooth Calabi-Yau hypersurfaces defined by potentials. This was verified by computer search, using the databases of Calabi-Yau threefold and fourfold hypersurfaces in [13].

References

[1] V. Alexeev. Boundedness and K^2 for log surfaces. *Internat. J. Math.* **5** (1994), 779–810. 1

[2] M. Artebani, S. Boissière, and A. Sarti. The Berglund-Hübsch-Chiodo-Ruan mirror symmetry for K3 surfaces. *J. Math. Pure Appl.* **102** (2014), 758–781. 9, 24, 25, 27

[3] M. Artin. Some numerical criteria for contractability of curves on algebraic surfaces. *Amer. J. Math.* **84** (1962), 485–496. 22

[4] V. Batyrev. On the stringy Hodge numbers of mirrors of quasi-smooth Calabi-Yau hypersurfaces. `arXiv:2006.15825` 12

[5] G. Balletti, A. Kasprzyk, and B. Nill. On the maximum dual volume of a canonical Fano polytope. *Forum Math. Sigma* **10** (2022), E109. 9

[6] A. Beauville. *Complex algebraic surfaces.* London Mathematical Society Student Texts 34, Cambridge Univ. Press, Cambridge (1996). 10

[7] C. Birkar, P. Cascini, C. Hacon, and J. McKernan. Existence of minimal models for varieties of log general type. *J. Amer. Math. Soc.* **23** (2010), 405–468. 5

[8] C. Birkar. Singularities on Fano fibrations and beyond. `arXiv:2305.18770` 1

[9] R. Blache. The structure of l.c. surfaces of Kodaira dimension zero. I. *J. Alg. Geom.* **4** (1995), 137–179. 20

[10] L. Borisov, L. Chen, and G. Smith. The orbifold Chow ring of toric Deligne-Mumford stacks. *J. Amer. Math. Soc.* **18** (2005), no. 1, 193–215. 9

[11] S. Brandhorst. The classification of purely non-symplectic automorphisms of high order on K3 surfaces. *J. Algebra* **533** (2019), 229–265. 20

[12] S. Brandhorst and T. Hofmann. Finite subgroups of automorphisms of K3 surfaces. *Forum Math. Sigma* **11** (2023), E54. 19, 20, 21, 22, 23

[13] G. Brown and A. Kasprzyk. Graded Ring Database. `http://www.grdb.co.uk/` 30

[14] P. Candelas. Max Kreuzer's contributions to the study of Calabi-Yau manifolds. *Strings, gauge fields, and the geometry behind: the legacy of Maximilian Kreuzer*, 161–172. World Scientific, Singapore (2013). 11

[15] W. Chen, G. Di Cerbo, J. Han, C. Jiang and R. Svaldi. Birational boundedness of rationally connected Calabi-Yau 3-folds. *Adv. Math.* **378** (2021), Paper No. 107541, 32 pp. 4

[16] W. Chen and Y. Ruan. A new cohomology theory of orbifold. *Comm. Math. Phys.* **248** (2004), 1–31. 11

[17] A. Chiodo and Y. Ruan. LG/CY correspondence: the state space isomorphism. *Adv. Math.* **227** (2011), 2157–2188. 14

[18] L. Esser, B. Totaro, and C. Wang. Varieties of general type with doubly exponential asymptotics. *Trans. Amer. Math. Soc. Ser. B* **10** (2023), 288–309. 1, 3, 6, 8, 12

[19] C. Hacon, J. McKernan, and C. Xu. On the birational automorphisms of varieties of general type. *Ann. of Math.* **177** (2013), 1077–1111. 5

[20] C. Hacon, J. McKernan and C. Xu. ACC for log canonical thresholds. *Ann. of Math.* **180** (2014), no. 2, 523–571. 1, 4

[21] A. R. Iano-Fletcher. Working with weighted complete intersections. *Explicit birational geometry of 3-folds*, 101–173. London Math. Soc. Lecture Notes Ser. 281, Cambridge Univ. Press, Cambridge (2000). 3

[22] C. Jiang and H. Liu. Boundedness of log pluricanonical representations of log Calabi-Yau pairs in dimension 2. *Algebra Number Theory* **15** (2021), 545–567. 1

[23] A. Kasprzyk. Bounds on fake weighted projective space. *Kodai Math. J.* **32** (2009), 197–208. 3

[24] J. Kollár. Log surfaces of general type: some conjectures. *Classification of algebraic varieties* (L'Aquila, 1992), 261–275. Contemp. Math. 162, Amer. Math. Soc. (1994). 2, 5

[25] J. Kollár. Singularities of pairs. *Algebraic geometry* (Santa Cruz, 1995), 221–287. Proc. Symp. Pure Math. 62, Amer. Math. Soc. (1997). 2

[26] J. Kollár and S. Mori. *Birational geometry of algebraic varieties.* Cambridge Univ. Press, Cambridge (1998). 3, 5

[27] J. Kollár. *Lectures on resolution of singularities.* Princeton Univ. Press, Princeton (2007). 5

[28] J. Kollár. *Singularities of the minimal model program.* Cambridge Univ. Press, Cambridge (2013). 4, 7, 21

[29] S. Kondo. Automorphisms of algebraic K3 surfaces which act trivially on Picard groups. *J. Math. Soc. Japan* **44** (1992), 75–98. 20

[30] M. Kreuzer and H. Skarke. Reflexive polyhedra, weights, and toric Calabi-Yau fibrations. *Adv. Theor. Math. Phys.* **4** (2000), 1209–1230. 11

[31] J. Liu. Remark on complements on surfaces. *Forum Math. Sigma* **11** (2023), E42. 2, 5

[32] N. Machida and K. Oguiso. On K3 surfaces admitting finite non-symplectic group actions. *J. Math. Sci. Univ. Tokyo* **5** (2):273–297, 1998. 2, 7, 10

[33] J. McKernan and Y. Prokhorov. Threefold thresholds. *Manuscripta Math.* **114** (2004), 281–304. 1

[34] Y. Prokhorov. Equivariant minimal model program. *Russian Math. Surveys* **76** (2021), 461–542. 5

[35] M. Reid. Young person's guide to canonical singularities. *Algebraic geometry* (Bowdoin, 1985), 345–414. Proc. Symp. Pure Math. 46, Amer. Math. Soc. (1987). 3

[36] Y. Xu. Some results about the index conjecture for log Calabi-Yau pairs. `arXiv:1905.00297`. 1

[37] T. Yasuda. Twisted jets, motivic measure and orbifold cohomology. *Compos. Math.* **140** (2004), 396–422. 11

[38] D. Zhang. Logarithmic Enriques surfaces. *J. Math. Kyoto Univ.* 31 (1991), no. 2, 419–466. 20

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