Klt varieties with conjecturally minimal volume

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We construct mildly singular (klt) complex projective varieties with ample canonical class and the smallest known volume. We also find exceptional Fano varieties with the smallest known volume. In fact, we conjecture that our examples have the minimum volume in every dimension n, and we give low-dimensional evidence to support this. Crudely, the volume is about $1/2^{2^n}$. These varieties improve on the examples by Chengxi Wang and me [32, Theorem 0.1]. Those examples had roughly the right asymptotics, but they were known not to be optimal.

By definition, the *volume* of a normal projective variety X measures the asymptotic growth of the plurigenera,

$$\mathrm{vol}(X) := \lim_{m\to\infty} h^0(X, mK_X)/(m^n/n!),$$

where $n = \dim(X)$. The volume is equal to the intersection number K_X^n if the canonical class K_X is ample; it is the basic discrete invariant for a variety of general type, analogous to the genus of a curve. (When X is klt and K_X is ample, this is literally the volume of X with its unique Kähler-Einstein metric of Ricci curvature -1, up to a constant factor [12, Theorem C].) By Hacon-M$^{\mathrm{c}}$Kernan-Xu, there is a positive lower bound for the volumes of all klt complex projective varieties with ample canonical class, depending only on the dimension [16, Theorem 1.3]. Finding an explicit bound is a central problem in the classification of algebraic varieties, wide open in dimensions at least 3. (Alexeev, Mori, and Liu gave a bound in dimension 2 [3, Theorem 4.8], [2, section 10].) We are showing that the bound must go to zero extremely fast, and conjecturally we give the exact bound in each dimension. In related work, Kollár found a klt *pair* with ample canonical class and standard coefficients (described in section 2) that conjecturally has minimal volume among such pairs [23], [15, Introduction]. If we restrict to varieties with milder singularities, Esser, Wang, and I constructed varieties in several classes with small volume, such as smooth varieties of general type or terminal Fano varieties [11].

The examples here are hypersurfaces in weighted projective spaces. In contrast to most previous work, including the examples by Wang and me [32], the new hypersurfaces are not quasi-smooth. As a result, the singularities are not always quotient singularities, and proving that they are klt is more subtle. In dimension 2, our example is Alexeev-Liu's klt surface with ample canonical class and volume 1/48983 [2, Theorem 1.4]. Their construction was different, but we find that their example is in fact a non-quasi-smooth hypersurface, namely $X_{438} \subset \mathbf{P}^3(219, 146, 61, 11)$.

We also consider an analogous problem for klt Fano varieties. The anticanonical volume of a klt Fano variety can be arbitrarily small in a given dimension. (For example, for any positive integer a, the weighted projective plane $Y = \mathbf{P}^2(2a+1, 2a, 2a-1)$ is a klt del Pezzo surface with $\mathrm{vol}(-K_Y) = 18a/(4a^2-1)$.) However, Birkar showed that exceptional Fano varieties form a bounded family in each dimension, and in particular there is a positive lower bound for their volumes [4, Theorem 1.3]. (By definition, a klt Fano variety X is *exceptional* if the pair (X, D) is klt for every effective $\mathbf{Q}$-divisor D that is $\mathbf{Q}$-linearly equivalent to $-K_X$. Equivalently, the *global log canonical threshold* (or *α-invariant*) of X is greater than 1, by [5, Theorem 1.7]. Non-exceptional Fano varieties can be analyzed in terms of lower-dimensional Fano pairs, and so exceptional Fano varieties can be considered the core of the classification problem for Fano varieties.) By the recent proof of the Yau-Tian-Donaldson conjecture for singular varieties, every exceptional Fano variety has a Kähler-Einstein metric [30, Theorem 1.4], [28, Theorem 1.6].

We construct here what we conjecture to be the exceptional Fano variety of minimum volume in every dimension n. Again, the volume is roughly $1/2^{2^n}$. It seems that the only known examples of exceptional Fano varieties in high dimensions are the quasi-smooth hypersurfaces found by Johnson and Kollár [20, Proposition 3.3]. We extend their argument to prove exceptionality of our examples. In fact, we compute the global log canonical threshold exactly. Crudely, it is about 2^{2^n}, hence greater than 1 as we want. The method, based on weighted multiplicities in place of the usual multiplicity of a variety at a point, should be useful for many other Fano varieties.

We give low-dimensional evidence that our exceptional Fano varieties have the minimum volume in each dimension. As in the case of ample canonical class, seeking optimal examples among all exceptional Fano varieties forces us to consider non-quasi-smooth hypersurfaces in weighted projective space. In dimension 2, our example (apparently new) is the exceptional del Pezzo surface $X_{354} \subset \mathbf{P}^3(177, 118, 49, 11)$, for which $\text{vol}(-K_X) = 1/31801$. This is smaller than the volume of any exceptional del Pezzo surface with Picard number 1, by Lacini's classification of those surfaces [26, 21]. (The surface of smallest volume in Lacini's list is LDP 15 with $(s, r) = (4, 1)$, part (1), blowing up above q along A eight times; then X has volume 1/3953.) The surface here has Picard number 2.

Finally, in every even dimension, we construct klt varieties with ample canonical class, and also klt Fano varieties, with the largest known bottom weight (Theorems 8.1 and 9.1). The *bottom weight* means the smallest positive integer m such that $H^0(X, mK_X) \neq 0$ in the K-ample case, or $H^0(X, -mK_X) \neq 0$ in the Fano case. The global log canonical threshold should be extremely large in these examples, perhaps even maximal (Question 8.2).

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1 Background

Our examples use *Sylvester's sequence*, defined by $s_0 = 2$ and $s_{n+1} = s_n(s_n - 1) + 1$. The sequence begins $2, 3, 7, 43, 1807, \ldots$. We have $s_{n+1} = s_0 \cdots s_n + 1$, and hence the numbers in Sylvester's sequence are pairwise coprime. More important for applications to extremal problems is that the sums of the reciprocals of these numbers converge quickly to 1:

$$\frac{1}{s_0} + \cdots + \frac{1}{s_{n-1}} = 1 - \frac{1}{s_n - 1}.$$

Here s_n grows doubly exponentially in n, and so this sum is very close to 1. In fact, for every positive integer n, Soundararajan showed that the sum above is the closest to 1 of all sums of n unit fractions that are less than 1 [31].

There is a constant $c \doteq 1.264$ such that s_i is the closest integer to $c^{2^{i+1}}$ for all $i \geq 0$ [14, equations 2.87 and 2.89]. For example, it follows that $s_i > 2^{2^{i-1}}$ for all $i \geq 0$. We write $a_i \sim b_i$ to mean that two sequences of positive real numbers are asymptotic, meaning that a_i/b_i converges to 1 as i goes to infinity.

We consider algebraic varieties over the complex numbers, although some of the paper would work in any characteristic. A reference for the singularities of the minimal model program, such as Kawamata log terminal (klt) and log canonical (lc), is [25]. We often use without comment that the klt or lc properties of a pair (X, Δ) are unchanged under finite coverings f of normal varieties with f étale in codimension 1 [25, Corollary 2.43]. (This would not be true for other singularity classes such as terminal or canonical.) As a result, we do not need to distinguish between the klt or lc property for a normal Deligne-Mumford stack and for its associated coarse moduli space, provided that the stabilizer groups are trivial in codimension 1.

For an effective $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor D on a klt variety X, the *log canonical threshold* $\text{lct}(X, D)$ is the supremum of the real numbers λ such that the pair $(X, \lambda D)$ is lc. For a klt Fano variety X, the *global log canonical threshold* (or *α-invariant*) $\text{glct}(X)$ is the supremum of the real numbers λ such that $(X, \lambda D)$ is lc for every effective $\mathbf{Q}$-divisor D with $D \sim_{\mathbf{Q}} -K_X$. It follows that a Fano variety with global log canonical threshold greater than 1 must be exceptional. In fact, $\text{glct}(X) > 1$ is equivalent to exceptionality, by Birkar [5, Theorem 1.7].

For positive integers $a_0, \ldots, a_n$, the weighted projective space $Y = \mathbf{P}^n(a_0, \ldots, a_n)$ means the quotient variety $(A^{n+1} - 0)/G_m$ over $\mathbf{C}$, where the multiplicative group G_m acts by $t(x_0, \ldots, x_n) = (t^{a_0}x_0, \ldots, t^{a_n}x_n)$ [17, section 6]. Starting in section 4, we switch to viewing weighted projective space as the quotient stack $\mathcal{Y} = [(A^{n+1} - 0)/G_m]$. Here $\mathcal{Y}$ is a smooth Deligne-Mumford stack with canonical class $K_{\mathcal{Y}} = O_{\mathcal{Y}}(-\sum a_j)$. We say that Y is *well-formed* if the stack $\mathcal{Y}$ has trivial stabilizer in codimension 1, or equivalently if $\gcd(a_0, \ldots, \widehat{a_j}, \ldots, a_n) = 1$ for each j. In the well-formed case, the canonical class of the variety Y is given by the same formula as for the stack.

Here $O(1)$ is a line bundle on the stack $\mathcal{Y}$. On the variety Y, with Y well-formed, $O_Y(1)$ is only the reflexive sheaf associated to a Weil divisor, in general; the divisor class $O_Y(m)$ is Cartier if and only if m is a multiple of every weight a_j. The intersection number $\int_{\mathcal{Y}} c_1(O(1))^n$ is $1/(a_0 \cdots a_n)$. More generally, for an integral closed substack Z of dimension r in $\mathcal{Y}$, its *degree* means $\int_Z c_1(O(1))^r$.

Let Y be a well-formed weighted projective space. A closed subvariety X of Y is called *quasi-smooth* if its affine cone in A^{n+1} is smooth outside the origin. (Equivalently, the inverse image of X in the stack $\mathcal{Y}$ is smooth over $\mathbf{C}$.) In particular, a quasi-smooth subvariety has only cyclic quotient singularities and hence is klt. Also, X is *well-formed* if Y is well-formed and the codimension of $X \cap Y^{\text{sing}}$ in X is at least 2. (For a well-formed weighted projective space Y, the singular locus of the variety Y corresponds to the locus where the stack $\mathcal{Y}$ has nontrivial stabilizer.)

For a well-formed normal hypersurface X of degree d in a weighted projective space Y, we have $K_X = O_X(d - \sum a_j)$. (We are not assuming quasi-smoothness of X.) Indeed, the canonical class of a normal variety is defined as a Weil divisor up to linear equivalence, and so we are free to delete closed subsets of codimension at least 2 from X in order to prove this formula. So we can delete the singular locus of X and the singular locus of Y from X and Y, and then $K_X = O_X(d - \sum a_j)$ is the usual adjunction formula for a smooth hypersurface in a smooth variety.

Note an ambiguity in the notion of "degree": if X is a hypersurface of degree d in $\mathcal{Y}$, meaning that it is defined by a weighted-homogeneous polynomial of degree d, then its degree as a substack of $\mathcal{Y}$ is $X \cdot c_1(O(1))^{n-1} = d/(a_0 \cdots a_n)$.

2 Klt varieties with ample canonical class

Theorem 2.1. *For each integer n at least 2, let*

$$a_{n+1} = \begin{cases} \frac{1}{4}(s_n^2 - s_n + 2) & \text{if } n \text{ is even} \\ \frac{1}{4}(s_n^2 - 3s_n + 4) & \text{if } n \text{ is odd.} \end{cases}$$

Let $a_n = (s_n - 2)a_{n+1} + (s_n - 1)$, $x = 1 + a_n + a_{n+1}$, $d = (s_n - 1)x$, and $a_i = d/s_i$ for $0 \leq i \leq n-1$. Then there is a hypersurface X of degree d in $\mathbf{P}^{n+1}(a_0, \ldots, a_{n+1})$ that is well-formed and klt, with $K_X = O_X(1)$. It has volume

$$\frac{1}{(s_n - 1)^{n-2} x^{n-1} a_n a_{n+1}},$$

which is asymptotic to $2^{2n+2}/s_n^{4n}$. In particular, this is less than $1/2^{2^n}$.

Explicitly, define X by the equation, for $n \geq 2$ even:

$$0 = x_0^2 + x_1^3 + \cdots + x_{n-1}^{s_{n-1}} + x_n^{s_n} x_{n+1} + x_1 \cdots x_n x_{n+1}^b,$$

where $b = (s_n^2 - 2s_n + 7)/2$. For $n \geq 3$ odd, define X by

$$0 = x_0^2 + x_1^3 + \cdots + x_{n-1}^{s_{n-1}} + x_n^{s_n} x_{n+1} + x_1 \cdots x_{n-1} x_n^2 x_{n+1}^b,$$

where now $b = (s_n^2 - 4s_n + 11)/2$. Since the number of monomials is equal to the number of variables, any linear combination of these monomials with all coefficients nonzero defines an isomorphic variety, by scaling the variables. One can check that the monomials shown are all the monomials of degree d, and hence that an open subset of all hypersurfaces of degree d are isomorphic to the one indicated; but we will not need those facts.

Note that X is not quasi-smooth.

Conjecture 2.2. For each integer n at least 2, the variety in Theorem 2.1 has the minimum volume among all klt projective n-folds with ample canonical class.

We know that there is some positive lower bound for the volume in each dimension, by Hacon-M$^{\text{c}}$Kernan-Xu [16, Theorem 1.3].

In dimension 2, our example is $X_{438} \subset \mathbf{P}^3(219, 146, 61, 11)$, with volume $1/48983 \doteq 2.0 \times 10^{-5}$. This is the smallest known volume for a klt surface with ample canonical class. This example was found earlier by Alexeev and Liu, without the description as a hypersurface [2, Theorem 1.4]. It has smaller volume than all quasi-smooth hypersurfaces of dimension 2 with $K_X = O_X(1)$, by Brown and Kasprzyk's computer classification [7, 8]. Namely, the best quasi-smooth hypersurface is $X_{316} \subset \mathbf{P}^3(158, 85, 61, 11)$, with volume $2/57035 \doteq 3.5 \times 10^{-5}$.

In dimension 3, our example is

$$X_{762090} \subset \mathbf{P}^4(381045, 254030, 108870, 17713, 431),$$

which has volume about 9.5×10^{-18}. This beats the best previously known 3-fold, the quasi-smooth hypersurface

$$X_{340068} \subset \mathbf{P}^4(170034, 113356, 47269, 9185, 223),$$

which has volume about 1.8×10^{-16}. (The latter example is optimal among quasi-smooth hypersurfaces with $K_X = O_X(1)$, by Brown and Kasprzyk's program, which can be downloaded from the Graded Ring Database. This example is part of the sequence of examples constructed by Wang and me [32, section 2].) Finally, in dimension 4, our new example has volume about 8.0×10^{-50}. Again, this beats the optimal quasi-smooth hypersurface with $K_X = O_X(1)$ in dimension 4, which has volume about 1.4×10^{-47} [8, ID 538926].

Finally, our klt variety with ample canonical class has volume quite close to that of Kollár's conjecturally optimal example in the broader setting of klt pairs with ample canonical class and standard coefficients (meaning coefficients of the form $(m-1)/m$ for positive integers m). That example is

$$(Y, \Delta) = \left(\mathbf{P}^n, \frac{1}{2}H_0 + \frac{2}{3}H_1 + \frac{6}{7}H_2 + \cdots + \frac{s_{n+1}-1}{s_{n+1}}H_{n+1}\right),$$

where $H_0, H_1, \ldots, H_{n+1}$ are $n+2$ general hyperplanes. The volume of $K_Y + \Delta$ is $1/(s_{n+2}-1)^n$, which is (crudely) about $1/2^{2^n}$. Our example in Theorem 2.1 has $\mathrm{vol}(X)/\mathrm{vol}(K_Y + \Delta)$ about 2^{2n+2}. (Precisely, $\log(\mathrm{vol}(X)/\mathrm{vol}(K_Y + \Delta))$ is asymptotic to $(2n+2)\log 2$ as n goes to infinity.) So $\mathrm{vol}(X)$ is bigger than $\mathrm{vol}(K_Y + \Delta)$, but not by much, since 2^{2n+2} is far smaller than 2^{2^n}. That is some further evidence for the optimality of Theorem 2.1.

Proof. (Theorem 2.1) To explain the choice of weights a_i, we first prove some properties of these numbers. First, we have $d - \sum a_i = 1$ (which will imply that $K_X = O_X(1)$), because $d - \sum_{i=0}^{n-1} a_i = d(1 - 1/s_0 - \cdots - 1/s_{n-1}) = d/(s_n - 1) = x = 1 + a_n + a_{n+1}$. Next, let us check that the monomials listed in the equation of X (above) have degree d. For n even, let $b = (s_n^2 - 2s_n + 7)/2$; then we have to show that $d = 2a_0 = 3a_1 = \cdots = s_{n-1}a_{n-1} = s_n a_n + a_{n+1} = a_1 + \cdots + a_n + b a_{n+1}$. All these equations except the last one are easy by our choice of weights. For the last one, note that

$$\begin{aligned}
d - a_1 - \cdots - a_n &= d\left(1 - \frac{1}{s_1} - \cdots - \frac{1}{s_{n-1}}\right) - a_n \\
&= d\left(\frac{1}{2} + \frac{1}{s_n - 1}\right) - a_n \\
&= \frac{s_n + 1}{2} x - a_n \\
&= \frac{s_n + 1}{2} + \frac{s_n - 1}{2} a_n + \frac{s_n + 1}{2} a_{n+1} \\
&= \frac{1}{2}(s_n^2 - s_n + 2) + \frac{1}{2}(s_n^2 - 2s_n + 3) a_{n+1} \\
&= \frac{1}{2}(s_n^2 - 2s_n + 7) a_{n+1},
\end{aligned}$$

as we want. For n odd, a similar calculation shows that the monomials in the equation of X have degree d.

We first show that the weighted projective space $Y = \mathbf{P}^{n+1}(a_0, \ldots, a_{n+1})$ is well-formed. That is, we have to show that $\gcd(a_0 \ldots, \widehat{a_j}, \ldots, a_{n+1}) = 1$ for each j. It suffices to show that $\gcd(a_{n+1}, x) = 1$, $\gcd(a_n, x) = 1$, and $\gcd(a_{n+1}, a_n, s_n - 1) = 1$. (This uses that $s_n - 1 = s_0 \cdots s_{n-1}$, where $s_0, \ldots, s_{n-1}$ are pairwise coprime.)

First, note that s_n is $7 \pmod 8$ if $n \geq 2$ is even and $3 \pmod 8$ if $n \geq 3$ is odd. This is immediate by induction from the recurrence $s_{n+1} = s_n(s_n - 1) + 1$. It follows that $s_n^2 - s_n + 2$ is $4 \pmod 8$ if $n \geq 2$ is even, and that $s_n^2 - 3s_n + 4$ is $4 \pmod 8$ if $n \geq 3$ is odd. So a_{n+1} is odd in both cases.

Next, let us show that $\gcd(a_{n+1}, x) = 1$. Suppose that a prime number p divides both a_{n+1} and x. Since a_{n+1} is odd, p is not 2. Since $x = 1 + a_n + a_{n+1}$, we have $a_n \equiv -1 \pmod p$. Since $a_n = (s_n - 2)a_{n+1} + (s_n - 1)$, we have $-1 = s_n - 1 \pmod p$, so p divides s_n. If $n \geq 2$ is even, $s_n^2 - s_n + 2 \equiv 2 \pmod p$, and this is not zero mod p. So a_{n+1} is not zero mod p, a contradiction. Likewise, for n odd, $s_n^2 - 3s_n + 4 \equiv 4 \pmod p$, and this is not zero mod p. So a_{n+1} is not zero mod p, a contradiction. Thus a_{n+1} is prime to x.

To show that $\gcd(a_n, x) = 1$, suppose that a prime number p divides both a_n and x. Since $x = 1 + a_n + a_{n+1}$, we have $a_{n+1} \equiv -1 \pmod p$. Since $a_n = (s_n - 2)a_{n+1} + (s_n - 1)$, we have $0 \equiv -(s_n - 2) + (s_n - 1) \equiv -1 \pmod p$, a contradiction. So a_n is prime to x.

It remains to show that $\gcd(a_n, a_{n+1}, s_n - 1) = 1$; in fact, we show that $\gcd(a_{n+1}, s_n - 1) = 1$. Let p be a prime number that divides a_{n+1} and $s_n - 1$. We have $s_n \equiv 1 \pmod p$, so $s_n^2 - s_n + 2 \equiv 2 \pmod p$ and $s_n^2 - 3s_n + 4 \equiv 2 \pmod p$. Since p is not 2, these two expressions are not zero mod p. It follows that a_{n+1} is not zero mod p, a contradiction. This completes the proof that Y is well-formed.

To show that X is well-formed, it remains to show that X does not contain any $(n-1)$-dimensional coordinate linear subspace of Y along which Y is singular. Since the equation of X includes the monomials $x_0^2, x_1^3, \ldots, x_{n-1}^{s_{n-1}}$, and also $x_n^{s_n} x_{n+1}$, X does not contain any positive-dimensional coordinate linear subspace of Y. Since $n \geq 2$, X is well-formed.

Next, we show that X is klt. First suppose that n is even. In this case, the equation defining X is $0 = x_0^2 + x_1^3 + \cdots + x_{n-1}^{s_{n-1}} + x_n^{s_n} x_{n+1} + x_1 \cdots x_n x_{n+1}^b$, where $b = (s_n^2 - 2s_n + 7)/2$. Since the number of monomials is equal to the number of variables, any linear combination of these monomials with nonzero coefficients defines a hypersurface isomorphic to X. So X is isomorphic to a general divisor in the linear system on Y spanned by these monomials. The base locus of this linear system is contained in the two last coordinate points, $[0,\ldots,0,1,0]$ and $[0,\ldots,0,1]$. Since we are in characteristic zero, Bertini's theorem on $A^{n+2}-0$ gives that a general divisor in this linear system is quasi-smooth outside those two points; so X is quasi-smooth outside those two points. In view of the monomial $x_n^{s_n}x_{n+1}$, X is also quasi-smooth at the point $[0,\ldots,0,1,0]$. So X is klt outside the point $[0,\ldots,0,1]$.

At the point $[0,\ldots,0,1]$, X is not quasi-smooth, but we will show that it is still klt. As just mentioned, we know that a general linear combination of the monomials in the equation of X defines a hypersurface X' isomorphic to X, by scaling the variables. So it suffices to show that a general hypersurface X' of that form is klt at $[0,\ldots,0,1]$. In coordinates $x_{n+1}=1$, the equation of X' is $0=c_0x_0^2+c_1x_1^3+\cdots+c_{n-1}x_{n-1}^{s_{n-1}}+c_nx_n^{s_n}+c_{n+1}x_1\cdots x_n$ for general complex numbers c_i. The open subset $x_{n+1}\neq 0$ of X' is the quotient by the finite cyclic group $\mu_{a_{n+1}}$ of the hypersurface with the same equation in A^{n+1}. Because the klt property is preserved by finite quotients, it suffices to show that such a general hypersurface S in A^{n+1} has canonical singularities (or equivalently, rational singularities).

Ishii and Prokhorov (following earlier work) described when the general hypersurface $S\subset A^{n+1}$ with equation spanned by a given set I of monomials has canonical singularities, as follows [18, Proposition 2.9]. By definition, the *Newton polyhedron* of a finite subset $I\subset \mathbf{R}^{n+1}$ is the convex hull of I plus the positive orthant, $(\mathbf{R}^{\geq 0})^{n+1}$.

Theorem 2.3. *Let I be a finite subset of $\mathbf{N}^{n+1}$, viewed as monomials in $\mathbf{C}[x_0,\ldots,x_n]$. Let $S\subset A^{n+1}_{\mathbf{C}}$ be the zero set of a general linear combination of these monomials. Assume that for every $0\leq i<j\leq n$, there is an monomial in I that contains neither x_i nor x_j; then the hypersurface S is normal. If the Newton polyhedron of I in $\mathbf{R}^{n+1}$ contains $(1,\ldots,1)$ in its interior, then the hypersurface S has canonical singularities. The converse holds if I contains no monomial of degree 1.*

As above, for $n\geq 2$ even, let S be the zero set in A^{n+1} of a general linear combination of $x_0^2,x_1^3,\ldots,x_{n-1}^{s_{n-1}},x_n^{s_n}$, and $x_1\cdots x_n$. The first condition in Theorem 2.3 (ensuring normality of S) is clear. Therefore, to show that S is canonical and hence X is klt, it suffices to show that the convex hull of the points $(2,0,\ldots,0)$, $(0,3,0,\ldots,0)$, $\ldots$, $(0,\ldots,0,s_n)$, $(0,1,\ldots,1)$ in $\mathbf{R}^{n+1}$ contains a point with all coordinates less than 1. In fact, we only need three of these points: namely, $(5/12)(2,0,\ldots,0)+(1/6)(0,3,0\ldots,0)+(5/12)(0,1,\ldots,1)$ has all coordinates less than 1. Thus X is klt when its dimension n is even.

For $n\geq 3$ odd, the equation defining X is $0=x_0^2+x_1^3+\cdots+x_{n-1}^{s_{n-1}}+x_n^{s_n}x_{n+1}+x_1\cdots x_{n-1}x_n^2x_{n+1}^b$, where now $b=(s_n^2-4s_n+11)/2$. As above, it is equivalent to consider a general linear combination of these monomials. As in the case of n even, X is quasi-smooth outside the point $[0,\ldots,0,1]$. To show that X is klt at that point, it suffices to show that the convex hull of the points $(2,0,\ldots,0)$, $(0,3,0,\ldots,0)$, $\ldots$, $(0,\ldots,0,s_n)$, $(0,1,\ldots,1,2)$ in $\mathbf{R}^{n+1}$ contains a point with all coordinates less than 1. Again, we only need three of these points: namely, $(5/12)(2,0,\ldots,0)+(1/6)(0,3,0\ldots,0)+(5/12)(0,1,\ldots,1,2)$ has all coordinates less than 1. Thus X is klt whether n is even or odd.

Since X is well-formed, the adjunction formula holds, meaning that $K_X = O_X(d - \sum a_j) = O_X(1)$. Therefore,

$$\begin{aligned}\operatorname{vol}(K_X) &= \frac{d}{a_0 \cdots a_{n+1}} \\ &= \frac{1}{(s_n - 1)^{n-2} x^{n-1} a_n a_{n+1}}.\end{aligned}$$

Here $a_{n+1} \sim s_n^2/4$ and $a_n \sim s_n^3/4$ (much bigger than a_{n+1}), so $x \sim s_n^3/4$. It follows that $\operatorname{vol}(K_X) \sim 2^{2n+2}/s_n^{4n}$. $\square$

3 Klt Fano varieties

Theorem 3.1. *For each integer n at least 2, let*

$$a_{n+1} = \begin{cases} \frac{1}{4}(s_n^2 - s_n + 2) & \text{if } n \text{ is even} \\ \frac{1}{4}(s_n^2 - 3s_n + 4) & \text{if } n \text{ is odd.} \end{cases}$$

Let $a_n = (s_n - 2)a_{n+1} - (s_n - 1)$, $x = -1 + a_n + a_{n+1}$, $d = (s_n - 1)x$, and $a_i = d/s_i$ for $0 \leq i \leq n-1$. Then there is a hypersurface X of degree d in $\mathbf{P}^{n+1}(a_0, \ldots, a_{n+1})$ that is a well-formed klt Fano variety, with $-K_X = O_X(1)$. The volume of $-K_X$ is

$$\frac{1}{(s_n - 1)^{n-2} x^{n-1} a_n a_{n+1}},$$

which is asymptotic to $2^{2n+2}/s_n^{4n}$. In particular, this is less than $1/2^{2^n}$.

Explicitly, define X by the equation, for $n \geq 2$ even:

$$0 = x_0^2 + x_1^3 + \cdots + x_{n-1}^{s_{n-1}} + x_n^{s_n} x_{n+1} + x_1 \cdots x_n x_{n+1}^b,$$

where $b = (s_n^2 - 2s_n - 1)/2$. For $n \geq 3$ odd, define X by

$$0 = x_0^2 + x_1^3 + \cdots + x_{n-1}^{s_{n-1}} + x_n^{s_n} x_{n+1} + x_1 \cdots x_{n-1} x_n^2 x_{n+1}^b,$$

where now $b = (s_n^2 - 4s_n + 3)/2$. Since the number of monomials is equal to the number of variables, any linear combination of these monomials with all coefficients nonzero defines an isomorphic variety, by scaling the variables. One can check that the monomials shown are all the monomials of degree d, and hence that an open subset of all hypersurfaces of degree d are isomorphic to the one indicated; but we will not need those facts.

Note that X is not quasi-smooth. We show in Theorems 6.1 and 7.1 that this Fano variety is exceptional.

Conjecture 3.2. For each integer n at least 2, the variety in Theorem 3.1 has the minimum anticanonical volume among all exceptional Fano n-folds.

Birkar showed that exceptional Fano varieties form a bounded family in each dimension, and so there is some positive lower bound for their volumes [4, Theorem 1.3].

In dimension 2, our example is the klt del Pezzo surface $X_{354} \subset \mathbf{P}^3(177, 118, 49, 11)$, for which $\mathrm{vol}(-K_X) = 1/31801 \doteq 3.1 \times 10^{-5}$. The lowest volume previously known for an exceptional del Pezzo surface occurs for Johnson-Kollár's quasi-smooth surface $X_{256} \subset \mathbf{P}^3(128, 69, 49, 11)$, with volume $2/37191 \doteq 5.4 \times 10^{-5}$ [19, Theorem 8]. (This volume is listed in the arXiv version of [9, Big Table].) Exceptionality of the latter surface follows from Johnson-Kollár's theorem that a well-formed quasi-smooth hypersurface $X_d \subset \mathbf{P}^{n+1}(a_0, \ldots, a_{n+1})$ with $d = -1 + \sum a_j$ (so $K_X = O_X(-1)$) and $a_0 \geq \cdots \geq a_{n+1}$ is exceptional if $d \leq a_n a_{n+1}$ [20, Proposition 3.3].

In dimension 3, our example is the klt Fano 3-fold

$$X_{758478} \subset \mathbf{P}^4(379239, 252826, 108354, 17629, 431),$$

with anticanonical volume about 9.6×10^{-18}. The lowest previously known volume of an exceptional Fano 3-fold occurs for Johnson-Kollár's quasi-smooth hypersurface

$$X_{336960} \subset \mathbf{P}^4(168480, 112320, 46837, 9101, 223),$$

which has volume about 1.9×10^{-16} [20, introduction]. Finally, our klt Fano 4-fold has volume about 8.0×10^{-50}. The smallest previously known volume of an exceptional Fano 4-fold is 1.4×10^{-47}, again for a certain quasi-smooth hypersurface [8, ID 1233322]. (To explain these previous records: the Graded Ring Database lists all quasi-smooth hypersurfaces with $K_X = O_X(-1)$ in dimension 4, in terms of 1597 infinite series and 1233322 sporadic cases. A summary can be found in [7, Theorem 1.3]. The database shows that the Fano 4-fold [8, ID 1233322] has the lowest volume among the sporadic cases, and then one checks by Johnson-Kollár's criterion that it is exceptional. The 3-dimensional quasi-smooth example mentioned above can be found by a similar search, using Brown and Kasprzyk's program.)

Proof. (Theorem 3.1) The proof is similar to that of Theorem 2.1, where the canonical class is ample. In particular, the weight a_{n+1} is the same in the two theorems, and the formulas for a_n and x differ only by sign changes. Modifying the calculation at the start of the proof of Theorem 2.1 by these sign changes shows that $d - \sum a_i = -1$ (rather than 1). Also, whether n is even or odd, we compute that the monomials in the equation for X have degree d.

As shown in the proof of Theorem 2.1, a_{n+1} is odd. The sign changes make no difference to the proof that $\gcd(a_{n+1}, x) = 1$, $\gcd(a_n, x) = 1$, and $\gcd(a_{n+1}, a_n, s_n - 1) = 1$. Therefore, $Y = \mathbf{P}^{n+1}(a_0, \ldots, a_{n+1})$ is well-formed.

To show that X is well-formed, it remains to show that X does not contain any $(n-1)$-dimensional coordinate linear subspace of Y along which Y is singular. Since the equation of X includes the monomials $x_0^2, x_1^3, \ldots, x_{n-1}^{s_{n-1}}$, and $x_n^{s_n} x_{n+1}$, X does not contain any positive-dimensional coordinate linear subspace of Y. Since $n \geq 2$, X is well-formed.

Next, we show that X is klt. First suppose that n is even. In this case, the equation defining X is $0 = x_0^2 + x_1^3 + \cdots + x_{n-1}^{s_{n-1}} + x_n^{s_n} x_{n+1} + x_1 \cdots x_n x_{n+1}^b$, where $b = (s_n^2 - 2s_n - 1)/2$. It follows that X is quasi-smooth and hence klt outside the point $[0, \ldots, 0, 1]$. At the point $[0, \ldots, 0, 1]$, X is not quasi-smooth, but we will show that it is still klt. In coordinates $x_{n+1} = 1$, the equation of X is $0 = x_0^2 + x_1^3 + \cdots + x_{n-1}^{s_{n-1}} + x_n^{s_n} + x_1 \cdots x_n$. We showed in the proof of Theorem 2.1 that this hypersurface in A^{n+1} has canonical singularities near the origin. Therefore, X (the quotient by $\mu_{a_{n+1}}$) is klt at $[0,\ldots,0,1]$, as we want.

For $n \geq 3$ odd, the equation defining X is $0 = x_0^2 + x_1^3 + \cdots + x_{n-1}^{s_{n-1}} + x_n^{s_n} x_{n+1} + x_1 \cdots x_{n-1} x_n^2 x_{n+1}^b$, where now $b = (s_n^2 - 4s_n + 3)/2$. Again, X is quasi-smooth and hence klt outside the last coordinate point $[0,\ldots,0,1]$. To show that X is klt at that point, we use coordinates $x_{n+1} = 1$ to write the equation of X as $0 = x_0^2 + x_1^3 + \cdots + x_{n-1}^{s_{n-1}} + x_n^{s_n} + x_1 \cdots x_{n-1} x_n^2$. We showed in the proof of Theorem 2.1 that this hypersurface in A^{n+1} has canonical singularities near the origin. Therefore, X (the quotient by $\mu_{a_{n+1}}$) is klt at $[0,\ldots,0,1]$, as we want, whether n is even or odd.

Since X is well-formed, the adjunction formula holds, meaning that $K_X = O_X(d - \sum a_j) = O_X(-1)$. Therefore,

$$\begin{aligned}\operatorname{vol}(-K_X) &= \frac{d}{a_0 \cdots a_{n+1}} \\ &= \frac{1}{(s_n - 1)^{n-2} x^{n-1} a_n a_{n+1}}.\end{aligned}$$

Here $a_{n+1} \sim s_n^2/4$ and $a_n \sim s_n^3/4$ (much bigger than a_{n+1}), so $x \sim s_n^3/4$. It follows that $\operatorname{vol}(-K_X) \sim 2^{2n+2}/s_n^{4n}$. ☐

4 Estimating the log canonical threshold in terms of the weighted tangent cone

Sections 4 to 7 will show that the klt Fano varieties in Theorem 3.1 are exceptional (Theorems 6.1 and 7.1). This follows from their global log canonical threshold (or α-invariant) being greater than 1. In fact, we compute the global log canonical threshold exactly. The method should be useful for many other examples.

Lemma 4.1. *Let $a_0, \ldots, a_n$ be positive integers with $n \geq 2$. Let X be a hypersurface in A^{n+1} over $\mathbf{C}$ that contains the origin. Let X_1 be the a-weighted tangent cone of X at the origin; thus X_1 is a hypersurface of some degree d in the weighted projective space $Y = \mathbf{P}^n(a_0, \ldots, a_n)$, viewed as a smooth Deligne-Mumford stack over $\mathbf{C}$. Assume that X_1 is normal. Let Δ be an effective $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor in X, and let $\Delta_1 \subset X_1$ be the weighted tangent cone of Δ. Then $\Delta_1 \sim_{\mathbf{Q}} O_{X_1}(e)$ for some positive rational number e, and $K_{X_1} + \Delta_1 \sim_{\mathbf{Q}} O_{X_1}(r)$ where $r = d + e - \sum a_j$. If $r \leq 0$ and (X_1, Δ_1) is log canonical, then (X, Δ) is log canonical near 0.*

This generalizes Kollár's description of which cones are lc, which (in effect) concerns the case $a_0 = \cdots = a_n = 1$ [25, Lemma 3.1]. We do not assume that X is a weighted cone. The proof is fairly straightforward once one is willing to use stack-theoretic weighted blow-ups. In retrospect, they are the right tool for the job.

Proof. Let $p\colon B \to X$ be the stack-theoretic weighted blow-up of $X \subset A^{n+1}$ at the origin with the given weights $a = (a_0, \ldots, a_n)$. Explicit coordinate charts can be found in [1, Section 3.4]. The weighted tangent cone X_1 is defined to be the exceptional divisor $E \subset B$; in particular, E is a hypersurface in the smooth stack $Y = \mathbf{P}^n(a_0, \ldots, a_n) = [(A^{n+1} - 0)/G_m]$. (Because we view Y as a stack, we can allow $a_0, \dots, a_n$ to have a common factor; that is, the weighted projective space Y need not be well-formed.)

We are assuming that $E = X_1$ is normal. It is a Cartier divisor in the stack B. By the adjunction formula, the canonical class $K_E = (K_B + E)|_E$ is given by $O_E(d - \sum_j a_j)$.

Next, write $X = \{f = 0\}$ for some regular function f near 0 in A^{n+1}. Let f_1 be the part of f with smallest weighted degree, so that $X_1 = \{f_1 = 0\}$ in Y. Since X_1 is a normal hypersurface in Y of dimension at least 1, it is irreducible. Since Δ is assumed to be $\mathbf{Q}$-Cartier on X, there is a positive integer m and a regular function g near 0 on A^{n+1} such that $m\Delta = \{f = g = 0\}$. After subtracting a multiple of f from g, we can assume that the part g_1 of g with smallest weighted degree is not a multiple of f_1. Then the weighted tangent cone $\Delta_1 \subset X_1$ is $1/m$ times the complete intersection $\{f_1 = g_1 = 0\}$ in Y. Let e be $1/m$ times the a-weighted degree of g_1; then $\Delta_1 \sim_{\mathbf{Q}} O_{X_1}(e)$ on X_1, as we want.

Therefore, $K_E + \Delta_1 \sim_{\mathbf{Q}} O_{X_1}(r) \sim_{\mathbf{Q}} -rE|_E$, where $r = d + e - \sum a_j \in \mathbf{Q}$. Write Δ_B for the birational transform $p_*^{-1}\Delta$ on B. Then $K_B + \Delta_B + (1+r)E \sim_{\mathbf{Q}} p^*(K_X + \Delta)$, using that both sides are trivial on E.

By [25, Definition 2.23], it follows that the discrepancy of (X, Δ) is given by

$$\operatorname{discrep}(X, \Delta) = \min(-1 - r, \operatorname{discrep}(B, (1+r)E + \Delta_B)).$$

(Here (X, Δ) is lc if and only if $\operatorname{discrep}(X, \Delta) \geq -1$.) So (X, Δ) is lc near 0 if $r \leq 0$ (as we assume) and $(B, (1+r)E + \Delta_B)$ is lc near E. Since $r \leq 0$, it suffices to show that $(B, E + \Delta_B)$ is lc. Since E is a normal Cartier divisor in B, inversion of adjunction says that this follows from $(E, \Delta_B|_E) = (X_1, \Delta_1)$ being lc [25, Proposition 4.5, Theorem 4.9]. □

The following corollary applies Lemma 4.1 to the case of a "weighted ordinary" hypersurface singularity, meaning that the weighted tangent cone is smooth. That covers many examples. The singularities in this paper are more complicated, however, and so we will have to go back to Lemma 4.1. Corollary 4.2 is a weighted version of an *Izumi-type inequality*, meaning a bound of the form $\operatorname{lct}_0(X, D) \geq c_X / \operatorname{mult}(D)$ (cf. [27, Theorem 3.2, Remark 3.3]).

Corollary 4.2. *Let $a_0 \geq \cdots \geq a_n$ be positive integers. Let X be a hypersurface in A^{n+1} over $\mathbf{C}$ that contains the origin. Let D be an effective $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor in X. Let X_1 be the weighted tangent cone of X at the origin; thus X_1 is a hypersurface of some degree d in the weighted projective space $Y = \mathbf{P}^n(a_0, \dots, a_n)$, viewed as a smooth Deligne-Mumford stack over $\mathbf{C}$. Suppose that the stack X_1 is smooth over $\mathbf{C}$. Let $D_1 \subset X_1$ be the weighted tangent cone of D. Let $b = -d + \sum_j a_j$. If $b > 0$, then the log canonical threshold of (X, D) near the origin satisfies:*

$$\operatorname{lct}_0(X, D) \geq \frac{\min(a_{n-1}a_n, bd)}{a_0 \cdots a_n \operatorname{mult}_a(D)}.$$

Here, given positive integers $a = (a_0, \dots, a_n)$, the *weighted multiplicity* (at the origin) of a closed subscheme $S \subset A^n$ over a field k, written $\operatorname{mult}_a(S)$, means the degree of the weighted tangent cone S_1 as a substack of the weighted projective space Y. Equivalently, define a decreasing filtration of the ring $O(S)$ of regular functions on S by: $F^bO(S)$ is the linear span of the monomials x^I with $\sum_{j=0}^n a_j i_j \geq b$. Let m be the dimension of S. Then the weighted multiplicity is the limit

$$\lim_{b\to\infty} \frac{\dim_k(O(S)/F^bO(S))}{b^m/m!}.$$

(This interpretation makes sense for any positive real weights $a_0, \ldots, a_n$.) When all weights are equal to 1, this is the usual multiplicity at S at the origin [13, Example 4.3.1].

Proof. (Corollary 4.2) Let $c = \min(a_{n-1}a_n, bd)/(a_0 \cdots a_n \deg(D_1))$. We want to show that (X, cD) is lc near 0. Let $\Delta = cD$ and $\Delta_1 = cD_1$ its weighted tangent cone. By Lemma 4.1, it suffices to show that $K_{X_1} + \Delta_1 \sim_{\mathbf{Q}} O_{X_1}(r)$ with $r \leq 0$ and that (X_1, Δ_1) is lc.

By the adjunction formula, we have $-K_{X_1} = O_{X_1}(b)$. To show that $r \leq 0$, it is equivalent to show that $\deg(\Delta_1) = \Delta_1 \cdot c_1(O(1))^{n-2} \leq (-K_{X_1}) \cdot c_1(O(1))^{n-2}$, where the intersection numbers are computed on the $(n-1)$-dimensional stack $X_1 \subset Y$. So $(-K_{X_1}) \cdot c_1(O(1))^{n-2} = bd/(a_0 \cdots a_n)$. Using this plus the fact that $\Delta_1 = cD_1$, the inequality above holds if $c \leq bd/(a_0 \cdots a_n \deg(D_1))$. That holds by our definition of c.

It remains to show that (X_1, Δ_1) is lc. We define the multiplicity at a point of an irreducible closed substack (or an effective algebraic cycle) in weighted projective space Y to be the multiplicity at a corresponding point of its inverse image in any orbifold chart $A^n \to [A^n/\mu_{a_i}] \cong \{x_i \neq 0\} \subset Y$. (This is independent of i, because the different orbifold charts are étale-locally isomorphic.) Johnson and Kollár proved the following bound [19, Proposition 11]. (The last sentence of Theorem 4.3 is not stated in their paper, but it is immediate from their argument.)

Theorem 4.3. *Let $a_0 \geq \cdots \geq a_n$ be positive integers. Let M be an irreducible closed substack of dimension r in the stack $\mathbf{P}^n(a_0, \ldots, a_n)$. Then the multiplicity of M at every point is at most $(a_0 \cdots a_r)\deg(M)$. If M is not contained in the hyperplane $x_n = 0$, then this bound can be improved to $(a_0 \cdots a_{r-1}a_n)\deg(M)$.*

Returning to the proof of Corollary 4.2, Theorem 4.3 gives that the $(n-2)$-dimensional cycle D_1 in Y has multiplicity at every point at most $(a_0 \cdots a_{n-2})\deg(D_1)$.

Now use the assumption that X_1 is a smooth stack. Then, for $0 \leq i \leq n$, the inverse image $X_{1,i}$ of X_1 in the ith orbifold chart is a smooth hypersurface in A^n. For a smooth variety S with an effective $\mathbf{Q}$-divisor T, the pair (S, T) is lc if T has multiplicity at most 1 at every point [25, Claim 2.10.4]. So the stack (X_1, cD_1) is lc if cD_1 has multiplicity at most 1 at each point. By the previous paragraph, it suffices to show that $c(a_0 \cdots a_{n-2})\deg(D_1) \leq 1$. This holds by the definition of c. So (X_1, Δ_1) is lc and hence (X, Δ) is lc. □

The proof of Corollary 4.2 works by bounding the unweighted multiplicity of D in A^{n+1}. At several points in this paper, it works better to bound a *weighted* multiplicity of D at the worst point of X, where information would be lost by going through Theorem 4.3. The idea is that D is given to us as a subspace of a weighted projective space; so we should use those weights in analyzing the singularities of D, as follows.

Lemma 4.4. *Let $a_0, \dots, a_{n+1}$ be positive integers (in any order). Let S be an irreducible closed substack of a weighted projective stack $Y = \mathbf{P}^{n+1}(a_0, \dots, a_{n+1})$. In coordinates $x_{n+1} = 1$, S corresponds to a subvariety of A^{n+1}. Consider the weights $a_0, \dots, a_n$ on A^{n+1}. Then the weighted multiplicity of S at the origin in A^{n+1} satisfies*

$$\operatorname{mult}_a(S) \le a_{n+1} \deg(S).$$

Proof. Consider the family of hypersurfaces $\{x_{n+1} = t\}$ in A^{n+2} as t varies. Write $C(S)$ for the affine cone over S in A^{n+2}. Consider the weighted multiplicity of $C(S) \cap \{x_{n+1} = t\}$ at the point $(0, \dots, 0, t)$. For $t \neq 0$, this is equal to $\operatorname{mult}_a(S)$. For $t = 0$, this is equal to $\deg(S \cap \{x_{n+1} = 0\}) = a_{n+1} \deg(S)$ if S is not contained in the hyperplane $\{x_{n+1} = 0\}$. Then upper semicontinuity of the weighted multiplicity gives that $\operatorname{mult}_a(S) \le a_{n+1} \deg(S)$. If S is contained in the hyperplane $\{x_{n+1} = 0\}$, then S does not contain the point $[0, \dots, 0, 1]$; so $\operatorname{mult}_a(S) = 0$ and the inequality again holds. □

5 The log canonical threshold for a certain singular hypersurface

In order to show that the klt Fano varieties in Theorem 3.1 are exceptional, we need to analyze their worst singular point, as follows. Specifically, we need to estimate the log canonical threshold of any divisor on this singular hypersurface. The proof involves an induction on these singularities of different dimensions.

Lemma 5.1. *Let n be a positive integer. Let X_n be the hypersurface in A^{n+1} defined by $0 = x_0^2 + x_1^3 + \cdots + x_n^{s_n} + x_1 \cdots x_n$. Let D be an effective $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor in X_n. Let $c_i = (s_{n+1} - 1)/s_i$ for $0 \le i \le n$. Write $\operatorname{mult}_c(D)$ for the c-weighted multiplicity of D at the origin. Then the log canonical threshold of D in X near the origin satisfies $\operatorname{lct}_0(X_n, D) \ge 1/\operatorname{mult}_c(D)$ if $n = 1$ and*

$$\operatorname{lct}_0(X_n, D) \ge \frac{2}{s_n^{n-1}(s_n+1)^2(s_n-1)^{n-3}\operatorname{mult}_c(D)}$$

if $n \ge 2$.

Proof. For $n = 1$, X_1 is a smooth curve, and the origin in X_1 has c-multiplicity 1. So every effective $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor D on X_1 has $\operatorname{lct}_0(X, D) \ge 1/\operatorname{mult}_c(D)$, as we want.

For any positive integer n, write λ_n for the constant in the lemma, so we are trying to show that $\operatorname{lct}_0(X_n, D) \ge \lambda_n/\operatorname{mult}_c(D)$. (In particular, let $\lambda_1 = 1$.) Suppose that $n \ge 2$ and the inequality holds for $n - 1$ in place of n. Let D be an effective $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor, and let $c_i = (s_{n+1} - 1)/s_i$ for $0 \le i \le n$. We have to show that if $\operatorname{mult}_c(D) = 1$, then $(X_n, \lambda_n D)$ is lc near the origin.

Consider the modified weights $w_i = c_i$ for $0 \le i \le n-1$ and $w_n = s_n(s_n+1)/2$. Since $w_n > c_n$, we have $\operatorname{mult}_w(D) \le \operatorname{mult}_c(D) = 1$. (The inequality is clear from the interpretation of weighted multiplicity in section 4.) To simplify the numbering, let $b_i = w_i/s_n$ for all $0 \le i \le n$. Since D has dimension $n - 1$, we have $\operatorname{mult}_b(D) = s_n^{n-1} \operatorname{mult}_w(D) \le s_n^{n-1}$. Here $b_i = (s_n - 1)/s_i$ for $0 \le i \le n-1$ and $b_n = (s_n + 1)/2$.

Note that $b_n > b_0 > \cdots > b_{n-1}$, contrary to our usual ordering. Finally, let $e = s_n - 1$.

The reason for considering the weights $b_0, \ldots, b_n$ on A^{n+1} is that the weighted tangent cone of the hypersurface X_n at the origin is klt; namely, it is the hypersurface X_{n-1} in $Y := \mathbf{P}^n(b_0, \ldots, b_n)$ of degree e defined by $0 = x_0^2 + x_1^3 + \cdots + x_{n-1}^{s_{n-1}} + x_1 \cdots x_n$. (Here only the monomial $x_n^{s_n}$ has disappeared.) The stack X_{n-1} is smooth outside the point $[x_0, \ldots, x_n] = [0, \ldots, 0, 1]$. The singularity at that point, in coordinates $x_n = 1$, is

$$X_{n-1} = \{0 = x_0^2 + \cdots + x_{n-1}^{s_{n-1}} + x_1 \cdots x_{n-1}\} \subset A^n,$$

in agreement with the notation of this lemma.

Let $D_{n-1} \subset X_{n-1}$ (inside Y) be the b-weighted tangent cone of D. Since D is $\mathbf{Q}$-Cartier, D_{n-1} is $\mathbf{Q}$-linearly equivalent to a rational multiple of $O_{X_{n-1}}(1)$. We have $\deg(D_{n-1}) = \operatorname{mult}_b(D) \leq s_n^{n-1}$, as shown above. By Lemma 4.4, it follows that in coordinates $\{x_n = 1\} \cong A^n$, and with weights $b_0, \ldots, b_{n-1}$ on A^n, we have

$$\begin{aligned} \operatorname{mult}_b(D_{n-1}) &\leq b_n \deg(D_{n-1}) \\ &\leq s_n^{n-1}(s_n + 1)/2, \end{aligned}$$

using that $b_n = (s_n + 1)/2$. Here $b_i = (s_n - 1)/s_i$ for $0 \leq i \leq n - 1$, and so our inductive assumption gives that $\operatorname{lct}_0(X_{n-1}, D_{n-1}) \geq \lambda_{n-1}/\operatorname{mult}_b(D_{n-1}) \geq 2\lambda_{n-1}/(s_n^{n-1}(s_n+1))$. This number is at least λ_n. Indeed, $\lambda_n \sim 2/s_n^{2n-2}$, and so λ_n divided by $2\lambda_{n-1}/(s_n^{n-1}(s_n + 1))$ is asymptotic to $1/2$. So it is less than 1 for large n, and a bit more calculation shows that it is less than 1 for all $n \geq 2$. (For $n = 2$, using that $\lambda_1 = 1$, this ratio is $3/4$, and for $n = 3$ this ratio is $28/33$.) Thus we have shown that the pair $(X_{n-1}, \lambda_n D_{n-1})$ is lc near the point $[x_0, \ldots, x_n] = [0, \ldots, 0, 1]$.

Since the stack X_{n-1} is smooth outside that point, we check easily that the pair $(X_{n-1}, \lambda_n D_{n-1})$ is lc on all of X_{n-1}. Namely, by Theorem 4.3, D_{n-1} has (unweighted) multiplicity at every point at most $b_0 \cdots b_{n-3} b_n \deg(D_{n-1}) \leq b_0 \cdots b_{n-3} b_n s_n^{n-1}$. Here $b_0 \cdots b_{n-3} = (s_n - 1)^{n-2}/(s_{n-2} - 1) \sim s_{n-2}^{4n-9}$, $b_n = (s_n + 1)/2 \sim s_{n-1}^4/2$, and $\lambda_n \sim 2/(s_{n-2}^{8n-8})$. So $u_n := \lambda_n b_0 \cdots b_{n-3} b_n s_n^{n-1} \sim 1/s_{n-2}$. This is less than 1 for n large. With a bit more calculation, we have $u_n \leq 1$ for all $n \geq 2$ (for example, $u_2 = 3/4$). So $\lambda_n D_{n-1}$ has multiplicity at most 1 at every point, for each $n \geq 2$. Therefore, the pair $(X_{n-1}, \lambda_n D_{n-1})$ is lc at points other than $[0, \ldots, 0, 1]$ as well as at that point.

By Lemma 4.1, the pair $(X_n, \lambda_n D)$ is lc near the origin if $(X_{n-1}, \lambda_n D_{n-1})$ is lc (as we have shown) and $K_{X_{n-1}} + \lambda_n D_{n-1} \sim_{\mathbf{Q}} O_{X_{n-1}}(r)$ with $r \leq 0$. Let us show that $r \leq 0$. By the adjunction formula, we have $-K_{X_{n-1}} = O_{X_{n-1}}(-e + \sum_j b_j) = O_X((s_n - 1)/2)$. To show that $r \leq 0$, it is equivalent to show that $\lambda_n \deg(D_{n-1}) = \lambda_n D_{n-1} \cdot c_1(O(1))^{n-2} \leq (-K_{X_1}) \cdot c_1(O(1))^{n-2}$. Here $(-K_{X_1}) \cdot c_1(O(1))^{n-2} = (s_n - 1)e/(2b_0 \cdots b_n) = 1/((s_n - 1)^{n-3}(s_n + 1))$. As a result, the inequality above holds if $\lambda_n (s_n - 1)^{n-3}(s_n + 1) \deg(D_{n-1}) \leq 1$. We have $\deg(D_{n-1}) = \operatorname{mult}_b(D) \leq s_n^{n-1}(s_n + 1)/2$. So the inequality above holds if $\lambda_n s_n^{n-1}(s_n + 1)^2(s_n - 1)^{n-3}/2 \leq 1$. In fact, λ_n has been chosen to make equality hold here. Therefore, the pair $(X_n, \lambda_n D)$ is lc near the origin, as we want. $\square$

6 Exceptionality of the klt Fano example in even dimensions

In order to show that the klt Fano variety X in Theorem 3.1 is exceptional, it suffices to show that its global log canonical threshold is greater than 1. It turns out that we can compute $\mathrm{glct}(X)$ exactly, and it is doubly exponentially large in terms of $n = \dim(X)$. In this section, we consider the case where n is even, which turns out to be simpler.

Theorem 6.1. *For each even number n at least 2, the klt Fano n-fold X in Theorem 3.1 is exceptional. More strongly, the global log canonical threshold of X is equal to $(s_n - 2)a_{n+1}/(s_n - 1) \sim s_n^2/4$. In particular, this is greater than 1.*

Proof. To recall the example: let $a_{n+1} = (s_n^2 - s_n + 2)/4$, $a_n = (s_n - 2)a_{n+1} - (s_n - 1)$, $x = -1 + a_n + a_{n+1}$, $d = (s_n - 1)x$, and $a_i = d/s_i$ for $0 \leq i \leq n-1$. Then X is a hypersurface of degree d in $Y := \mathbf{P}^{n+1}(a_0, \ldots, a_{n+1})$. Since $-d + \sum_j a_j = 1$, we have $-K_X = O_X(1)$. Let $\sigma_n = (s_n - 2)a_{n+1}/(s_n - 1)$. We have to show that for every effective $\mathbf{Q}$-divisor $D \sim_{\mathbf{Q}} -K_X$, the pair $(X, \sigma_n D)$ is lc, and that this bound is optimal. As a cycle on the stack Y, D has degree $d/(a_0 \cdots a_{n+1})$.

To see that the bound σ_n is optimal, let E be the hyperplane section $X \cap \{x_{n+1} = 0\}$; then we can take $D = (1/a_{n+1})E$. I claim that $(X, \sigma_n D)$ is not klt, or equivalently that $(X, ((s_n - 2)/(s_n - 1))E)$ is not klt. Here E is given by the equations $\{0 = x_{n+1},\ 0 = x_0^2 + \cdots + x_{n-1}^{s_{n-1}}\}$. We read off that E is irreducible. The stack E is singular at the point $[x_0, \ldots, x_n, x_{n+1}] = [0, \ldots, 0, 1, 0]$, where X is smooth. In coordinates $x_n = 1$, the singularity of E is étale-locally isomorphic to the Fermat-type hypersurface singularity $0 = x_0^2 + \cdots + x_{n-1}^{s_{n-1}}$ in A^n, which is known to have log canonical threshold equal to $\min(1, 1/2 + \cdots + 1/s_{n-1}) = (s_n - 2)/(s_n - 1)$ [24, Example 8.15]. That is, $(X, ((s_n - 2)/(s_n - 1))E)$ is lc but not klt at the point $[0, \ldots, 0, 1, 0]$, as we want.

It remains to show that the pair $(X, \sigma_n D)$ is lc for every effective $\mathbf{Q}$-divisor $D \sim_{\mathbf{Q}} -K_X$. The discrepancy of a $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor D on X at a given irreducible divisor over X is an affine-linear function of D [25, Lemma 2.5]. By considering a log resolution of $(X, D_1 + D_2)$, it follows that if the pairs (X, D_1) and (X, D_2) are lc, then so is $(X, (1-t)D_1 + tD_2)$ for any $t \in [0, 1]$. Since we have already handled the case where D is $1/a_{n+1}$ times the irreducible divisor E, it suffices to show that $(X, \sigma_n D)$ is lc when $D \sim_{\mathbf{Q}} -K_X$ and the support of D does not contain E.

In this case, no irreducible component of D is contained in the hyperplane $x_{n+1} = 0$. So Theorem 4.3 gives that D has multiplicity at every point at most $a_0 \cdots a_{n-2} a_{n+1} \deg(D) = d/(a_{n-1}a_n) = s_{n-1}/a_n$. Therefore, $\sigma_n D$ has multiplicity at every point at most $s_{n-1}(s_n - 2)a_{n+1}/((s_n - 1)a_n) \sim 1/s_{n-1}$. This is less than 1. So $(X, \sigma_n D)$ is lc at all smooth points of the stack X [25, Claim 2.10.4], hence at all points other than $[x_0, \ldots, x_n, x_{n+1}] = [0, \ldots, 0, 1]$. In order to handle that point, we will switch to analyzing a certain weighted multiplicity of D.

In coordinates $x_{n+1} = 1$, X becomes the hypersurface $0 = x_0^2 + x_1^3 + \cdots + x_{n-1}^{s_{n-1}} + x_n^{s_n} + x_1 \cdots x_n$ in A^{n+1}. We want to show that $(X, \sigma_n D)$ is lc near the origin. Consider the weights $a_0, \ldots, a_n$ on A^{n+1}. Then the a-weighted multiplicity of D at the origin in A^{n+1} satisfies

$$\begin{aligned}\operatorname{mult}_a(D) &\leq a_{n+1}\deg(D)\\ &= d/(a_0\cdots a_n),\end{aligned}$$

by Lemma 4.4.

Consider the modified weights $w_i = a_i = d/s_i$ for $0 \leq i \leq n-1$, and $w_n = d/s_n$. Here w_n is not an integer, but we can still define multiplicity for positive rational weights by scaling. We have $w_n = (s_n - 1)x/s_n = x - (x/s_n)$. Also, $x = -1 + a_n + a_{n+1} = (s_n - 1)a_{n+1} - s_n < s_n(a_{n+1} - 1)$. So $x/s_n < a_{n+1} - 1$, and hence $w_n = x - (x/s_n) > x - (a_{n+1} - 1) = a_n$. It follows that $\operatorname{mult}_w(D) \leq \operatorname{mult}_a(D) \leq d/(a_0 \cdots a_n)$.

Next, let $c_i = s_n w_i/x$ for all $0 \leq i \leq n$; then $c_i = (s_{n+1} - 1)/s_i$ for all $0 \leq i \leq n$. Since D has dimension $n-1$, we have $\operatorname{mult}_c(D) = (x/s_n)^{n-1}\operatorname{mult}_w(D) \leq x^{n-1}d/(s_n^{n-1}a_0\cdots a_n) = 1/(s_n^{n-1}(s_n-1)^{n-2}a_n)$. By Lemma 5.1, it follows that (X, e_nD) is lc near the point $[0, \ldots, 0, 1]$, where we let

$$\begin{aligned}e_n &= \frac{2}{s_n^{n-1}(s_n+1)^2(s_n-1)^{n-3}} \cdot s_n^{n-1}(s_n-1)^{n-2}a_n\\ &= \frac{2(s_n-1)a_n}{(s_n+1)^2}.\end{aligned}$$

It remains to show that this number e_n is at least σ_n, for every even number $n \geq 2$. The ratio e_n/σ_n is $2(s_n-1)^2a_n/((s_n+1)^2(s_n-2)a_{n+1})$. Since $a_{n+1} \sim s_n^2/4$ and $a_n \sim s_n^3/4$, this ratio is asymptotic to 2, so it is greater than 1 for n large. With a bit more calculation, we find that $e_n/\sigma_n > 1$ for every even number $n \geq 2$. (For example, $e_2/\sigma_2 = 441/440$.) That completes the proof that $\operatorname{glct}(X) = \sigma_n$. Since this is greater than 1, the klt Fano variety X is exceptional. □

7 Exceptionality of the klt Fano example in odd dimensions

We now prove the exceptionality of our klt Fano example in odd dimensions.

Theorem 7.1. *For each odd number n at least 3, the klt Fano n-fold X in Theorem 3.1 is exceptional. More strongly, the global log canonical threshold of X is equal to $(s_n - 2)a_{n+1}/(s_n - 1) \sim s_n^2/4$. In particular, this is greater than 1.*

Proof. To recall the example: let $a_{n+1} = (s_n^2 - 3s_n + 4)/4$, $a_n = (s_n - 2)a_{n+1} - (s_n - 1)$, $x = -1 + a_n + a_{n+1}$, $d = (s_n - 1)x$, and $a_i = d/s_i$ for $0 \leq i \leq n-1$. Then X is a hypersurface of degree d in $Y := \mathbf{P}^{n+1}(a_0, \ldots, a_{n+1})$. Since $-d + \sum_j a_j = 1$, we have $-K_X = O_X(1)$. Let $\sigma_n = (s_n - 2)a_{n+1}/(s_n - 1)$. We have to show that for every effective $\mathbf{Q}$-divisor $D \sim_{\mathbf{Q}} -K_X$, the pair $(X, \sigma_n D)$ is lc, and that this bound is optimal. As a cycle on the stack Y, D has degree $d/(a_0 \cdots a_{n+1})$.

To see that the bound σ_n is optimal, let E be the hyperplane section $X \cap \{x_{n+1} = 0\}$; then we can take $D = (1/a_{n+1})E$. I claim that $(X, \sigma_n D)$ is not klt, or equivalently that $(X, ((s_n - 2)/(s_n - 1))E)$ is not klt. Here E is given by the equations $\{0 = x_{n+1},\ 0 = x_0^2 + \cdots + x_{n-1}^{s_{n-1}}\}$, which in particular shows that E is irreducible. The stack E is singular at the point $[x_0, \ldots, x_n, x_{n+1}] = [0, \ldots, 0, 1, 0]$, where X is smooth. In coordinates $x_{n+1} = 1$, the singularity of E is étale-locally isomorphic to the Fermat-type hypersurface singularity $0 = x_0^2 + \cdots + x_{n-1}^{s_{n-1}}$ in A^n, which is known to have log canonical threshold equal to $\min(1, 1/2 + \cdots + 1/s_{n-1}) = (s_n - 2)/(s_n - 1)$ [24, Example 8.15]. That is, $(X, ((s_n - 2)/(s_n - 1))E)$ is lc but not klt at the point $[0, \ldots, 0, 1, 0]$, as we want.

It remains to show that the pair $(X, \sigma_n D)$ is lc for every effective $\mathbf{Q}$-divisor $D \sim_{\mathbf{Q}} -K_X$. The discrepancy of a $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor D on X at a given irreducible divisor over X is an affine-linear function of D [25, Lemma 2.5]. By considering a log resolution of $(X, D_1 + D_2)$, it follows that if the pairs (X, D_1) and (X, D_2) are lc, then so is $(X, (1 - t)D_1 + tD_2)$ for any $t \in [0, 1]$. Since we have already handled the case where D is $1/a_{n+1}$ times the irreducible divisor E, it suffices to show that $(X, \sigma_n D)$ is lc when $D \sim_{\mathbf{Q}} -K_X$ and the support of D does not contain E.

In this case, no irreducible component of D is contained in the hyperplane $x_{n+1} = 0$. So Theorem 4.3 gives that D has multiplicity at every point at most $a_0 \cdots a_{n-2} a_{n+1} \deg(D) = d/(a_{n-1} a_n) = s_{n-1}/a_n$. Therefore, $\sigma_n D$ has multiplicity at every point at most $s_{n-1}(s_n - 2)a_{n+1}/((s_n - 1)a_n) \sim 1/s_{n-1}$. This is less than 1. So $(X, \sigma_n D)$ is lc at all smooth points of the stack X [25, Claim 2.10.4], hence at all points other than $[x_0, \ldots, x_n, x_{n+1}] = [0, \ldots, 0, 1]$. In order to handle that point, we will switch to analyzing a certain weighted multiplicity of D.

In coordinates $x_{n+1} = 1$, X becomes the hypersurface $0 = x_0^2 + x_1^3 + \cdots + x_{n-1}^{s_{n-1}} + x_n^{s_n} + x_1 \cdots x_{n-1} x_n^2$ in A^{n+1}. We want to show that $(X, \sigma_n D)$ is lc near the origin. Consider the weights $a_0, \ldots, a_n$ on A^{n+1}. Then the a-weighted multiplicity of D at the origin in A^{n+1} satisfies

$$\begin{aligned} \operatorname{mult}_a(D) &\leq a_{n+1} \deg(D) \\ &= d/(a_0 \cdots a_n), \end{aligned}$$

by Lemma 4.4.

Let $w_i = a_i$ for $0 \leq i \leq n - 1$, and let $w_n = (s_n + 1)x/4$. Since $w_n > a_n$, we have $\operatorname{mult}_w(D) \leq \operatorname{mult}_a(D) \leq d/(a_0 \cdots a_n)$. To simplify the numbering, let $c_i = w_i/x$ for all $0 \leq i \leq n$; then $\operatorname{mult}_c(D) = x^{n-1} \operatorname{mult}_w(D) \leq x^{n-1} d/(a_0 \cdots a_n)$. Going back through the definitions, this means that $c_i = (s_n - 1)/s_i$ for $0 \leq i \leq n - 1$ and $c_n = (s_n + 1)/4$. Also, let $e = s_n - 1$, so that $d = (s_n - 1)x = ex$. Then we can rewrite our bound as $\operatorname{mult}_c(D) \leq e/(c_0 \cdots c_{n-1} a_n)$. (Note that the largest numbers from $c_0, \ldots, c_n$ are c_0, c_1, c_n, contrary to our usual convention.)

The reason for choosing the weights $c_0, \ldots, c_n$ is that with these weights, the weighted tangent cone at the origin to the hypersurface $X \subset A^{n+1}$ is klt: it is the hypersurface $S \subset \mathbf{P}^n(c_0, \ldots, c_n)$ of degree e defined by

$$0 = x_0^2 + x_1^3 + \cdots + x_{n-1}^{s_{n-1}} + x_1 \cdots x_{n-1} x_n^2.$$

The stack S is smooth outside the point $[x_0, \ldots, x_n] = [0, \ldots, 0, 1]$. The singularity of S at that point, in coordinates $x_n = 1$, is $0 = x_0^2 + x_1^3 + \cdots + x_{n-1}^{s_{n-1}} + x_1 \cdots x_{n-1}$. This is exactly the singularity of dimension $n - 1$ in Lemma 5.1.

Let F be the weighted tangent cone of D at the origin in A^{n+1}, so that F is an effective $\mathbf{Q}$-divisor in $S \subset \mathbf{P}^n(c_0, \ldots, c_n)$. Since D is $\mathbf{Q}$-Cartier, F is $\mathbf{Q}$-linearly equivalent to a rational multiple of $O_S(1)$. By inversion of adjunction as in the proof of Lemma 4.1, the pair $(X, \sigma_n D)$ is lc if $(S, \sigma_n F)$ is lc and $K_S + \sigma_n F \sim_{\mathbf{Q}} O_S(r)$ with $r \leq 0$.

We first check that $r \leq 0$. By the adjunction formula, we have $-K_S = O_S(-e + \sum_j c_j) = O_X((s_n - 3)/4)$. To show that $r \leq 0$, it is equivalent to show that $\sigma_n \deg(F) = \sigma_n F \cdot c_1(O(1))^{n-2} \leq (-K_S) \cdot c_1(O(1))^{n-2}$. Here $(-K_S) \cdot c_1(O(1))^{n-2} = (s_n - 3)e/(4c_0 \cdots c_n)$. As a result, the inequality above holds if $\sigma_n \leq (s_n - 3)e/(4c_0 \cdots c_n \deg(F))$. We have $\deg(F) = \mathrm{mult}_c(D) \leq e/(c_0 \cdots c_{n-1}a_n)$. So the inequality above holds if $\sigma_n \leq g_n := (s_n - 3)a_n/(4c_n)$. Here $a_n \sim s_n^3/4$ and $c_n \sim s_n/4$, and so $g_n \sim s_n^3$, whereas $\sigma_n \sim s_n^2/4$. With a bit more calculation, we see that g_n is greater than σ_n for each odd $n \geq 3$. (For example, $g_3 \doteq 16026.4$ and $\sigma_3 \doteq 420.7$.)

It remains to show that $(S, \sigma_n F)$ is lc. Since the stack S is smooth outside the point $[x_0, \ldots, x_n] = [0, \ldots, 0, 1]$, it is easy to show that $(S, \sigma_n F)$ is lc outside that point. Namely, F has degree at most $e/(c_0 \cdots c_{n-1}a_n)$. By Johnson-Kollár's bound (Theorem 4.3), in every orbifold chart $\{x_i \neq 0\}$ and at every point, F has multiplicity at most $c_0 \cdots c_{n-3}c_n \deg(F)$ (if $n \geq 5$) or $c_0c_1 \deg(F)$ (if $n = 3$). So F has multiplicity at every point at most $ec_n/(c_{n-2}c_{n-1}a_n)$ if $n \geq 5$, or $e/(c_2a_n)$ if $n = 3$. So $\sigma_n F$ has multiplicity at every point at most $\sigma_n ec_n/(c_{n-2}c_{n-1}a_n)$ if $n \geq 5$, or $\sigma_3 e/(c_2a_n) \doteq 0.17$ if $n = 3$. In particular, this is less than 1 for $n = 3$.

For $n \geq 5$, we have $e \sim s_n \sim s_{n-2}^4$, $c_{n-2} \sim s_n/s_{n-2} \sim s_{n-2}^3$, $c_{n-1} \sim s_n/s_{n-1} \sim s_{n-2}^2$, $c_n \sim s_n/4 \sim s_{n-2}^4/4$, $a_n \sim s_n^3/4 \sim s_{n-2}^{12}/4$, and $\sigma_n \sim s_n^2/4 \sim s_{n-2}^8/4$. So $e\sigma_n c_n/(c_{n-2}c_{n-1}a_n) \sim 1/(4s_{n-2})$, which is less than 1 for n large. With a bit more calculation, it is less than 1 for every odd number $n \geq 5$. (For example, for $n = 5$, it is about 0.024.) So, for each odd number $n \geq 3$, $\sigma_n F$ has multiplicity less than 1 everywhere. Since the stack S is smooth outside the point $[x_0, \ldots, x_n] = [0, \ldots, 0, 1]$, it follows that $(S, \sigma_n F)$ is lc outside that point.

In coordinates $x_n = 1$, F corresponds to a codimension-2 cycle on A^n. Using weights $c_0, \ldots, c_{n-1}$ on A^n, Lemma 4.4 gives that the weighted multiplicity of F at the origin in A^n satisfies

$$\begin{aligned} \mathrm{mult}_c(F) &\leq c_n \deg(F) \\ &\leq ec_n/(c_0 \cdots c_{n-1}a_n) \\ &= \frac{s_n + 1}{4(s_n - 1)^{n-2}a_n}. \end{aligned}$$

The weights $c_0, \ldots, c_{n-1}$ are those considered in Lemma 5.1 to analyze the hypersurface $X_{n-1} \subset A^n$. That lemma gives that $(S, \sigma_n F)$ is lc near the point $[x_0, \ldots, x_n] = [0 \ldots, 0, 1]$ if

$$\sigma_n \leq \frac{2}{s_{n-1}^{n-2}(s_{n-1} + 1)^2(s_{n-1} - 1)^{n-4} \,\mathrm{mult}_c(F)},$$

hence if

$$\sigma_n \leq \frac{8(s_n - 1)^{n-2}a_n}{s_{n-1}^{n-2}(s_{n-1} + 1)^2(s_{n-1} - 1)^{n-4}(s_n + 1)}.$$

I claim that this fraction f_n is greater than σ_n for every odd number $n \geq 3$. We have $a_n \sim s_n^3/4$, and so $f_n \sim s_n^2$, whereas $\sigma_n \sim s_n^2/4$. In particular, $f_n > \sigma_n$ for n sufficiently large. With a bit more calculation, we find that $f_n > \sigma_n$ for all odd $n \geq 3$. (For example, $f_3 \doteq 1803.0$ and $\sigma_3 \doteq 420.7$.) That completes the proof that $\mathrm{glct}(X) = \sigma_n$. Since this is greater than 1, the klt Fano variety X is exceptional. $\square$

8 Klt Fano varieties with large bottom weight

The *bottom weight* of a Fano variety X means the smallest positive integer m such that $H^0(X, -mK_X) \neq 0$. The following klt Fano variety has the largest known bottom weight in even dimensions n at least 4, asymptotic to $\frac{5}{9}s_n^2$ (hence, crudely, about 2^{2^n}). We know that there is some upper bound for the bottom weight of klt Fano varieties in each dimension, by Birkar's theorem on boundedness of complements [4, Theorem 1.1].

In particular, Theorem 8.1 beats the examples by Wang and me of klt Fano varieties with large bottom weight [32, Theorem 5.1]. The example here is not quasi-smooth.

Theorem 8.1. *For each even integer n at least 4, let $a_{n+1} = \frac{1}{36}(20s_n^2 - 295s_n + 113)$ and $a_n = \frac{1}{36}(20s_n^2 - 55s_n + 17)$. Let $x = -1 + a_n + a_{n+1}$, $d = (s_n - 1)x$, and $a_i = d/s_i$ for $0 \leq i \leq n-1$. Then a general hypersurface X of degree d in $\mathbf{P}^{n+1}(a_0, \ldots, a_{n+1})$ is well-formed and is a klt Fano variety, with $-K_X = O_X(1)$. Its bottom weight is a_{n+1}, which is asymptotic to $\frac{5}{9}s_n^2$.*

Moraga conjectured that every Fano type variety of dimension n has an N-complement for some $N \leq (2s_n - 3)(s_n - 1)$ [29, Conjecture 4.1]. Theorem 8.1 implies that this bound, if true, would be optimal up to a constant factor.

It would be interesting to compute the global log canonical threshold for these examples. The global log canonical threshold of a Fano variety is at most the bottom weight, and it seems to be close to the bottom weight when the bottom weight is large.

Question 8.2. For each even number n at least 4, does the variety in Theorem 8.1 have the largest bottom weight and the largest glct among all klt Fano n-folds?

I speculate that the optimal examples in odd dimensions will also have bottom weight and glct asymptotic to $\frac{5}{9}s_n^2$.

The best examples I know in low dimensions are as follows. I conjecture that the klt del Pezzo surface with largest bottom weight and largest glct is $X_{154} \subset \mathbf{P}^3(77, 45, 19, 14)$, for which $\text{glct}(X) = 21/2 = 10.5$; this is a non-quasi-smooth hypersurface, apparently new. (The first known klt del Pezzo surface with bottom weight 14 was Kim-Park's quasi-smooth complete intersection $X_{64,70} \subset \mathbf{P}^4(45, 32, 25, 19, 14)$, which has glct equal to $28/3 \doteq 9.33$ [22, Table 2].)

I conjecture that the klt Fano 3-fold with largest bottom weight and largest glct is another non-quasi-smooth hypersurface, introduced here:

$$X_{65418} \subset \mathbf{P}^4(32709, 21806, 9233, 884, 787),$$

with equation $0 = x_0^2 + x_1^3 + x_2^7 x_4 + x_1 x_2 x_3^{38} x_4 + x_3 x_4^{82}$. The largest previously known bottom weight of a klt Fano 3-fold occurs for Johnson-Kollár's quasi-smooth hypersurface

$$X_{37584} \subset \mathbf{P}^4(18792, 12528, 5311, 547, 407)$$

[20, Introduction].

Finally, the klt Fano 4-fold in Theorem 8.1 has bottom weight 1799223. The largest previously known bottom weight of a klt Fano 4-fold is 1094225, which occurs for the quasi-smooth hypersurface [8, ID 1228436].

Proof. (Theorem 8.1) By the properties of the Sylvester sequence, we have $d - \sum a_i = -1$. Also, we compute that the equation of X includes at least the monomials $0 = x_0^2 + x_1^3 + \cdots + x_{n-1}^{s_{n-1}} + x_1 x_n^b x_{n+1} + x_2 \cdots x_{n-1} x_n^{12} x_{n+1}^c$, where $b = (4s_n - 31)/3$ and $c = (5s_n - 5)/3$.

We first check that a_{n+1} and a_n are integers. The denominator 36 factors as $2^2 3^2$. Since n is even and at least 2, we have $s_n \equiv -1 \pmod 8$ by induction from the definition of the Sylvester sequence, as in the proof of Theorem 2.1. So $20s_n^2 - 295s_n + 113 \equiv 20(-1)^2 - 295(-1) + 113 \equiv 4 \pmod 8$. It follows that a_{n+1} is integral at 2 and odd. Let $e = a_n - a_{n+1} = \frac{1}{3}(20s_n - 8)$. We see that e is integral at 2 and even, and so a_n is integral at 2 and odd. Next, we have $s_n \equiv -2 \pmod 9$ since $n \geq 2$. Write $s_n = 9t - 2$ for an integer t. Then $a_{n+1} = \frac{1}{36}(20s_n^2 - 295s_n + 113) = \frac{1}{4}(180t^2 - 375t + 87) \equiv 0 \pmod 3$. Also, $e = \frac{1}{3}(20s_n - 8) = 60t - 16 \equiv -1 \pmod 3$. So a_{n+1} and a_n are integers, both are odd, and $a_{n+1} \equiv 0 \pmod 3$ while $a_n \equiv -1 \pmod 3$. It is also immediate from the definitions that a_{n+1} and a_n are nonzero modulo 5.

Let us show that the weighted projective space $Y = \mathbf{P}^{n+1}(a_0, \ldots, a_{n+1})$ is well-formed. That is, we have to show that $\gcd(a_0 \ldots, \widehat{a_j}, \ldots, a_{n+1}) = 1$ for each j. Let $x = -1 + a_n + a_{n+1}$. It suffices to show that $\gcd(a_{n+1}, x) = 1$, $\gcd(a_n, x) = 1$, and $\gcd(a_{n+1}, a_n, s_n - 1) = 1$.

We first show that $\gcd(a_{n+1}, x) = 1$. Let p be a prime number that divides both a_{n+1} and x. We know that a_{n+1} is odd and not a multiple of 5, and $x \equiv -1 + a_n + a_{n+1} \equiv -1 - 1 + 0 \equiv -2 \pmod 3$; so we must have $p > 5$. Since $x = -1 + 2a_{n+1} + e$, we have $e \equiv 1 \pmod p$. That is, $\frac{1}{3}(20s_n - 8) \equiv 1 \pmod p$, and so $20s_n \equiv 11 \pmod p$. (Since 3 is invertible in the field $\mathbf{Z}/p$, the fraction $\frac{1}{3}$ makes sense as an element of $\mathbf{Z}/p$.) It follows that $s_n \equiv 11/20 \pmod p$. So $0 \equiv a_{n+1} \equiv (1/36)(20(11/20)^2 - 295(11/20) + 113) \equiv -6/5 \pmod p$. So p is 2 or 3, contradiction. So $\gcd(a_{n+1}, x) = 1$.

Next, we show that $\gcd(a_n, x) = 1$. Let p be a prime number that divides a_n and x. Since a_n is not a multiple of 2, 3, or 5, we must have $p > 5$. Since $x = -1 + a_n + a_{n+1}$, we have $a_{n+1} \equiv 1 \pmod p$ and hence $e \equiv -1 \pmod p$. That is, $\frac{1}{3}(20s_n - 8) \equiv -1 \pmod p$, so $20s_n \equiv 5 \pmod p$, and hence $s_n \equiv 1/4 \pmod p$. So $0 \equiv a_n \equiv (1/36)(20(1/4)^2 - 55(1/4) + 17) \equiv 1/8 \pmod p$, a contradiction. (Since 36 is invertible in the field $\mathbf{Z}/p$, the fraction $1/36$ makes sense in $\mathbf{Z}/p$.) So $\gcd(a_n, x) = 1$.

Finally, we show that $\gcd(a_n, a_{n+1}, s_n - 1) = 1$ (and in fact $\gcd(a_n, s_n - 1) = 1$). Let p be a prime number that divides a_n and $s_n - 1$. In particular, $p > 5$ because p divides a_n. We have $s_n \equiv 1 \pmod p$, and so $0 \equiv a_n \equiv (1/36)(20(1)^2 - 55(1) + 17) \equiv -1/2$, contradiction. This completes the proof that Y is well-formed.

To show that X is well-formed, it remains to show that X does not contain any $(n-1)$-dimensional coordinate linear subspace of Y along which the variety Y is singular (that is, where the corresponding smooth stack has nontrivial stabilizer group). Since the equation of X includes the monomials $x_0^2, x_1^3, \ldots, x_{n-1}^{s_{n-1}}$, X contains at most one positive-dimensional coordinate linear subspace of Y, the projective line Z given by $0 = x_0 = \cdots = x_{n-1}$. Since $n \geq 4$, it follows that X is well-formed. Also, the hypersurface X is normal, by Serre's criterion, using that X is quasi-smooth outside the curve Z. Then adjunction applies: we have $K_X = O_X(d - \sum a_i) = O_X(-1)$, as we want.

Next, we show that a general hypersurface X of degree d in Y is klt. As we have said, X is quasi-smooth (and hence klt) outside the projective line Z. It remains to show that X is klt in the open subsets $x_n \neq 0$ and $x_{n+1} \neq 0$.

In coordinates $x_n = 1$, the equation of X includes the monomials $0 = x_0^2 + x_1^3 + \cdots + x_{n-1}^{s_{n-1}} + x_1 x_{n+1} + x_2 \cdots x_{n-1} x_{n+1}^c$, where $c = (5s_n - 5)/3$. By Theorem 2.3, it suffices to show that the convex hull of the points $(2, 0, \ldots, 0)$, $(0, 3, 0, \ldots, 0)$, $\ldots$, $(0, \ldots, 0, s_{n-1}, 0)$, $(0, 1, 0, \ldots, 0, 1)$, $(0, 0, 1, \ldots, 1, c)$ in $\mathbf{R}^{n+1}$ contains a point with all coordinates less than 1. In fact, we only need two of these points: namely, $(1/3)(2, 0, \ldots, 0) + (2/3)(0, 1, 0, \ldots, 0, 1)$ has all coordinates less than 1. Thus X is klt in the open set $x_n \neq 0$.

It remains to analyze X in coordinates $x_{n+1} = 1$. The equation of X includes the monomials $0 = x_0^2 + x_1^3 + \cdots + x_{n-1}^{s_{n-1}} + x_1 x_n^b + x_2 \cdots x_{n-1} x_n^{12}$. So it suffices to show that the convex hull of the points $(2, 0, \ldots, 0)$, $(0, 3, 0, \ldots, 0)$, $\ldots$, $(0, \ldots, 0, s_{n-1}, 0)$, $(0, 1, 0, \ldots, 0, b)$, $(0, 0, 1, \ldots, 1, 12)$ in $\mathbf{R}^{n+1}$ contains a point with all coordinates less than 1. Indeed, the point $(1/2 - \epsilon)(2, 0, \ldots, 0) + (1/3 - \epsilon)(0, 3, 0, \ldots, 0) + (1/12 + 3\epsilon)(0, 0, 7, 0, \ldots, 0) + (1/12 - \epsilon)(0, 0, 1, \ldots, 1, 12)$ has all coordinates less than 1 if $0 < \epsilon < 1/60$. This completes the proof that X is klt. $\square$

9 Klt varieties with ample canonical class and large bottom weight

The *bottom weight* of a projective variety X with K_X ample means the smallest positive integer m such that $H^0(X, mK_X) \neq 0$. The following klt variety with ample canonical class has the largest known bottom weight in even dimensions n at least 4, asymptotic to $\frac{5}{9} s_n^2$ (hence, crudely, about 2^{2^n}). (In other words, we are exhibiting klt varieties with ample canonical class that have many vanishing plurigenera.) We know that there is some upper bound for the bottom weight in each dimension, by Hacon-M$^{\mathrm{c}}$Kernan-Xu [16, Theorem 1.3].

In particular, Theorem 9.1 beats the examples by Wang and me of klt varieties with ample canonical class and large bottom weight [32, Remark 4.2]. The example here is not quasi-smooth.

Theorem 9.1. *For each even integer n at least 4, let $a_{n+1} = \frac{1}{36}(20s_n^2 - 415s_n + 161)$ and $a_n = \frac{1}{36}(20s_n^2 - 175s_n + 65)$. Let $x = 1 + a_n + a_{n+1}$, $d = (s_n - 1)x$, and $a_i = d/s_i$ for $0 \leq i \leq n-1$. Then a general hypersurface X of degree d in $\mathbf{P}^{n+1}(a_0, \ldots, a_{n+1})$ is well-formed and is a klt variety with $K_X = O_X(1)$. Its bottom weight is a_{n+1}, which is asymptotic to $\frac{5}{9} s_n^2$.*

The proof is completely parallel to that of Theorem 8.1 and hence is omitted. The proof uses that the equation of X includes the monomials $0 = x_0^2 + \cdots + x_{n-1}^{s_{n-1}} + x_1 x_n^{13} x_{n+1}^b + x_2 \cdots x_{n-1} x_n^c x_{n+1}^7$, where $b = (4s_n - 19)/3$ and $c = (5s_n - 50)/3$. Here X is not quasi-smooth.

The variety in Theorem 9.1 should also have large *global log canonical threshold* $\mathrm{glct}(X) := \mathrm{lct}(X, K_X)$. (For varieties with ample canonical class, this invariant was first studied by J. Chen, M. Chen, and C. Jiang [10, Section 2.5].) The global log canonical threshold of a variety with ample canonical class is at most the bottom weight, and it seems to be close to the bottom weight when the bottom weight is large.

Question 9.2. For each even number n at least 4, does the variety in Theorem 9.1 have the largest bottom weight and the largest glct among all klt projective n-folds with ample canonical class?

I speculate that the optimal examples in odd dimensions will also have bottom weight asymptotic to $\frac{5}{9}s_n^2$.

The best examples I know in low dimensions are as follows. I conjecture that the klt surface with ample canonical class of largest bottom weight and largest glct is $X_{182} \subset \mathbf{P}^3(91, 53, 23, 14)$; this is a non-quasi-smooth hypersurface, apparently new. By Brown and Kasprzyk's programs, the largest bottom weight for a quasi-smooth hypersurface with $K_X = O_X(1)$ is 13, which occurs for $X_{316} \subset \mathbf{P}^3(158, 101, 43, 13)$ and $X_{159} \subset \mathbf{P}^3(73, 43, 29, 13)$ [8].

I conjecture that the klt 3-fold with ample canonical class of largest bottom weight and largest glct is another non-quasi-smooth hypersurface, introduced here:

$$X_{72954} \subset \mathbf{P}^4(36477, 24318, 10422, 943, 793),$$

with equation $0 = x_0^2 + x_1^3 + x_2^7 + x_0 x_3^{37} x_4^2 + x_1 x_2 x_3 x_4^{47}$. The largest previously known bottom weight of a klt 3-fold with ample canonical class occurs for the quasi-smooth hypersurface

$$X_{18174} \subset \mathbf{P}^4(8854, 5889, 2457, 507, 466)$$

[8].

Finally, the klt 4-fold with ample canonical class in Theorem 9.1 has bottom weight 1793201. The largest previously known bottom weight of a klt 4-fold with ample canonical class is 1127113, which occurs for the quasi-smooth hypersurface [8, ID 534198].

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