The failure of Kodaira vanishing for Fano varieties, and terminal singularities that are not Cohen-Macaulay

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The Kodaira vanishing theorem says that for an ample line bundle L on a smooth projective variety X over a field of characteristic zero,

$$H^i(X, K_X + L) = 0$$

for all $i > 0$. (Here K_X denotes the canonical bundle, and we use additive notation for line bundles.) This result and its generalizations are central to the classification of algebraic varieties. For example, Kodaira vanishing can sometimes be used to show that there are only finitely many varieties with given intrinsic invariants, up to deformation equivalence. Unfortunately, Raynaud showed that Kodaira vanishing fails already for surfaces in any characteristic $p > 0$ [21].

For the minimal model program (MMP) in positive characteristic, it has been important to find out whether Kodaira vanishing holds for special classes of varieties, notably for Fano varieties (varieties with $-K_X$ ample). By taking cones, this is related to the question of whether the singularities arising in the MMP (klt, canonical, and so on) have the good properties (such as Cohen-Macaulayness or rational singularities) familiar from characteristic zero.

For example, Kodaira vanishing holds for smooth del Pezzo surfaces in any characteristic [4, Theorem II.1.6]. Also, klt surface singularities in characteristic $p > 5$ are strongly F-regular and hence Cohen-Macaulay; this is the key reason why the MMP for 3-folds is only known in characteristic $p > 5$ (or zero) [10, Theorem 3.1]. There are in fact some striking counterexamples in characteristics 2 and 3. Maddock found a regular (but not smooth) del Pezzo surface X over an imperfect field of characteristic 2 with $H^1(X, O) \neq 0$, which violates Kodaira vanishing [20]. And Cascini-Tanaka and Bernasconi found klt 3-folds over algebraically closed fields of characteristic 2 or 3 which are not Cohen-Macaulay [3, Theorem 1.3], [2, Theorem 1.2].

So far, the only known example of a smooth Fano variety on which Kodaira vanishing fails has been a 6-fold in characteristic 2 discovered by Haboush and Lauritzen [8, 19]. In this paper, however, we find that Kodaira vanishing fails for smooth Fano varieties in every characteristic $p > 0$. One family of examples has dimension 5 for $p = 2$ and $2p - 1$ for $p \geq 3$ (Theorem 3.1). (Thus, in characteristics 2 and 3, this is a 5-fold rather than a 6-fold.)

Our examples are projective homogeneous varieties with non-reduced stabilizer, as in the Haboush-Lauritzen example. Projective homogeneous varieties are smooth and rational in any characteristic $p > 0$, but most of them are not Fano (apart from the familiar flag varieties, where the stabilizer subgroup is reduced). A point that seems to have been overlooked is that certain infinite families of "nontrivial" homogeneous varieties are Fano. We disprove Kodaira vanishing for some of these varieties.

Kovács found that for the Haboush-Lauritzen Fano 6-fold X in characteristic 2, Kodaira vanishing fails already for the ample line bundle $-2K_X$; explicitly, we have $H^1(X, -K_X) \neq 0$ [16]. By taking a cone over X, he gave an example of a canonical singularity which is not Cohen-Macaulay, on a 7-fold in characteristic 2. Yasuda had earlier constructed quotient singularities of any characteristic $p > 0$ which are canonical but not Cohen-Macaulay [25, Proposition 6.9], [9, Remark 5.3].

We find an even better phenomenon among Fano varieties in every characteristic $p > 2$. Namely, there is a smooth Fano variety X in characteristic p such that $-K_X$ is divisible by 2, $-K_X = 2A$, and Kodaira vanishing fails for the ample line bundle $3A$; explicitly, we have $H^1(X, A) \neq 0$ (Theorem 2.1). Here X has dimension $2p+1$. By taking a cone over X, we give a first example of a terminal singularity which is not Cohen-Macaulay. Moreover, we have such examples in every characteristic $p > 2$.

After these results were announced, Takehiko Yasuda showed that there are also quotient singularities of any characteristic $p > 0$ which are terminal but not Cohen-Macaulay [26], summarized in section 4. Inspired by Yasuda's examples (which are quotients by a finite group acting linearly), we construct a new example in the lowest possible dimension: a terminal singularity *of dimension 3* over $\mathbf{F}_2$ which is not Cohen-Macaulay (Theorem 5.1). Our singularity is the quotient of a smooth variety by a non-linear action of the group $\mathbf{Z}/2$, and such quotients should be a rich source of further examples. The paper concludes with some open questions.

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1 Projective homogeneous varieties with non-reduced stabilizer group

In this section, we describe the projective homogeneous varieties with non-reduced stabilizer group. More details are given by Haboush and Lauritzen [8]. They assume that the base field is algebraically closed. We will construct projective homogeneous varieties over any field, but all the properties we consider can be checked after passage to an algebraically closed field.

Let G be a simply connected split semisimple group over a field k of characteristic $p > 0$. Let T be a split maximal torus in G and B a Borel subgroup containing T. Let $\Phi \subset X(T)$ be the set of roots of G with respect to T, and define the subset Φ^+ of positive roots to be the roots of G that are not roots of B. Let Δ be the associated set of simple roots. Choose a numbering of the simple roots, $\Delta = \{\alpha_1, \ldots, \alpha_l\}$, where l is the rank of G (the dimension of T). For each root β, there is an associated root subgroup $U_\beta \subset G$ isomorphic to the additive group G_a, the unique G_a subgroup normalized by T on which T acts by β.

Let f be a function from Δ to the set of natural numbers together with ∞. The function f determines a subgroup scheme P of G containing B, as follows. Every positive root β can be written uniquely as a linear combination of simple roots with nonnegative coefficients. The *support* of β means the set of simple roots whose coefficient in β is positive. Extend the function $f\colon \Delta \to \mathbf{N} \cup \infty$ to a function on all the positive roots, by defining

$$f(\beta) = \inf_{\alpha \in \mathrm{supp}(\beta)} f(\alpha).$$

For a natural number r, let α_{p^r} be the subgroup scheme of G_a defined by $x^{p^r} = 0$. Let P_{red} be the parabolic subgroup generated by B and the root subgroup U_β for each simple root β with $f(\beta) = \infty$. Finally, define the subgroup scheme P (with underlying reduced subgroup P_{red}) as the product (in any order) of P and the subgroup scheme α_{p^r} of U_β for each positive root β with $f(\beta) = r < \infty$. Wenzel showed that if $p \geq 5$, or if G is simply laced, then every subgroup scheme of G containing B is of this form for some function f [23, Theorem 14]. For our purpose, it is enough to use these examples of subgroup schemes.

The natural surjection $G/P_{\mathrm{red}} \to G/P$ is finite and purely inseparable. The homogeneous variety G/P is smooth and projective over k (even though P is not smooth). Lauritzen showed that G/P has a cell decomposition over k, coming from the Bruhat decomposition of G/P_{red} [18]. In particular, G/P is rational over k, and $H^i(G/P, \Omega^j) = 0$ for $i \neq j$.

If the function f takes only one value r apart from ∞, then P is the subgroup scheme generated by P_{red} and the rth Frobenius kernel of G. In that case, G/P is isomorphic to G/P_{red} as a variety. (In terms of such an isomorphism, the surjection $G/P_{\mathrm{red}} \to G/P$ is the rth power of the Frobenius endomorphism of G/P_{red}.) By contrast, for more general functions f, there can be intriguing differences between the properties of G/P and those of the familiar flag variety G/P_{red}.

The Picard group $\mathrm{Pic}(G/P)$ can be identified with a subgroup of $\mathrm{Pic}(G/P_{\mathrm{red}})$, or of $\mathrm{Pic}(G/B) = X(T)$, by pullback. For each root α in $X(T)$, write $\alpha^\vee$ for the corresponding coroot in $Y(T) = \mathrm{Hom}(X(T), \mathbf{Z})$. Then $X(T)$ has a basis given by the fundamental weights $\omega_1, \ldots, \omega_l$, which are characterized by:

$$\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}.$$

The subgroup $\mathrm{Pic}(G/P_{\mathrm{red}})$ is the subgroup generated by the ω_i with $f(i) < \infty$. And $\mathrm{Pic}(G/P)$ is the subgroup generated by $p^{f(i)}\omega_i$, for each i with $f(i) < \infty$ [8, section 2, Corollary 7]. A line bundle on G/P is ample if and only if its pullback to G/P_{red} is ample, which means that the coefficient of ω_i is positive for every $i \in \{1, \ldots, l\}$ with $f(i) < \infty$. Moreover, Lauritzen showed that every ample line bundle on a homogeneous variety G/P is very ample (that is, it has enough sections to embed G/P into projective space) [17, Theorem 1].

The anticanonical bundle of $X = G/P$ is given by [8, section 3, Proposition 7]

$$-K_X = \sum_{\substack{\beta \in \Phi^+ \\ f(\beta) < \infty}} p^{f(\beta)} \beta.$$

Haboush and Lauritzen showed that G/P is never Fano when $P_{\mathrm{red}} = B$ and $p \geq 5$, except when the function f is constant, in which case X is isomorphic to the full flag variety G/B [8, section 4, proof of Theorem 3]. By contrast, inseparable images of some partial flag varieties can be Fano without being isomorphic to the partial flag variety. The example we use in this paper is: let $G = SL(n)$ for $n \geq 3$ over a field k of characteristic $p > 0$. As is standard, write the weight lattice as $X(T) = \mathbf{Z}\{L_1, \ldots, L_n\}/(L_1 + \cdots + L_n = 0)$ [7, section 15.1]. The positive roots are $L_i - L_j$ for $1 \leq i < j \leq n$. The fundamental weights are given by $\omega_i = L_1 + \cdots + L_i$ and the simple roots are $\alpha_i = L_i - L_{i+1}$, for $1 \leq i \leq n-1$. Thus $\alpha_i = -\omega_{i-1} + 2\omega_i - \omega_{i+1}$ for $1 \leq i \leq n-1$, with the convention that ω_0 and ω_n are zero. Let P be the subgroup scheme associated to the function

$$f(\alpha_i) = \begin{cases} 1 & \text{if } i = 1. \\ 0 & \text{if } i = 2 \\ \infty & \text{if } 3 \leq i \leq n-1. \end{cases}$$

Then G/P_{red} is the flag manifold $\text{Fl}(1, 2, n)$, of dimension $2n-3$ and Picard number 2, and so $X := G/P$ also has dimension $2n-3$ and Picard number 2. By the formula above, X has anticanonical bundle

$$\begin{aligned} -K_X &= p(L_1 - L_2) + \sum_{i=3}^{n}(L_2 - L_i) + \sum_{i=3}^{n}(L_1 - L_i) \\ &= (p + n - 2)L_1 + (-p + n - 2)L_2 - 2(L_3 + \cdots + L_n) \\ &= 2p\,\omega_1 + (n - p)\omega_2. \end{aligned}$$

Thus $-K_X$ is ample if and only if $2p > 0$ and $n - p > 0$, which means that $p < n$.

2 Terminal cones that are not Cohen-Macaulay

Theorem 2.1. *Let p be a prime number at least 3. Then there is a smooth Fano variety X over $\mathbf{F}_p$ such that $-K_X$ is divisible by 2, $-K_X = 2A$, and $H^1(X, A) \neq 0$. Here X has dimension $2p + 1$ and Picard number 2.*

Moreover, for $p \geq 5$, the Euler characteristic $\chi(X, A)$ is negative.

Thus Kodaira vanishing fails for the ample line bundle $3A$. These are the first examples of smooth Fano varieties in characteristic greater than 2 for which Kodaira vanishing fails. Haboush and Lauritzen exhibited a smooth Fano 6-fold in characteristic 2 for which Kodaira vanishing fails. (This is [8, section 6, Example 4] or [19, section 2]. Both papers give examples of the failure of Kodaira vanishing in any characteristic, but the variety they consider is Fano only in characteristic 2.)

In the cases where $\chi(X, A)$ is negative, one amusing consequence is that the smooth Fano variety X does not lift to characteristic zero, even over a ramified extension R of the p-adic integers. Indeed, given such a lift $\mathcal{X}$, the result $H^2(X, O) = 0$ from section 1 implies that there is no obstruction to lifting a line bundle from X to the formal scheme $\mathcal{X}$ [11, Corollary 8.5.5]. Applying this to the ample line bundle A, Grothendieck's algebraization theorem implies that $\mathcal{X}$ extends to a projective scheme over R on which A is ample [11, Corollary 8.5.6]. But then $\chi(X, A)$ would be nonnegative, by Kodaira vanishing in characteristic zero.

Corollary 2.2. *Let p be a prime number at least 3. Then there is an isolated terminal singularity over $\mathbf{F}_p$ which is not Cohen-Macaulay. One can take its dimension to be $2p + 2$.*

These were the first examples of terminal singularities that are not Cohen-Macaulay. After these results were announced, Yasuda gave lower-dimensional examples, described in section 4. That in turn inspired the 3-dimensional example in this paper (Theorem 5.1).

Proof. (Corollary 2.2) In the notation of Theorem 2.1, let Y be the affine cone over the smooth Fano variety X with respect to the ample line bundle A, meaning

$$Y = \operatorname{Spec} \oplus_{m \geq 0} H^0(X, mA).$$

A cone Y is terminal if and only if the ample line bundle A is $\mathbf{Q}$-linearly equivalent to $a(-K_X)$ for some $0 < a < 1$, as is the case here (with $a = 1/2$) [15, 3.1]. Also, a cone Y is Cohen-Macaulay if and only if $H^i(X, mA) = 0$ for all $0 < i < \dim(X)$ and all $m \in \mathbf{Z}$ [15, 3.11]. Since $H^1(X, A) \neq 0$ and X has dimension $2p + 1 > 1$, Y is not Cohen-Macaulay. □

Proof. (Theorem 2.1) Let $n = p + 2$, and let $X = G/P$ be the smooth projective homogeneous variety for $G = SL(n)$ over $\mathbf{F}_p$ associated to the function f from section 1. Thus X has dimension $2n - 3 = 2p + 1$, and

$$\begin{aligned} -K_X &= 2p\,\omega_1 + (n-p)\omega_2 \\ &= 2p\,\omega_1 + 2\omega_2. \end{aligned}$$

Because $2p$ and 2 are positive, $-K_X$ is ample. Because $\operatorname{Pic}(X)$ has a basis given by $p\,\omega_1$ and ω_2, $-K_X$ is divisible by 2 in $\operatorname{Pic}(X)$, $-K_X = 2A$, with $A := p\,\omega_1 + \omega_2$. As discussed in section 1, the ample line bundle A on X is in fact very ample.

In the notation of section 1, let Q be the parabolic subgroup of G associated to the function

$$h(\alpha_i) = \begin{cases} \infty & \text{if } 1 \leq i \leq n-1 \text{ and } i \neq 2 \\ 0 & \text{if } i = 2. \end{cases}$$

Because $f \leq h$, P is contained in Q. The morphism $G/P \to G/Q$ is a $\mathbf{P}^1$-bundle, and G/Q is the Grassmannian $\operatorname{Gr}(2, n)$. In more detail, G/P is the Frobenius twist of the obvious $\mathbf{P}^1$-bundle over this Grassmannian. We analyze the cohomology of A using this $\mathbf{P}^1$-bundle, as Haboush and Lauritzen did in a similar situation [8, section 6].

Write α for the simple root α_1, the one with $f(\alpha) = 1$. For any line bundle on G/P, identified with a weight λ, $\langle \lambda, \alpha^\vee \rangle$ is a multiple of p, and the degree of λ on the $\mathbf{P}^1$ fibers of $G/P \to G/Q$ is $\langle \lambda, \alpha^\vee \rangle / p$. For the line bundle A, we have $\langle A, \alpha^\vee \rangle = p$, and so A has degree 1 on the fibers of $G/P \to G/Q$.

Consider the commutative diagram

$$\begin{tikzcd} G/B \arrow[r] \arrow[dr, "\gamma"'] & G/P \arrow[d, "\pi"] \\ & G/Q. \end{tikzcd}$$

Since the line bundle A has degree 1 on the $\mathbf{P}^1$ fibers of $G/P \to G/Q$, it has no higher cohomology on the fibers, and so $H^i(G/P, A) \cong H^i(G/Q, \pi_*(A))$ for all i. Moreover, since $h^0(\mathbf{P}^1, O(1)) = 2$, $\pi_*(A)$ is a vector bundle of rank 2 on G/Q.

Next, the morphism $\gamma\colon G/B \to G/Q$ has fibers isomorphic to Q/B, which is a projective homogeneous variety (with reduced stabilizer groups). Explicitly, Q/B is the fiber of the map $\mathrm{Fl}(1, 2, \ldots, n) \to \mathrm{Gr}(2, n)$, which is $\mathbf{P}^1$ times the flag manifold of $SL(n-2)$. Therefore, $H^i(Q/B, O) = 0$ for $i > 0$, and hence $R\gamma_*(O_{G/B}) = O_{G/Q}$. By the projection formula, it follows that $H^i(G/Q, \pi_*(A)) \cong H^i(G/B, \gamma^*\pi_*A)$ for all i. This and the previous isomorphism are isomorphisms of G-modules.

Finally, $\gamma^*\pi_*A$ is a G-equivariant rank-2 vector bundle on G/B, and so it corresponds to a 2-dimensional representation M of B [12, section II.4.2]. Every representation of B is an extension of 1-dimensional representations $k(\mu)$, corresponding to weights μ of T [12, section II.2.1(2)]. In this case, a direct computation [8, section 6, Proposition 2] shows that M is an extension

$$0 \to k(\lambda - p\alpha) \to M \to k(\lambda) \to 0,$$

where $\lambda := p\,\omega_1 + \omega_2$ is the weight corresponding to the line bundle A.

As a result, the long exact sequence of cohomology on G/B takes the form:

$$0 \to H^0(G/B, \lambda - p\alpha) \to H^0(G/P, A) \to H^0(G/B, \lambda) \to H^1(G/B, \lambda - p\alpha) \to \cdots .$$

At this point, we could compute the Euler characteristic $\chi(G/P, A) = \chi(G/B, \lambda) + \chi(G/B, \lambda - p\alpha)$ by the Riemann-Roch theorem on G/B (essentially the Weyl character formula), and see that $\chi(G/P, A)$ is negative for $p \geq 5$. That would suffice to disprove Kodaira vanishing for A. However, we choose to give a more detailed analysis of the cohomology of A, which will apply to the case $p = 3$ as well.

Let $\rho \in X(T)$ be half the sum of the positive roots; this is also the sum of the fundamental weights. As is standard in Lie theory, consider the "dot action" of the Weyl group W of G on the weight lattice:

$$w \cdot \mu = w(\mu + \rho) - \rho.$$

Let $s_\beta \in W$ be the reflection associated to a root β, $s_\beta(\mu) = \mu - \langle \mu, \beta^\vee \rangle \beta$.

We use the following result of Andersen on the cohomology of line bundles on the flag variety [12, Proposition II.5.4(d)]:

Theorem 2.3. *For a reductive group G in characteristic $p > 0$, a simple root β, and a weight μ with $\langle \mu, \beta^\vee \rangle$ of the form $sp^m - 1$ for some $s, m \in \mathbf{N}$ with $0 < s < p$, there is an isomorphism of G-modules for each integer i:*

$$H^i(G/B, \mu) \cong H^{i+1}(G/B, s_\beta \cdot \mu).$$

For the weight λ and simple root α considered above, we have $\langle \lambda - \alpha, \alpha^\vee \rangle = p - 2$ and $s_\alpha \cdot (\lambda - \alpha) = \lambda - p\alpha$. Here $p - 2$ is in the range where Theorem 2.3 applies, and so we have

$$H^i(G/B, \lambda - \alpha) \cong H^{i+1}(G/B, s_\alpha \cdot (\lambda - \alpha)).$$

Explicitly, $\lambda - \alpha = (p - 2)\omega_1 + 2\omega_2$, which is a dominant weight (like λ). By Kempf's vanishing theorem, it follows that the line bundles λ and $\lambda - \alpha$ on G/B have cohomology concentrated in degree zero [12, Proposition II.4.5]. Assembling all these results, we have an exact sequence of G-modules:

$$0 \to H^0(G/P, A) \to H^0(G/B, \lambda) \to H^0(G/B, \lambda - \alpha) \to H^1(G/P, A) \to 0,$$

and $H^i(G/P, A) = 0$ for $i \geq 2$.

In view of Kempf's vanishing theorem, the dimensions of the *Schur modules* $H^0(G/B, \lambda)$ and $H^0(G/B, \lambda - \alpha)$ are given by the Weyl dimension formula, as in characteristic zero. For $SL(n)$, the formula says [7, Theorem 6.3(1)]: for a dominant weight $\mu = a_1\omega_1 + \cdots + a_{n-1}\omega_{n-1}$,

$$h^0(G/B, \mu) = \prod_{1 \leq i < j \leq n} \frac{a_i + \cdots + a_{j-1} + j - i}{j - i}.$$

We read off that

$$h^0(G/B, \lambda) = \binom{2p+2}{p}(p+1)$$

and

$$h^0(G/B, \lambda - \alpha) = \binom{2p+1}{p}\frac{(p+2)(p-1)}{2}.$$

To compare these numbers, compute the ratio:

$$\frac{h^0(G/B, \lambda - \alpha)}{h^0(G/B, \lambda)} = \frac{(p-1)(p+2)^2}{4(p+1)^2}.$$

This is greater than 1 if $p \geq 5$ (since then $p - 1 \geq 4$). By the previous paragraph's exact sequence, it follows that the Euler characteristic $\chi(G/P, A)$ is negative for $p \geq 5$. Since A has no cohomology in degrees at least 2, we must have $H^1(G/P, A) \neq 0$, as we want.

For $p = 3$, $h^0(G/B, \lambda)$ is 224 whereas $h^0(G/B, \lambda - \alpha)$ is 175, and so the dimensions would allow the G-linear map $\varphi\colon H^0(G/B, \lambda) \to H^0(G/B, \lambda - \alpha)$ to be surjective. But in fact it is not surjective (and hence $H^1(G/P, A)$ is not zero), as we now show.

For a dominant weight μ, write $L(\mu)$ for the simple G-module with highest weight μ. For a reductive group G in any characteristic and a dominant weight μ, Chevalley showed that the socle (maximal semisimple submodule) of the Schur module $H^0(G/B, \mu)$ is simple, written $L(\mu)$. Moreover, this construction gives a one-to-one correspondence between the simple G-modules and the dominant weights [12, Corollary II.2.7].

The Steinberg tensor product theorem describes all simple G-modules in terms of those whose highest weight has coefficients less than p [12, Corollary II.3.17]. In particular, since $\lambda = p\,\omega_1 + \omega_2$, the theorem says that

$$L(\lambda) \cong L(\omega_1)^{[1]} \otimes L(\omega_2),$$

writing $M^{[1]}$ for the Frobenius twist of a G-module M. For $SL(n)$ in any characteristic, the simple module associated to the fundamental weight ω_i is the exterior power $\Lambda^i(V)$ with V the standard n-dimensional representation [12, section II.2.15]. So $L(\lambda)$ has dimension $\binom{n}{2}n = \binom{p+2}{2}(p+2)$. Also, because $\lambda > \lambda - \alpha$ in the partial ordering of the weight lattice given by the positive roots, the weight λ does not occur in the G-module $H^0(G/B, \lambda - \alpha)$. It follows that the G-linear map $\varphi\colon H^0(G/B, \lambda) \to H^0(G/B, \lambda - \alpha)$ is zero on the simple submodule $L(\lambda)$.

For $p = 3$, $H^0(\lambda)/L(\lambda)$ has dimension $224 - 50 = 174$, whereas $H^0(\lambda - \alpha)$ has dimension 175. It follows that φ is not surjective. Equivalently, $H^1(G/P, A) \neq 0$, as we want. $\square$

3 Lower-dimensional failure of Kodaira vanishing for Fano varieties

In this section, we give slightly lower-dimensional examples of smooth Fano varieties X in any characteristic $p > 0$ for which Kodaira vanishing fails: dimension $2p - 1$ rather than $2p + 1$ for $p \geq 3$, and dimension 5 for $p = 2$. (In particular, the examples in characteristics 2 or 3 have dimension 5, which is smaller than the dimension 6 of Haboush-Lauritzen's smooth Fano variety in characteristic 2 where Kodaira vanishing fails.) In return for this improvement, we consider ample line bundles that are not rational multiples of $-K_X$.

Theorem 3.1. *Let p be a prime number. Then there is a smooth Fano variety X over $\mathbf{F}_p$ and a very ample line bundle A on X such that $H^1(X, K_X + A) \neq 0$. Here X has dimension $2p - 1$ for $p \geq 3$ and dimension 5 for $p = 2$. Also, X has Picard number 2.*

Moreover, in the examples with $p \neq 3$, the Euler characteristic $\chi(X, K_X + A)$ is negative.

Proof. First assume $p \geq 3$. At the end, we will give the example for $p = 2$.

As in the proof of Theorem 2.1, let X be the homogenous variety over $\mathbf{F}_p$ defined in section 1, but now with n equal to $p + 1$ rather than $p + 2$. Thus X is a smooth projective homogenous variety for $SL(n)$ of dimension $2n - 3 = 2p - 1$. The anticanonical bundle of X is

$$\begin{aligned} -K_X &= 2p\,\omega_1 + (n-p)\omega_2 \\ &= 2p\,\omega_1 + \omega_2. \end{aligned}$$

Because $2p$ and 1 are positive, $-K_X$ is ample. (In this case, $-K_X$ is not divisible by 2.)

The Picard group of X has a basis consisting of $p\,\omega_1$ and ω_2. Therefore, the weight $\lambda := 3p\,\omega_1 + \omega_2$ corresponds to another ample line bundle A on X. In fact, A is very ample, as discussed in section 1. The weight $\mu := K_X + \lambda$ is equal to $p\,\omega_1$. Because the simple root $\alpha := \alpha_1 = 2\omega_1 - \omega_2$ has $\langle \mu, \alpha^\vee \rangle = p$ and $\langle \mu - \alpha, \alpha^\vee \rangle = p - 2$ (which is less than p), the same argument as in the proof of Theorem 2.1 gives an exact sequence of G-modules:

$$0 \to H^0(G/P, K_X + A) \to H^0(G/B, \mu) \to H^0(G/B, \mu - \alpha) \to H^1(G/P, K_X + A) \to 0,$$

and $H^i(G/P, K_X + A) = 0$ for $i \geq 2$.

Write V for the n-dimensional representation $H^0(G/B, \omega_1)$ of $G = SL(n)$. Since $\mu = p\,\omega_1$, $H^0(G/B, \mu)$ is the symmetric power $S^p(V)$, which has dimension $\binom{n+p-1}{n-1} = \binom{2p}{p}$ [12, Proposition I.5.12 and section II.2.16]. Also, $\mu - \alpha$ is equal to $(p - 2)\omega_1 + \omega_2$, which is also dominant, and the Weyl dimension formula gives that $h^0(G/B, \mu - \alpha)$ is $\binom{2p-1}{p-1}(p-1)$. It follows that the ratio $h^0(G/B, \mu - \alpha)/h^0(G/B, \mu)$ is equal to $(p - 1)/2$. Now suppose that $p \geq 5$. Then $H^0(G/B, \mu - \alpha)$ has bigger dimension than $H^0(G/B, \mu)$. By the exact sequence above, it follows that the Euler characteristic $\chi(X, K_X + A)$ is negative. Since $K_X + A$ has no cohomology in degrees at least 2, it follows that $H^1(X, K_X + A) \neq 0$ for $p \geq 5$, as we want.

Next, let $p = 3$. Then $G = SL(4)$, and the G-modules $H^0(G/B, \mu) = S^3V$ and $H^0(G/B, \mu - \alpha)$ both have dimension 20. However, because $\mu > \mu - \alpha$ in the partial order of the weight lattice given by the positive roots, the weight μ occurs in $H^0(G/B,\mu)$ and not in $H^0(G/B,\mu-\alpha)$. Therefore, the G-linear map $\varphi\colon H^0(G/B,\mu)\to H^0(G/B,\mu-\alpha)$ is not an isomorphism, and hence not surjective. By the exact sequence above, it follows that $H^1(X,K_X+A)\neq 0$.

Finally, let $p=2$. In this case, let n be $p+2=4$ (not $p+1$ as above). Let X be the homogeneous variety for $SL(n)$ over $\mathbf{F}_2$ described in section 1. Then X is a smooth Fano variety of dimension $2n-3=5$ and Picard number 2.

The Picard group of X is generated by $p\omega_1=2\omega_1$ and ω_2. The anticanonical bundle $-K_X$ is $2p\,\omega_1+(n-p)\omega_2=4\omega_1+2\omega_2$. Let A be the ample line bundle $6\omega_1+\omega_2$ on X. It is in fact very ample, as discussed in section 1. Let $\mu=K_X+A=2\omega_1-\omega_2$. Because of the negative coefficient, $H^0(X,\mu)=0$ (for example by the inclusion $H^0(X,\mu)\subset H^0(G/B,\mu)$ given by pullback). So Kodaira vanishing would imply that $H^i(X,\mu)=0$ for all i.

To disprove that, we compute the Euler characteristic. As in the proof of Theorem 2.1, X is a $\mathbf{P}^1$-bundle over the Grassmannian $\mathrm{Gr}(2,n)$. The degree of a line bundle ν on the $\mathbf{P}^1$ fibers is $\langle \nu,\alpha^\vee\rangle/p$, where α is the simple root α_1 and $p=2$. So μ has degree 1 on those fibers. As in the proof of Theorem 2.1, it follows that there is a long exact sequence of G-modules:

$$\to H^i(G/B,\mu-p\alpha)\to H^i(X,\mu)\to H^i(G/B,\mu)\to H^{i+1}(G/B,\mu-p\alpha)\to$$

Therefore, in terms of Euler characteristics,

$$\chi(X,\mu)=\chi(G/B,\mu)+\chi(G/B,\mu-p\alpha).$$

Here $G=SL(4)$ and $\mu=2\omega_1-\omega_2$, and so $\mu-p\alpha=-2\omega_1+\omega_2$. The Weyl dimension formula says that

$$\begin{aligned}&\chi(G/B,a_1\omega_1+a_2\omega_2+a_3\omega_3)\\&\qquad=\frac{(a_1+1)(a_2+1)(a_3+1)(a_1+a_2+2)(a_2+a_3+2)(a_1+a_2+a_3+3)}{12}.\end{aligned}$$

It follows that $\chi(G/B,\mu)=0$ and $\chi(G/B,\mu-p\alpha)=-1$, and hence $\chi(X,\mu)=-1$. Because this is negative, Kodaira vanishing fails on the smooth Fano 5-fold X in characteristic 2.

To prove the full statement of Theorem 3.1, we want to show more specifically that $H^1(X,K_X+A)=H^1(X,\mu)$ is not zero. It suffices to show that $H^i(X,\mu)=0$ for $i>1$. By the exact sequence above, this follows if the line bundles $\mu=2\omega_1-\omega_2$ and $\mu-p\alpha=-2\omega_1+\omega_2$ on G/B have no cohomology in degrees greater than 1. Because $\langle\mu,\alpha_2^\vee\rangle=-1$ (that is, μ has degree -1 on the fibers of one of the $\mathbf{P}^1$-fibrations of G/B), μ actually has no cohomology in any degree [12, Proposition II.5.4(a)]. Next, the trivial line bundle on G/B has cohomology only in degree zero by Kempf's vanishing theorem, and $s_\alpha\cdot 0=-2\omega_1+\omega_2=\mu-p\alpha$. Beause $\langle 0,\mu^\vee\rangle=0$ is of the form sp^m-1 for some $s,m\in\mathbf{N}$ with $0<s<p$, Theorem 2.3 gives that

$$H^i(G/B,O)\cong H^{i+1}(G/B,\mu-p\alpha)$$

for all i. Thus $\mu-p\alpha$ has cohomology only in degree 1, as we want. □

4 Quotient singularities

In this section, we describe Yasuda's examples of quotient singularities of any characteristic $p > 0$ which are terminal but not Cohen-Macaulay [26]. Again, the dimension increases with p, but more slowly than in the examples above.

For $p \geq 5$, his construction is as follows. Let G be the cyclic group $\mathbf{Z}/p$ and k a field of characteristic p. For each $1 \leq n \leq p$, there is a unique indecomposable representation V of G over k of dimension n, with a generator of G acting by a single Jordan block. Assume that $p \geq n \geq 4$. By [26, Corollary 1.4], $X := V/G$ is klt if and only if $n(n-1)/2 \geq p$, and X is terminal if and only if $n(n-1)/2 > p$. On the other hand, because the fixed point set V^G has dimension 1, which has codimension at least 3 in V, X is not Cohen-Macaulay, by Ellingsrud-Skjelbred [5] or Fogarty [6]. By a similar construction (using decomposable representations of $\mathbf{Z}/p$), Yasuda finds non-Cohen-Macaulay terminal quotient singularities of dimension 6 in characteristic 2 and of dimension 5 in characteristic 3.

5 A 3-dimensional terminal singularity that is not Cohen-Macaulay

Inspired by Yasuda's examples [26], we now give the first example of a terminal 3-fold singularity which is not Cohen-Macaulay. The base field can be taken to be any field of characteristic 2, say $\mathbf{F}_2$. Write $G_m = A^1 - \{0\}$ for the multiplicative group.

Theorem 5.1. *Let X be the 3-fold $(G_m)^3/(\mathbf{Z}/2)$ over the field $\mathbf{F}_2$, where the generator σ of $\mathbf{Z}/2$ acts by*

$$\sigma(x_1, x_2, x_3) = \left(\frac{1}{x_1}, \frac{1}{x_2}, \frac{1}{x_3}\right).$$

Then X is terminal but not Cohen-Macaulay.

Proof. Let Y be the 3-fold $(G_m)^3$ over the field $k = \mathbf{F}_2$. Clearly $G = \mathbf{Z}/2$ acts freely outside the point $(1, 1, 1)$ in Y. By Fogarty, when the group $G = \mathbf{Z}/p$ acts on a regular scheme Y in characteristic p such that the fixed point set has an irreducible component of codimension at least 3, Y/G is not Cohen-Macaulay [6].

It remains to show that $X = Y/G$ is terminal. Most of the work is to construct an explicit resolution of singularities of X. We do that by performing G-equivariant blow-ups of Y until the quotient variety becomes smooth over k. In characteristic 0, the quotient of a smooth variety by a cyclic group of prime order is smooth if and only if every irreducible component of the fixed point set has codimension at most 1. That fails for actions of $\mathbf{Z}/p$ in characteristic p, but there is a useful substitute by Király and Lütkebohmert [13, Theorem 2]:

Theorem 5.2. *Let G be a cyclic group of prime order which acts on a regular scheme X. If the fixed point scheme X^G is a Cartier divisor in X, then the quotient space X/G is regular.*

Since we work with varieties over the perfect field $k = \mathbf{F}_2$, being regular is the same as being smooth over k.

In checking the properties of the blow-up, it is helpful to observe that the singularity $X = Y/G$ has an enormous automorphism group. Namely, $GL(3, \mathbf{Z})$ acts by automorphisms of the torus $Y = (G_m)^3$, and this commutes with the action of $G = \mathbf{Z}/2$ (which corresponds to the diagonal matrix -1 in $GL(3, \mathbf{Z})$). Therefore, $GL(3, \mathbf{Z})$ acts on X (through its quotient $PGL(3, \mathbf{Z})$, clearly). The formal completion of X at its singular point has an action of an even bigger group, $PGL(3, \mathbf{Z}_2)$.

Remark 5.3. By analogy with the study of infinite discrete automorphism groups of projective varieties, this example suggests that it could be interesting to study profinite groups acting on formal completions of singularities in characteristic p.

We now begin to blow up. Identify Y with $Y_0 = (A^1 - \{1\})^3$ over k by $y_i = x_i + 1$ for $i = 1, 2, 3$. Then G acts on Y_0 by

$$\sigma(y_1, y_2, y_3) = \left(\frac{y_1}{1+y_1}, \frac{y_2}{1+y_2}, \frac{y_3}{1+y_3}\right).$$

We do this so that the point fixed by G is $(y_1, y_2, y_3) = (0, 0, 0)$. Let Y_1 be the blow-up of Y_0 at this point. Thus

$$Y_1 = \{((y_1, y_2, y_3), [w_1, w_2, w_3]) \in Y_0 \times \mathbf{P}^2 : y_1 w_2 = y_2 w_1, y_1 w_3 = y_3 w_1, y_2 w_3 = y_3 w_2\}.$$

The group G acts on Y_1 by

$$\sigma((y_1, y_2, y_3), [w_1, w_2, w_3]) = \left(\frac{y_1}{1+y_1}, \frac{y_2}{1+y_2}, \frac{y_3}{1+y_3}\right), \left[\frac{w_1}{1+y_1}, \frac{w_2}{1+y_2}, \frac{w_3}{1+y_3}\right].$$

Because $GL(3, \mathbf{Z})$ fixes the origin in Y_0, the action of $GL(3, \mathbf{Z})$ on Y_0 lifts to an action on the blow-up Y_1. (This includes the obvious action of the symmetric group S_3.)

To compute the fixed point scheme of G on Y_1, work in the open subset U_1 with $w_1 = 1$; this will suffice, by the S_3-symmetry of Y_1. We can view U_1 as the open subset of $A^3 = \{(y_1, w_2, w_3)\}$ defined by $y_1 \neq 1$, $y_1 w_2 \neq 1$, and $y_1 w_3 \neq 1$ (using that $y_2 = y_1 w_2$ and $y_3 = y_1 w_3$). The action of G on U_1 is given by

$$\sigma(y_1, w_2, w_3) = \left(\frac{y_1}{y_1+1}, \frac{w_2(y_1+1)}{y_1 w_2 + 1}, \frac{w_3(y_1+1)}{y_1 w_3 + 1}\right).$$

So the fixed point scheme $(U_1)^G$ is defined by the equations

$$y_1 = \frac{y_1}{y_1+1},\ w_2 = \frac{w_2(y_1+1)}{y_1 w_2 + 1},\ w_3 = \frac{w_3(y_1+1)}{y_1 w_3 + 1}.$$

Equivalently, $y_1^2 = 0$, $y_1 w_2(w_2 + 1) = 0$, and $y_1 w_3(w_3 + 1) = 0$. Thus the scheme $(U_1)^G$ is not a Cartier divisor; it is equal to the Cartier divisor $y_1 = 0$ (the exceptional divisor) except at the points $(y_1, w_2, w_3) = (0, 0, 0), (0, 0, 1), (0, 1, 0)$, and $(0, 1, 1)$.

In view of the S_3-symmetry of Y_1, it follows that the fixed point scheme $(Y_1)^G$ is equal to the exceptional divisor $E \cong \mathbf{P}^2$ with multiplicity 1 except at 7 points on that divisor, where $(y_1, y_2, y_3) = (0, 0, 0)$ and $[w_1, w_2, w_3]$ is one of $[1, 0, 0]$, $[0, 1, 0]$, $[0, 0, 1]$, $[1, 1, 0]$, $[1, 0, 1]$, $[1, 1, 0]$, or $[1, 1, 1]$. Note that $GL(3, \mathbf{Z})$ acts through its quotient group $GL(3, \mathbf{F}_2)$ on the divisor E, and it permutes these 7 points transitively. Therefore, to resolve the singularities of Y_1/G, it will suffice to blow Y_1 up at the point $[1,0,0]$; the blow-ups at the rest of the 7 points work exactly the same way.

The point we are blowing up is in the open set U_1 of Y_1 defined above, namely $(y_1, w_2, w_3) = (0,0,0)$ in

$$U_1 = \{(y_1, w_2, w_3)\} \in A^3 : y_1 \neq 1,\ y_1 w_2 \neq 1,\ \text{and } y_1 w_3 \neq 1\}.$$

The resulting blow-up Y_2 is:

$$\{((y_1, w_2, w_3), [v_1, v_2, v_3]) \in U_1 \times \mathbf{P}^2 : y_1 v_2 = w_2 v_1, y_1 v_3 = w_3 v_1, w_2 v_3 = w_3 v_2\}.$$

The group G acts on Y_2 by:

$$\sigma((y_1, w_2, w_3), [v_1, v_2, v_3]) = \left(\frac{y_1}{y_1+1}, \frac{w_2(y_1+1)}{y_1 w_2 + 1}, \frac{w_3(y_1+1)}{y_1 w_3 + 1}\right), \left[\frac{v_1}{y_1+1}, \frac{v_2(y_1+1)}{y_1 w_2 + 1}, \frac{v_3(y_1+1)}{y_1 w_3 + 1}\right].$$

To compute the fixed point scheme of G on Y_2, work first in the open set $v_1 = 1$. In those coordinates, G acts by:

$$\sigma(y_1, v_2, v_3) = \left(\frac{y_1}{y_1+1}, \frac{v_2(y_1+1)^2}{y_1^2 v_2 + 1}, \frac{v_3(y_1+1)^2}{y_1^2 v_3 + 1}\right)$$

(using that $w_2 = y_1 v_2$ and $w_3 = y_1 v_3$). The fixed point scheme of G on this open set is given by the equations $y_1^2 = 0$, $y_1^2 v_2(v_2+1) = 0$, and $y_1^2 v_3(v_3+1) = 0$, which just say that $y_1^2 = 0$. Thus the fixed point scheme $(Y_2)^G$ is a Cartier divisor in this open set: 2 times the new exceptional divisor E_1.

We next compute the fixed point scheme $(Y_2)^G$ in the open set $v_2 = 1$; we will not need to consider the remaining open set $v_3 = 1$ separately, in view of the symmetry between v_2 and v_3 in the action of G on Y_2. Namely, G acts on the open set $v_2 = 1$ by

$$\sigma(w_2, v_1, v_3) = \left(\frac{w_2(w_2 v_1 + 1)}{w_2^2 v_1 + 1}, \frac{v_1(w_2^2 v_1 + 1)}{(w_2 v_1 + 1)^2}, \frac{v_3(w_2^2 v_1 + 1)}{w_2^2 v_1 v_3 + 1}\right)$$

(using that $y_1 = w_2 v_1$ and $w_3 = w_2 v_3$). The fixed point scheme of G on this open set is given by the equations

$$w_2^2 v_1(w_2+1) = 0,\ w_2^2 v_1^2(v_1+1) = 0,\ \text{and } w_2^2 v_1 v_3(v_3+1) = 0.$$

Note that we defined Y_2 by blowing up only one of the 7 points listed earlier in Y_1, the one with $(y_1, w_2, w_3) = (0,0,0)$; since we are not concerned with the other 6 points here, we can assume that $w_2 \neq 1$. Then the first equation defining $(Y_2)^G$ gives that $w_2^2 v_1 = 0$, and that implies the other two equations. That is, we have shown that $(Y_2)^G$ is a Cartier divisor in the open set $v_2 = 1$ near the exceptional divisor E_1: it is $E_0 + 2E_1$, where E_0 is the proper transform of the exceptional divisor E in Y_1 (given by $v_1 = 0$ in these coordinates).

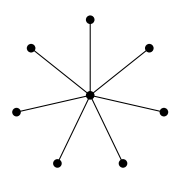
By the symmetry between v_2 and v_3 in the equations for Y_2, the same calculation applies to the open set $v_3 = 1$. Thus we have shown that the fixed point scheme $(Y_2)^G$ is Cartier near the exceptional divisor E_1.

From now on, write Y_2 for the blow-up of Y_1 at all 7 points listed above. By the previous calculation together with the $GL(3, \mathbf{Z})$-symmetry of Y_2, the fixed point scheme $(Y_2)^G$ is the Cartier divisor

$$E_0 + 2\sum_{j=1}^{7} E_j,$$

where $E_1, \ldots, E_7$ are the 7 exceptional divisors of $Y_2 \to Y_1$. By Theorem 5.2, it follows that Y_2/G is smooth over k. Thus Y_2/G is a resolution of singularities of $X = Y_0/G$.

Write $F_0, F_1, \ldots, F_7$ for the images in Y_2/G of the exceptional divisors $E_0, E_1, \ldots, E_7$. Note that although G fixes each divisor E_j in Y_2, the morphism $E_j \to F_j$ is a finite purely inseparable morphism, not necessarily an isomorphism. (Indeed, $G = \mathbf{Z}/2$ is not linearly reductive in characteristic 2. So if G acts on an affine scheme T preserving a closed subscheme S, the morphism $S/G \to T/G$ need not be a closed immersion. Equivalently, the G-equivariant surjection $O(T) \to O(S)$ need not yield a surjection $O(T)^G \to O(S)^G$.) In any case, our construction shows that the dual complex of the resolution $Y_2/G \to X$ is a star, with one edge from the vertex F_0 to each of the other 7 vertices $F_1, \ldots, F_7$.



This generalizes Artin's observation that the analogous singularity one dimension lower, $(G_m)^2/(\mathbf{Z}/2)$ in characteristic 2, is a D_4 surface singularity. That is, the dual graph of its minimal resolution is again a star, with one central vertex connected to 3 other vertices [1, p. 64].

The divisor class K_X is Cartier on $X = Y_0/G$, because G preserves the volume form $(dx_1/x_1) \wedge (dx_2/x_2) \wedge (dx_3/x_3)$ on the torus $Y_0 \cong (G_m)^3$. So we can write

$$K_{Y_2/G} = \pi^* K_X + \sum_{j=0}^{7} a_j F_j$$

for some (unique) integers a_j, where the sum runs over all exceptional divisors F_j of $Y_2/G \to X$. The variety X is terminal if and only if the *discrepancy* a_j is positive for all j [15, Corollary 2.12]. (Note that this characterization of terminal singularities applies to any resolution of singularities; there is no need for $\sum_j F_j$ to be a normal crossing divisor.) Here and below, we write π for all the relevant contractions, which in the formula above means $\pi\colon Y_2/G \to Y_0/G = X$.

The analogous formula for Y_2 is easy, because Y_2 is obtained from Y_0 by blowing up points. First, since Y_1 is the blow-up of the smooth 3-fold Y_0 at a point,

$$K_{Y_1} = \pi^* K_{Y_0} + 2E.$$

Next, Y_2 is the blow-up of Y_1 at 7 points on the exceptional divisor E, and so

$$\begin{aligned} K_{Y_2} &= \pi^* K_{Y_1} + 2\sum_{j=1}^{7} E_j \\ &= \pi^* K_{Y_0} + 2E_0 + 4\sum_{j=1}^{7} E_j, \end{aligned}$$

using that $\pi^* E = E_0 + \sum_{j=1}^{7} E_j$.

Write f for the quotient map $Y_0 \to Y_0/G$ or $Y_2 \to Y_2/G$. It remains to compute the *ramification index* of each divisor E_j in Y_2 (the positive integer e_j such that $f^* F_j = e_j E_j$) and the coefficient c_j of E_j in the ramification divisor (meaning that $K_{Y_2} = f^* K_{Y_2/G} + \sum_j c_j E_j$). Another name for c_j is the *valuation of the different* $v_L(\mathcal{D}_{L/K})$, where L is the function field $k(Y_2)$, $K = k(Y_2/G)$, and v_L is the valuation of L associated to the divisor E_j. Here $e_j f_j = 2$, where f_j is the degree of the field extension $k(E_j)$ over $k(F_j)$ (which is purely inseparable in the case at hand).

We want to compute these numbers without actually finding equations for the quotient variety Y_2/G. This can be done using the *Artin* and *Swan ramification numbers* of the G-action on Y_2, defined as:

$$\begin{aligned} i(\sigma) &= \inf_{a \in O_L} v_L(\sigma(a) - a) \\ s(\sigma) &= \inf_{a \in L^*} v_L(\sigma(a) a^{-1} - 1). \end{aligned}$$

Here O_L is the ring of integers of $L = k(Y_2)$ with respect to the valuation v_L associated to a given divisor E_j. We have already computed $i(\sigma)$ for each E_j: it is the multiplicity of E_j in the fixed point scheme $(Y_2)^G$, which is 1 for E_0 and 2 for E_j with $1 \leq j \leq 7$. Then, more generally for an action of $\mathbf{Z}/p$ on a normal scheme of characteristic p that fixes an irreducible divisor, we have $s(\sigma) > 0$, and either $i(\sigma) = s(\sigma) + 1$, in which case $e = p$ and $f = 1$, or $i(\sigma) = s(\sigma)$, in which case $e = 1$ and $f = p$ [24, section 2.1]. The first case is called *wild* ramification, and the second is called *fierce* ramification. In both cases, the valuation of the different $v_L(\mathcal{D}_{L/K})$ is equal to $(p-1)i(\sigma)$.

In particular, returning to our example with $p = 2$, we have computed $i(\sigma)$ for each divisor E_j (the multiplicity of E_j in the fixed point scheme $(Y_2)^G$), and (by the formula above for $v_L(\mathcal{D}_{L/K})$) this computes the ramification divisor of $f\colon Y_2 \to Y_2/G$. Namely,

$$K_{Y_2} = f^* K_{Y_2/G} + E_0 + 2\sum_{j=1}^{7} E_j.$$

The next step is to compute the ramification index of f along each exceptional divisor E_j. For E_0, we have $i(\sigma) = 1$ (the multiplicity of E_0 in the fixed point scheme $(Y_2)^G$). Then the results above imply that $Y_2 \to Y_2/G$ is fiercely ramified along E_0, and so $s(\sigma) = 1$. In particular, $e_0 = 1$, meaning that $f^* F_0 = E_0$.

For E_j with $1 \leq j \leq 7$, we have $i(\sigma) = 2$ (the multiplicity of E_j in the fixed point scheme $(Y_2)^G$), which implies that $s(\sigma)$ is 1 or 2 by the results above. To resolve this ambiguity, note that it suffices to compute $s(\sigma)$ for E_1, because the automorphism group $GL(3, \mathbf{Z})$ of Y_2 (commuting with G) permutes $E_1, \ldots, E_7$ transitively. And we showed that E_1 is defined in the coordinate chart $v_1 = 1$ by the equation $y_1 = 0$, on which G acts by $\sigma(y_1) = y_1/(y_1+1)$. So $v_L(\sigma(y_1)y_1^{-1} - 1) = v_L(y_1/(y_1+1)) = 1$. Since $s(\sigma) = \inf_{a \in L^*} v_L(\sigma(a)a^{-1} - 1)$, it follows that $s(\sigma)$ is 1, not 2. Thus $Y_2 \to Y_2/G$ is wildly (rather than fiercely) ramified along E_j for $1 \le j \le 7$. In particular, $f^*F_j = 2E_j$.

Thus, we have shown that

$$K_{Y_2} = f^*K_{Y_2/G} + E_0 + 2\sum_{j=1}^{7} E_j$$

and that $f^*F_0 = E_0$ and $f^*F_j = 2E_j$ for $1 \le j \le 7$. Since $f\colon Y_0 \to Y_0/G$ is étale in codimension 1, we have $K_{Y_0} = f^*K_{Y_0/G}$. It follows that

$$\begin{aligned} f^*K_{Y_2/G} &= K_{Y_2} - E_0 - 2\sum_{j=1}^{7} E_j \\ &= \left(\pi^*K_{Y_0} + 2E_0 + 4\sum_{j=1}^{7} E_j\right) - E_0 - 2\sum_{j=1}^{7} E_j \\ &= \pi^* f^* K_{Y_0/G} + E_0 + 2\sum_{j=1}^{7} E_j \\ &= f^*\left(\pi^*K_{Y_0/G} + F_0 + \sum_{j=1}^{7} F_j\right). \end{aligned}$$

Therefore,

$$K_{Y_2/G} = \pi^*K_{Y_0/G} + F_0 + \sum_{j=1}^{7} F_j.$$

Because the coefficient of every exceptional divisor F_j is positive, and Y_2/G is a resolution of singularities of Y_0/G, $X = Y_0/G$ is terminal. $\square$

6 Open questions

One question suggested by these examples is whether, for each positive integer n, there is a number $p_0(n)$ such that Fano varieties of dimension n in characteristic $p \ge p_0(n)$ satisfy Kodaira vanishing. (One could ask this for smooth Fanos, or in greater generality.) This is related to the fundamental question of whether the smooth Fano varieties of given dimension form a bounded family over $\mathbf{Z}$, as they do in characteristic zero by Kollár-Miyaoka-Mori [14, Corollary V.2.3].

A related question is whether, for each positive integer n, there is a number $p_0(n)$ such that klt singularities of dimension n in characteristic $p \ge p_0(n)$ are Cohen-Macaulay. This was shown by Hacon and Witaszek for $n = 3$, although no explicit value for $p_0(3)$ is known [9, Theorem 1.1].

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